POLYGORIALS
SPECIAL "FACTORIALS" OF POLYGONAL NUMBERS

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Abstract. We consider, define and demonstrate a special class of numbers, called "polygorials", which are a generalized form of factorial derived from polygonal numbers of size k.

Long interested in polygonal numbers, I recently returned to them after reading Nikomachus' treatment in book ii of his $A\rho\iota\theta\mu\eta\tau\iota\kappa\eta$ $E\iota\sigma\alpha\gamma\omega\gamma\eta$ [Hoche]. In the process, I became interested in the idea of a "factorial," as it were, formed as the product of the series of k-gonal numbers; in the following, I present what emerged from pursuing that interest.

Before continuing, I will for the purpose of foundation attempt to quickly review

1. Polygonal Numbers

The first, simplest form is $k = 3$, which produces the *triangular numbers*, so named because each was formed by an arrangement of pebbles in the shape of an equilateral triangle (Figure 1).

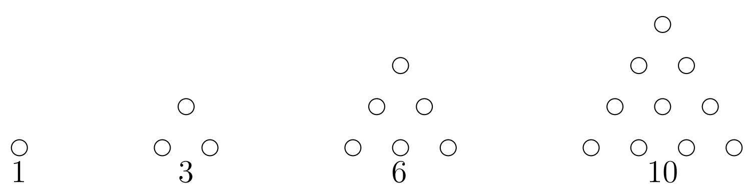

Figure 1. Triangular Numbers

Nikomachus noted that every n^{th} *polygonal number* is formed by taking the sum of every $k-2^{\text{th}}$ number from 1 to n. Another way of saying it is to note that every successive figure, P_n^k, is formed by adding to P_{n-1}^k a *gnomon* having $(k-2)$ sides of n points per side. Since the connecting points of each such side overlap, the total number of points in a gnomon is equal to $(k-2)n-(k-3)$. Using this, we can say

$$P_n^k = \sum_{i=1}^{n} \left((k-2)\,i-(k-3)\right) \tag{1.1}$$

The triangular numbers, then, can be generated by

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$$T_n = \sum_{i=1}^{n} \left((3-2)\, i - (3-3) \right) = \sum_{i=1}^{n} i = \frac{n\,(n+1)}{2} \tag{1.2}$$

or, by taking the sum of sequential integers, 1 to n. In the same way, the *squares* (Figure 2) can be generated by taking the sum of every second integer from 1 to n.

$$S_n = \sum_{i=1}^{n} \left((4-2)\, i - (4-3) \right) = \sum_{i=1}^{n} (2i-1) = n^2 \tag{1.3}$$

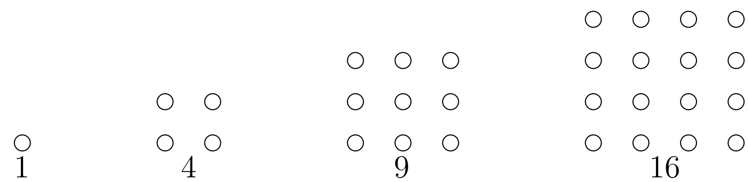

Figure 2. Square Numbers

The *pentagonal numbers* (Figure 3), by taking the sum of every third.

$$\text{Pent}_n = \sum_{i=1}^{n} \left((5-2)\, i - (5-3) \right) = \sum_{i=1}^{n} (3i-2) = \frac{n\,(3n-1)}{2} \tag{1.4}$$

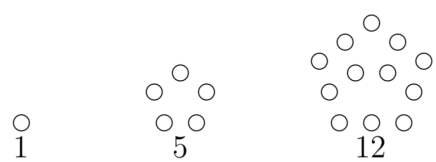

Figure 3. Pentagonal Numbers

The same holds true for any k-gonal number however far you would like to take them.

Remark 1.1. There are naturally other ways to approach the generation of polygonal numbers. For instance, in book two, cap. xii, Nikomachus observed that every square is the sum of two successive triangles, such that $S_n = T_{n-1} + T_n$. He then noted that every pentagon, Pent_n, is $T_{n-1} + S_n$. Given the triangular definition of the squares, we can say that $\text{Pent}_n = T_n + 2T_{n-1}$. Going on, he stated that, for *hexagonalnumbers*, $\text{Hex}_n = \text{Pent}_n + T_{n-1}$; expanding by the triangular form of Pent_n, we can say $\text{Hex}_n = T_n + 3T_{n-1}$. It holds that every polygonal number P_n^k equals $P_n^{k-1} + P_{n-1}^3$. This allows us to create an alternate formula for (1.1) based on triangular numbers:

$$P_n^k = T_n + (k-3)\, T_{n-1} \tag{1.5}$$

Or, explicitly—given $T_n = n\,(n+1)/2$:

$$P_n^k = \frac{n\,(nk - 2n - k + 4)}{2} \tag{1.6}$$

2. Factorials

A *factorial*, designated by $n!$, is simply

$$\prod_{i=1}^{n} i \tag{2.1}$$

So, whereas a triangular number is the sum of the first n integers greater than zero, a factorial is their product. The idea which captured my interest is what I have termed

3. Polygorials

The name is a simple coinage intended to reflect the two primary items involved: *polygo*nal numbers and fact*orials*. The idea is that instead of taking the product of sequential integers, for a *polygorial* one takes the product of sequential polygonal numbers for some base k. For instance, $\mathcal{P}_5^3$ is 2700 since $P_1^3 = 1, P_2^3 = 3, P_3^3 = 6, P_4^3 = 10, P_5^3 = 15$ and $1 \cdot 3 \cdot 6 \cdot 10 \cdot 15 = 2700$. We can define $\mathcal{P}_n^k$ with

$$\mathcal{P}_n^k = \prod_{i=1}^{n} P_i^k \tag{3.1}$$

Exploring this, we can make a few observations about the sequences produced. For example, if we evaluate $\mathcal{P}_n^3$ for $0 \le n \le 17$,[1] we get

$$1, 1, 3, 18, 180, 2700, 56700, 1587600, 57153600, 2571912000, 141455160000, 9336040560000, 728211163680000, 66267215894880000, 6958057668962400000, 834966920275488000000, 113555501157466368000000, 17373991677092354304000000$$

which we find to be A006472 [EIS]. We can generate these directly with

$$\mathcal{P}_n^3 = 2^{-n} n!^2 (n+1) \tag{3.2}$$

The *tetragorials*—square polygorials, $\mathcal{P}_n^4$—are

$$1, 1, 4, 36, 576, 14400, 518400, 25401600, 1625702400, 131681894400, 13168189440000, 1593350922240000, 229442532802560000, 38775788043632640000, 7600054456551997440000, 1710012252724199424000000, 437763136697395052544000000, 126513546505547170185216000000$$

[1] For each example we will use this same range for n.

which is A001044. Cloitre [EIS] noted that when M_n is a symmetrical $n \times n$ matrix such that $M_n(i,j) = 1/max(i,j)$, the determinant of M_n, for $n > 0$, is $1/\mathcal{P}_n^4$.

$$M_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \det(M_4) = \frac{1}{576}$$

Murthy [EIS] points out that for M_n $(n > 0)$ as an $n \times n$ matrix with $M_n(i,j) = ij$, $\mathcal{P}_n^4$ is the product of the n^{th} antidiagonal.

$$M_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 16 & 20 \\ 5 & 10 & 15 & 20 & 25 \end{pmatrix}$$

$$\begin{aligned} \mathcal{P}_1^4 &= 1, \\ \mathcal{P}_2^4 &= 2 \cdot 2 = 4, \\ \mathcal{P}_3^4 &= 3 \cdot 4 \cdot 3 = 36, \\ \mathcal{P}_4^4 &= 4 \cdot 6 \cdot 6 \cdot 4 = 576, \\ \mathcal{P}_5^4 &= 5 \cdot 8 \cdot 9 \cdot 8 \cdot 5 = 14400 \end{aligned}$$

(See Appendix B for a generalization of this applicable to all $\mathcal{P}_n^k$.)

Further, while we generate our terms for $\mathcal{P}_n^4$ with

$$\mathcal{P}_n^4 = n!^2 \tag{3.3}$$

Penson [EIS] notes that, as an integral representation as the n^{th} moment of a positive function on a positive half-axis,

$$\mathcal{P}_n^4 = \int_0^\infty x^n \; 2K_0(2\sqrt{x}) \; \mathrm{d}x \tag{3.4}$$

where $K_n(x)$ is the modified Bessel function of the second kind [WM].

The first few *pentagorials*—pentagonal polygorials, $\mathcal{P}_n^5$—are

$$\begin{aligned} &1, 1, 5, 60, 1320, 46200, 2356200, 164934000, 15173928000, \\ &1775349576000, 257425688520000, 45306921179520000, \\ &9514453447699200000, 2350070001581702400000, \\ &674470090453948588800000, 222575129849803034304000000, \\ &83688248823525940898304000000, 35567505749998524881779200000000 \end{aligned}$$

which sequence has entered the OEIS as A0084939 [EIS]. The values can be generated directly by

$$\mathcal{P}_n^5 = \frac{n!}{2^n} 3^n \left(\frac{2}{3}\right)_n$$

where $(x)_n$ is the Pochhammer symbol [WM]; or, expanding, with the gamma function [WM],

$$\mathcal{P}_n^5 = \frac{n!}{2^n} \frac{3^n \Gamma\left(n + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \tag{3.5}$$

Remark 3.1. Evaluating $\Gamma(n + 2/3)$ for a number of n, I noticed it produces the sequence $x/3^n \Gamma(2/3)$, where x is a value in the series $1, 2, 10, 80, 880, 12320, 209440, \ldots$ Checking, I found the latter to be A008544 [EIS]. This allows us to restate (3.5) as

$$\mathcal{P}_n^5 = \frac{n!}{2^n} \prod_{i=0}^{n-1} (3i + 2)$$

Hexagorials, $\mathcal{P}_n^6$, begin

$$1, 1, 6, 90, 2520, 113400, 7484400, 681080400, 81729648000,$$
$$12504636144000, 2375880867360000, 548828480360160000,$$
$$151476660579404160000, 49229914688306352000000,$$
$$18608907752179801056000000, 8094874872198213459360000000,$$
$$4015057936610313875842560000000,$$
$$2252447502438386084347676160000000$$

which is A000680 [EIS]. Penson [EIS] again gives an integral form as

$$\mathcal{P}_n^6 = \int_0^\infty \frac{x^n \mathbf{e}^{-\sqrt{2x}}}{\sqrt{2x}} \, \mathrm{d}x$$

which is equal to our own

$$\mathcal{P}_n^6 = \frac{(2n)!}{2^n} \tag{3.6}$$

Remark 3.2. This can also be expressed as

$$\mathcal{P}_n^6 = \frac{n!}{2} \frac{2^{n+1} \Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}}$$

Enumerating $\Gamma(n + 1/2)$ for a number of n produces $x\sqrt{\pi}/2^n$ where x is a value in the series $1, 1, 3, 15, 105, 945, 10395, \ldots$: A001147 [EIS]. This gives us

$$\mathcal{P}_n^6 = n! \prod_{i=1}^{n} (2i - 1)$$

We find the first several *heptagorials* to be

$$1, 1, 7, 126, 4284, 235620, 19085220, 2137544640, 316356606720,\\ 59791398670080, 14050978687468800, 4018579904616076800,\\ 1374354327378698265600, 553864793933615401036800,\\ 259762588354865623086259200, 140271797711627436466579968000,\\ 86407427390362500863413260288000,\\ 60225976891082663101799042420736000$$

which sequence has entered the OEIS as A084940. These can be generated directly by

$$\mathcal{P}_n^7 = \frac{n!}{2^n}\frac{\Gamma\left(n+\frac{2}{5}\right)\Gamma\left(\frac{3}{5}\right)5^n}{\pi\csc\left(\frac{2\pi}{5}\right)} \tag{3.7}$$

Remark 3.3. For $n \geq 0$, $\Gamma(n+2/5)$ produces $x/5^n\pi\csc(2\pi/5)/\Gamma(3/5)$ where $x = 1, 2, 14, 168, 2856, 62832, 1696464, \ldots$; or, where x is A047055 [EIS]. This lets us rephrase (3.7) as

$$\mathcal{P}_n^7 = \frac{n!}{2^n}\prod_{i=0}^{n-1}(5i+2)$$

For $\mathcal{P}_n^8$, the *octagorials*, we have

$$1, 1, 8, 168, 6720, 436800, 41932800, 5577062400, 981562982400,\\ 220851671040000, 61838467891200000, 21086917550899200000,\\ 8603462360766873600000, 4138265395528866201600000,\\ 2317428621496165072896000000, 1494741460865026472017920000000,\\ 1100129715196659483405189120000000,\\ 916408052758817349676522536960000000$$

which sequence is A084941. The numbers can be generated directly with

$$\mathcal{P}_n^8 = \frac{n!}{2}\frac{\Gamma\left(n+\frac{1}{3}\right)\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)3^n}{\pi} \tag{3.8}$$

Remark 3.4. Here we see an apparent variation on the pattern emerging in previous remarks. If we evaluate $\Gamma(n+1/3)$ for a number of n, we get $x/3^{n+1}\pi\sqrt{3}/\Gamma(2/3)$, where x is a multiple of A047657 [EIS]; namely, where $x = 2/2^n$ A047657. But as (3.8) used $n!/2$ instead of the earlier $n!/2^n$, the multiplier applied to A047657 merely brings us back to our earlier format (since $n!/2 \cdot 2/2^n = n!/2^n$):

$$\mathcal{P}_n^8 = \frac{n!}{2^n}\prod_{i=0}^{n-1}(6i+2)$$

Evaluating the *enneagorials*, $\mathcal{P}_n^9$, we arrive at

$$1, 1, 9, 216, 9936, 745200, 82717200, 12738448800, 2598643555200, \\ 678245967907200, 220429939569840000, 87290256069656640000, \\ 41375581377017247360000, 23128949989752641274240000, \\ 15056946443328969469530240000, 11292709832496727102147680000000, \\ 9666559616617198399438414080000000, \\ 9366896268502065249055823243520000000$$

which is now A084942 in the OEIS. It can be generated with

$$\mathcal{P}_n^9 = \frac{n!}{2^n} 7^n \left(\frac{2}{7}\right)_n$$

or

$$\mathcal{P}_n^9 = \frac{n!}{2^n} \frac{7^n \Gamma\left(n + \frac{2}{7}\right)}{\Gamma\left(\frac{2}{7}\right)} \tag{3.9}$$

As with the earlier cases, it may be rewritten as

$$\mathcal{P}_n^9 = \frac{n!}{2^n} \prod_{i=0}^{n-1} (7i + 2) \tag{3.10}$$

We find the opening of the *decagorial* sequence to be

$$1, 1, 10, 270, 14040, 1193400, 150368400, 26314470000, 6104957040000, \\ 1813172240880000, 670873729125600000, 302564051835645600000, \\ 163384587991248624000000, 104075982550425373488000000, \\ 77224379052415627128096000000, 66026844089815361194522080000000, \\ 64442199831659792525853550080000000, 71208630813984070741068172838400000000$$

(A084943 [EIS]). These numbers can be generated by

$$\mathcal{P}_n^{10} = \frac{n!}{2} \frac{4^n \sqrt{2}\, \Gamma\left(n + \frac{1}{4}\right)\, \Gamma\left(\frac{3}{4}\right)}{\pi} \tag{3.11}$$

Remark 3.5. For $n \geq 0$, $\Gamma(n+1/4)$ produces $x/4^n\pi\sqrt{2}/\Gamma(3/4)$ where $x = 1, 1, 5, 45, 585, 9945, 208845, \ldots$; or, where x is A007696 [EIS], which lets us rephrase (3.11) as

$$\mathcal{P}_n^{10} = n! \prod_{i=0}^{n-1} (4i + 1)$$

4. Concluding Remarks

When recording the alternate formulations included earlier, I noticed the products in the remarks (3.1), (3.3), (3.4) and equation (3.10), and wondered if their pattern could be applied to all $\mathcal{P}_n^k$. In fact, it can. If we say

$$\mathcal{A}_n^k = \prod_{i=0}^{n-1} ((k-2)\, i + 2) \tag{4.1}$$

we will find that

$$\frac{\mathcal{P}_n^k}{\mathcal{A}_n^k} = \frac{n!}{2^n} \tag{4.2}$$

This allows us to restate (3.1) without relying on P_n^k, like so:

$$\mathcal{P}_n^k = \frac{n!}{2^n} \prod_{i=0}^{n-1} ((k-2)\, i + 2) \tag{4.3}$$

which we can also write as

$$\mathcal{P}_n^k = \frac{n!}{2^n} (k-2)^n \left(\frac{2}{k-2}\right)_n \tag{4.4}$$

or

$$\mathcal{P}_n^k = \frac{n!}{2^n} \frac{(k-2)^n \, \Gamma\left(\frac{nk-2n+2}{k-2}\right)}{\Gamma\left(\frac{2}{k-2}\right)} \tag{4.5}$$

Appendix A. Miscellaneous Observations

Here follow a number of collected observations regarding the family of sequences discussed in the preceding pages.

$$\mathcal{P}_3^k = 6P_{k-1}^3 = 6T_{k-1} = \text{A028896}_{k-1} \tag{A.1}$$

$$\sigma_{n,k}^p = \sum_{i=1}^{k} \mathcal{P}_n^i \tag{A.2}$$

$$\sigma_{3,k}^p = k^3 - k = \text{A007531}_k \tag{A.3}$$

$$\mathcal{P}_4^k = 12kP_{k-1}^5 = 12k\text{Pent}_{k-1} \tag{A.4}$$

$$\mathcal{P}_5^k = 60\,(2k-3)\,kP_{k-1}^5 = 60\,(2k-3)\,k\text{Pent}_{k-1} \tag{A.5}$$

$$\mathcal{P}_n^4 = \frac{\text{A048617}_n}{2} \tag{A.6}$$

If M_n is defined to be a symmetrical $n \times n$ matrix such that $M_n(i,j) = 1/min(i,j)$, then for $n \geq 0$,

$$\mathcal{P}_n^4 = \frac{-(-1)^n n}{det(M_n)} = n\text{A010790}_{n-1} \tag{A.7}$$

$$\mathcal{P}_n^6 = n\text{A007019}_{n-1} \tag{A.8}$$

Curious if there was an integer or multiplex $x : 1$ ratio for $\mathcal{P}_n^k/\mathcal{P}_k^n$, I established the function

$$R_{n,k} = \frac{\mathcal{P}_n^k}{\mathcal{P}_k^n} = \frac{n!\left(\frac{k}{2}-1\right)^n\left(\frac{2}{k-2}\right)_n}{k!\left(\frac{n}{2}-1\right)^k\left(\frac{2}{n-2}\right)_k} \tag{A.9}$$

Examining this for $3 \le k \le 80$, $k \le n$, only $k = 3$ proved to be an integer sequence. Specifically,

$$R_{n,3} = \frac{(n+1)!^2}{3 \cdot 2^n \prod_{i=n-1}^{n+1} i}, n \ge 3 \tag{A.10}$$

whose first 18 values are

$$1, 5, 45, 630, 12600, 340200, 11907000, 523908000, 28291032000, 1838917080000,$$
$$141596615160000, 12743695364400000, 1325344317897600000, 157715973829814400000,$$
$$21291656467024944000000, 3236331782987791488000000,$$
$$550176403107924552960000000, 103983340187397740509440000000$$

which is A085356. Thanks to **SuperSeeker** [SS], the following property was discovered:

If you take M_n to be a diagonal $n \times n$ matrix, where the diagonal consists of the values for P_1^3 to P_n^3 with the remaining entries set as 1, n of the above sequence is equal to half the determinant of M_{n-1}, or

$$R_{n,3} = \frac{det\left(M_{n-1}\right)}{2} = \frac{\text{A067550}_{n-1}}{2}, n \ge 2 \tag{A.11}$$

We can also define this with

$$R_{n,3} = \frac{\mathcal{P}_{n+1}^4}{3 \cdot 2^n \sum_{i=1}^{n} \mathcal{P}_3^i} = \frac{\mathcal{P}_{n+1}^4}{3 \cdot 2^n \sigma_{3,n}^p} \tag{A.12}$$

$$\frac{\mathcal{P}_n^6}{\mathcal{P}_n^3} = C_n, \text{ the Catalan numbers} = \text{A000108}_n \tag{A.13}$$

$$\frac{\mathcal{P}_{n+1}^4}{\mathcal{P}_n^3} = 2^n(n+1) = \text{A001787}_{n+1} \tag{A.14}$$

$$\frac{\mathcal{P}_{n+1}^6}{\mathcal{P}_n^3} = \frac{(2n+1)!}{n!^2} = \text{A002457}_n = \frac{\text{A009445}}{\mathcal{P}_n^4} \tag{A.15}$$

$$\frac{\mathcal{P}^3_{2n}}{\mathcal{P}^3_n} = \text{A036770}_n \tag{A.16}$$

$$\begin{aligned}\frac{\mathcal{P}^6_{2n}}{\mathcal{P}^2_n} &= 16^n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{2n} \\ &= \frac{16^n \Gamma\left(n+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}\right)}{\pi} \\ &= \text{A060706}_n\end{aligned} \tag{A.17}$$

Appendix B. Antidiagonals

When discussing the tetragorials, we noted Murthy's observations [EIS] (in A001044) regarding the product of the n^{th} antidiagonal of a specific symmetrical matrix, M_n. We find that we can generalize that idea to apply to all polygorials as follows: given a symmetrical $n \times n$ matrix, M_n, for $n > 0$, where

$$M_n(i,j) = \begin{cases} P^k_i & \text{if } i = j, \\ \sqrt{P^k_i}\sqrt{P^k_j} & \text{otherwise,} \end{cases}$$

the product of the antidiagonal will equal $\mathcal{P}^k_n$. Since M_n is symmetrical, we know $M_n(i,j) = M_n(j,i)$, and from that we know that the product of an antidiagonal of n terms will be equal to the product of its first $\lfloor n/2 \rfloor$ terms squared (and multiplied by its $\lceil n/2 \rceil^{\text{th}}$ term for odd n). Since the values for $M_n(i,j)$ when $i \neq j$ are themselves merely the products of the roots of P^k_i and P^k_j, their squares reproduce the original values, explaining the antidiagonal's equality with $\mathcal{P}^k_n$ as the product of P^k_i for $1 \leq i \leq n$.

Example 1. For $k = 3$,

$$M_5 = \begin{pmatrix} 1 & \sqrt{3} & \sqrt{6} & \sqrt{10} & \sqrt{15} \\ \sqrt{3} & 3 & \sqrt{3}\sqrt{6} & \sqrt{3}\sqrt{10} & \sqrt{3}\sqrt{15} \\ \sqrt{6} & \sqrt{3}\sqrt{6} & 6 & \sqrt{6}\sqrt{10} & \sqrt{6}\sqrt{15} \\ \sqrt{10} & \sqrt{3}\sqrt{10} & \sqrt{6}\sqrt{10} & 10 & \sqrt{10}\sqrt{15} \\ \sqrt{15} & \sqrt{3}\sqrt{15} & \sqrt{6}\sqrt{15} & \sqrt{10}\sqrt{15} & 15 \end{pmatrix}$$

$$\mathcal{P}^3_5 = \sqrt{15}\sqrt{3}\sqrt{10}\ 6\sqrt{3}\sqrt{10}\sqrt{15} = 3 \cdot 6 \cdot 10 \cdot 15 = 2700$$

Example 2. For $k = 5$,

$$M_5 = \begin{pmatrix} 1 & \sqrt{5} & 2\sqrt{3} & \sqrt{22} & \sqrt{35} \\ \sqrt{5} & 5 & 2\sqrt{3}\sqrt{5} & \sqrt{5}\sqrt{22} & \sqrt{5}\sqrt{35} \\ 2\sqrt{3} & 2\sqrt{3}\sqrt{5} & 12 & 2\sqrt{3}\sqrt{22} & 2\sqrt{3}\sqrt{35} \\ \sqrt{22} & \sqrt{5}\sqrt{22} & 2\sqrt{3}\sqrt{22} & 22 & \sqrt{22}\sqrt{35} \\ \sqrt{35} & \sqrt{5}\sqrt{35} & 2\sqrt{3}\sqrt{35} & \sqrt{22}\sqrt{35} & 35 \end{pmatrix}$$

$$\mathcal{P}^5_5 = \sqrt{35}\sqrt{5}\sqrt{22}\ 12\sqrt{5}\sqrt{22}\sqrt{35} = 5 \cdot 12 \cdot 22 \cdot 35 = 46200$$

References

HMF. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover Press, New York, NY, 1974.

GKP. R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, second edition, Addison-Wesley, Reading, MA, 1994.

Hoche. R. Hoche, ed., *Νικομαχου Γερασηνου Πυθαγορικου Αριθμητικη Εισαγωγη*, B. G. Teubner, Leipzig, 1866, caps. 8–12, pp. 87–99.

EIS. N.J.A. Sloane, Integer sequences A000108, A00147, A000442, A000680, A001044, A001453, A001787, A002457, A006472, A007019, A007531, A007696, A008544, A009445, A010790, A028896, A036770, A047055, A047657, A048617, A060706, A067550 and new to the present paper, A084939 through A084942 and A085356, in "The On-Line Encyclopedia of Integer Sequences," http://www.research.att.com/ njas/sequences/ (last checked on 20030621).

SS. N.J.A. Sloane, **SuperSeeker**, additional information available online at the OEIS website http://www.research.att.com/ njas/sequences/ol.html (last checked on 20030621).

WM. E.W. Weisstein, the entries for the Bessel Function of the Second Kind, the Gamma Function and the Pochhammer Symbol in *Eric Weisstein's World of Mathematics*; web version, http://mathworld.wolfram.com/GammaFunction.html (last checked on 20030621), http://mathworld.wolfram.com/PochhammerSymbol.html (which was checked: 20030621), http://mathworld.wolfram.com/ModifiedBesselFunctionoftheSecondKind.html (last checked: 20030621).

CRC. D. Zwillinger, *CRC Standard Mathematical Tables and Formulæ*, thirtieth edition, CRC Press, Boca Raton, FL, 1995.

(Concerned with the sequences: A000108, A000142, A000165, A000217, A000290, A000326, A000384, A000442, A000566, A000567, A000680, A001044, A001106, A001107, A001147, A001453, A001787, A001813, A002457, A006472, A006882, A007019, A007531, A007661, A007662, A007696, A008544, A009445, A010790, A028896, A036770, A047055, A047657, A048617, A051682, A051624, A051865, A051866, A051867, A051868, A051869, A051870, A051871, A051872, A051873, A051874, A051875, A051876, A060706, A067550, A084939, A084940, A084941, A084942, A084943, A084944, A084946, A084947, A084948, A084949, A085356.)

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