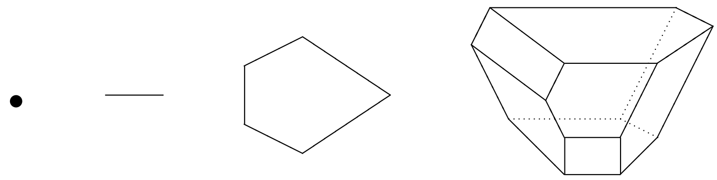
THE MULTIPLE FACETS OF THE ASSOCIAHEDRON

JEAN-LOUIS LODAY

ABSTRACT. This is a survey of some of the nice properties of the associahedron (also called Stasheff polytope) from several points of views: topological, geometrical, combinatorial and algebraic.

1. INTRODUCTION

The associahedron $\mathcal{K}^n$ is a polytope of dimension n whose vertices are in one to one correspondence with the parenthesizings of the word $x_0x_1 \dots x_{n+1}$.



$\mathcal{K}^0$ $\mathcal{K}^1$ $\mathcal{K}^2$ $\mathcal{K}^3$

The set of vertices is in one to one correspondence with the planar binary trees having $n+2$ leaves, this is why it is often denoted by K_{n+2}. This polytope has numerous properties: geometrical, combinatorial and algebraic, that we will survey.

First, we construct the associahedron as the convex hull of its vertices by giving the explicit coordinates of the vertices from the combinatorial properties of the trees. As a cellular complex the associahedron is isomorphic to the Stasheff polytope which plays a key-role in the characterization of spaces homotopically equivalent to loop spaces. Next, we describe a partial order structure on the set of vertices of the associahedron. This structure can be read off the euclidean realization of the polytope. Then, we compare the associahedron to the permutohedron,

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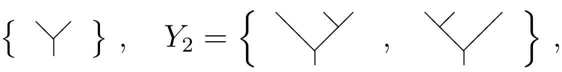
whose vertices are in bijection with the permutations. Algebraically one can endow the vector space spanned by the planar binary trees the structure of an associative algebra. This structure can be built from the partial order. In fact this algebra inherits a finer structure, a *dendriform* algebra structure. Moreover it is the free dendriform algebra on one generator. One of the advantages of this feature is to permit us to give a meaning to the notion of series indexed by trees (instead of being indexed by integers). Finally we show a close relationship between the associahedron and the inversion of integral series. Among the many other subjects related to the associahedron one finds renormalization theory, noncommutative geometry and moduli spaces.

This survey on some properties of the associahedron is essentially a translation of a previous text "Les multiples facettes de l'associaèdre" which was written in french. The corresponding talk was delivered during the Clay Research Academy (April 2005). I thank the Clay Institute for support and Vida Salahi for her help. Many thanks to David Ellwood for his generous multi-facets help.

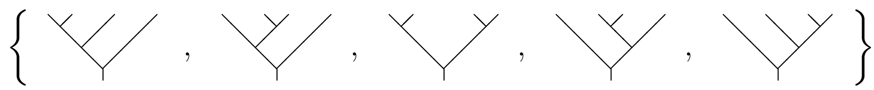
2. Euclidean geometry

Let Y_n be the set of planar binary trees with n internal vertices, that is with $n+1$ leaves. So one gets

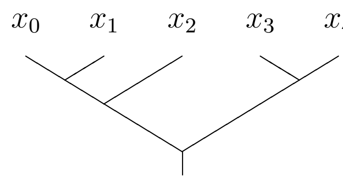
$Y_0 = \{ \, | \, \}$, $Y_1 =$, $Y_2 =$



$Y_3 =$

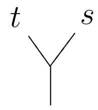


Observe that there is a bijection between the set of trees and the set of parenthesizings of a word with $n+1$ letters:

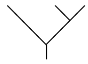
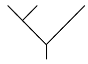


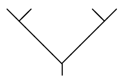
$$(((x_0x_1)x_2)(x_3x_4))$$

The *grafting* of $t \in Y_p$ and of $s \in Y_q$ is the tree denoted $t \vee s \in Y_n$ obtained by glueing the root of t and the root of s to a new vertex, and thereby creating a new root:

$t \vee s =$ 

So we get $n = p + 1 + q$ since the vertices of $t \vee s$ are those of t, those of s and the new vertex. For $n > 0$ any tree t can be written uniquely as $t = t^l \vee t^r$. This property can be taken as a definition of the set Y_n (union of all the sets $Y_p \times Y_q$ for $n = p + 1 + q$). Examples :

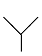
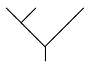
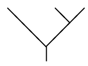
$| \vee | =$  , $| \vee$  $=$  ,  $\vee | =$  ,

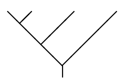
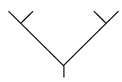
 $\vee$  $=$  .

To any tree $t \in Y_n, n > 0$, we associate a point $M(t) \in \mathbb{R}^n$ with integral coordinates as follows. Let us number the leaves of t from left to right by $0, 1, \ldots, n$. So one can number the internal vertices from 1 to n (the vertex number i is in between the leaves $i - 1$ and i). Let a_i be the number of leaves on the left side of the vertex i and let b_i be the number of leaves on the right side. Observe that these numbers depend only on the subtree determined by the vertex i. We define:

$$M(t) := (a_1 b_1, \ldots, a_i b_i, \ldots, a_n b_n) \in \mathbb{R}^n \ .$$

In low dimension we get:

$M($  $) = (1)$, $M($  $) = (1, 2)$, $M($  $) = (2, 1)$,

$M\Big($  $\Big) = (1, 2, 3)$, $M\Big($  $\Big) = (1, 4, 1)$.

For the tree corresponding to the parenthesizing $(((x_0x_1)x_2)(x_3x_4))$ (described above) one gets the following point $(1 \times 1, 2 \times 1, 3 \times 2, 1 \times 1) = (1, 2, 6, 1)$.

Denote by H_n the hyperplane of $\mathbb{R}^n$ whose equation is

$$x_1 + \cdots + x_n = \frac{n(n+1)}{2}.$$

2.1. **Lemma.** *For any tree $t \in Y_n$ the point $M(t)$ belongs to the hyperplane H_n.*

Proof. This is immediate for $n = 1$. Under the decomposition $t = t^l \vee t^r$ where $t^l \in Y_p$ and $t^r \in Y_q$ one gets

$$M(t) = (M(t^l), (p+1)(q+1), M(t^r)),$$

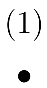
since t^l has $p+1$ leaves and t^r has $q+1$ leaves. By induction we get:

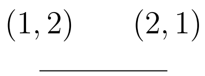
$$\sum_{i=1}^{i=n} a_i b_i = \frac{p(p+1)}{2} + (p+1)(q+1) + \frac{q(q+1)}{2} = \frac{n(n+1)}{2}$$

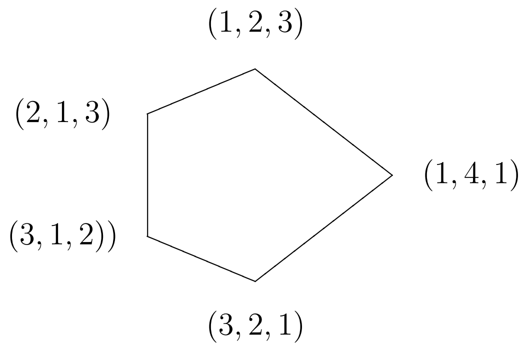
since $n = p + 1 + q$. □

2.2. **Definition.** *For n fixed the* associahedron *or* Stasheff polytope *is the convex hull, denoted $\mathcal{K}^{n-1}$ (or sometimes K_{n+1} in the literature), of the points $M(t)$ in the hyperplane H_n for $t \in Y_n$, cf.* [Lo3].

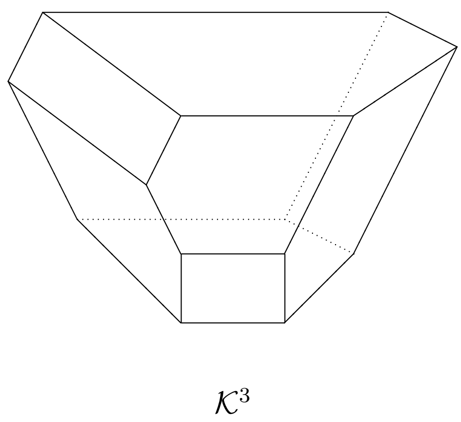
The associahedron $\mathcal{K}^n$ is a convex polytope of dimension n:







$\mathcal{K}^0$ $\mathcal{K}^1$ $\mathcal{K}^2$



$\mathcal{K}^3$

It is interesting to determine the affine subspaces which contain the faces of this polytope. Since $\mathcal{K}^{n-1}$ is contained in the hyperplane H_n these subspaces are of codimension 2 and so they are the intersection of H_n with another hyperplane.

Let i and k be two positive integers satisfying $i+k \leq n+1$. Denote by $F_{i,k}$ the hyperplane whose equation is

$$x_i + x_{i+1} + \cdots + x_{i+k-1} = \frac{k(k+1)}{2} \ .$$

So one gets $F_{1,n} = H_n$. One can show that $F_{i,k} \cap H_n$ contains a face of $\mathcal{K}^{n-1}$ and that any face of $\mathcal{K}^{n-1}$ is contained in an affine subspace of this type. As a polytope such a face is isomorphic (up to homothety) to $\mathcal{K}^{k-1} \times \mathcal{K}^{n-k-1}$.

Exercice. Find the face of $\mathcal{K}^3$ which contains the vertices with coordinates $(3,1,2,4), (3,2,1,4), (4,1,2,3), (4,2,1,3)$.

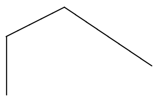
2.3. **Symmetry.** There is an obvious involution on the set Y_n: start with a tree t and symmetrize it through the axis passing through the root. From the definition of the coordinates of $M(t)$ it is clear that the euclidean realization of $\mathcal{K}^{n-1}$ is invariant globally under the involution $(x_1, x_2, \ldots, x_n) \mapsto (x_n, \ldots, x_2, x_1)$.

2.4. **Constructing $\mathcal{K}^{n+1}$ out of $\mathcal{K}^n$.** It is fairly simple to construct the simplex (resp. the hypercube) out of the simplex (resp. the hypercube) of lower dimension. For the associahedron it is slightly more involved, but we can do it by using the following recipe. First, we start with the associahedron $\mathcal{K}^n$, which is a ball and whose boundary is a cellular sphere. The cells of the boundary are of the form $\mathcal{K}^p \times \mathcal{K}^q$ where $p+q = n-1$. Second, we "enlarge" each cell $\mathcal{K}^p \times \mathcal{K}^q$ into a cell of dimension n by replacing it by $\mathcal{K}^{p+1} \times \mathcal{K}^q$. We leave the reader the task of understanding how to glue these polytopes together to make a new cell complex. Thirdly, we take the cone over this enlargement and we discover that we have constructed $\mathcal{K}^{n+1}$.

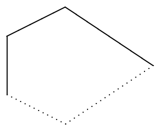
Example $n = 1$:

– $\mathcal{K}^1$

– $\mathcal{K}^1$ enlarged

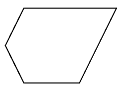


– Cone over $\mathcal{K}^1$ enlarged $= \mathcal{K}^2$

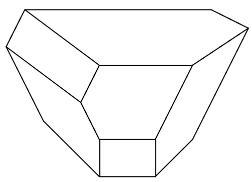


Example $n = 2$:

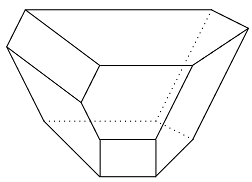
– $\mathcal{K}^2$



– $\mathcal{K}^2$ enlarged



– Cone over $\mathcal{K}^2$ enlarged $= \mathcal{K}^3$

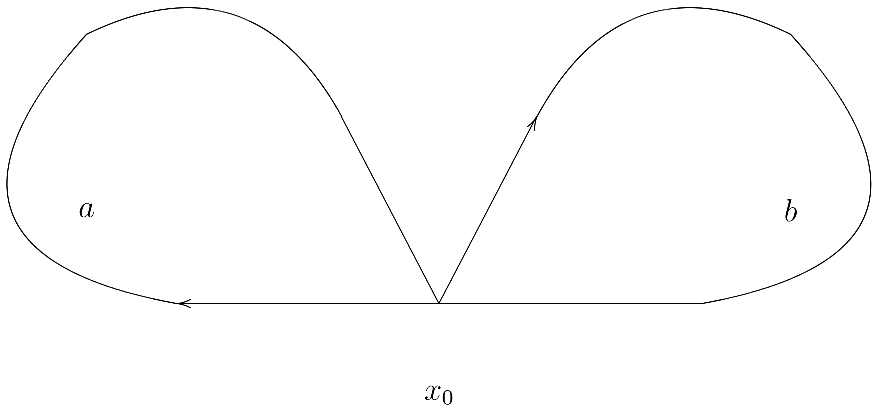


This construction permits us to "simplicialize" the associahedron, that is to decompose it as a union of simplices by induction.

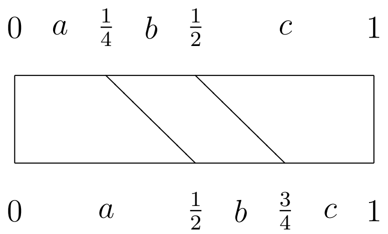
Show that for $\mathcal{K}^n$ the number of n-simplices is $(n+1)^{n-1}$. In a future article article [Lo4] we will unravel the relationship between simplicialization of the associahedron and parking functions.

3. Topology

A loop a in a pointed topological space (X, x_0) is a continuous map from the interval $I = [0, 1]$ into X which sends 0 and 1 to the base-point x_0. On can compose two loops a and b to obtain a new loop denoted ab. It is obtained by running a from 0 to $\frac{1}{2}$ and b from $\frac{1}{2}$ to 1.



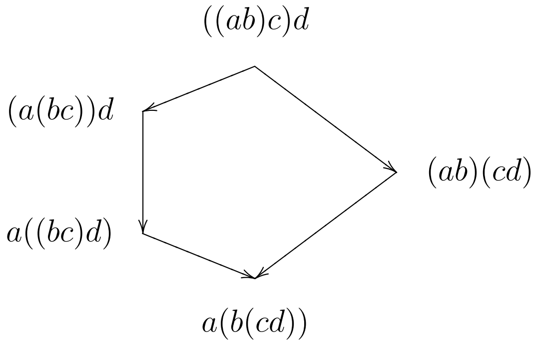
Starting with 3 loops a, b, c the products $(ab)c$ and $a(bc)$ are not equal. Indeed, though their image in X are the same, the parametrization is not. In the first case one runs a from 0 to $\frac{1}{4}$, b from $\frac{1}{4}$ to $\frac{1}{2}$ and c from $\frac{1}{2}$ to 1. In the second case one runs a from 0 to $\frac{1}{2}$, b from $\frac{1}{2}$ to $\frac{3}{4}$ and c from $\frac{3}{4}$ to 1. It is easy to check that these two loops are *homotopic* :



More precisely there exists a continuous map $F : I \times I \to X$ such that $F(0, -) : I \to X$ is the loop $(ab)c$ and such that $F(1, -) : I \to X$ is the loop $a(bc)$. In the loopspace ΩX of the pointed space (X, x_0) there is path $F : I \to \Omega X$ from the point $(ab)c$ to the point $a(bc)$:

$$(ab)c \rightsquigarrow a(bc)$$

What happens with 4 loops a, b, c, d ? Then there are 5 different ways of composing them, one for each parenthesizing of the word $abcd$, that is one for each planar binary tree with four leaves. Not only the corresponding points are connected by paths (homotopies), but the two possible compositions of these homotopies, which go from $((ab)c)d$ to $a(b(cd))$ are themselves homotopic (among the paths with fixed end points). At this point one sees the appearance of a cellular decomposition of the ball D^2 (the disc) and of its boundary (the circle):

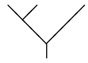
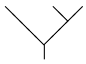


Jim Stasheff has proved in 1966 (cf. [Sta1]) that this phenomenon is quite general. More precisely he showed the existence of a certain cellular decomposition of the ball D^n for all n which captures the existence of homotopies between homotopies as soon as one starts with $n+2$ loops and one tries to compare all the parenthesizings of $a_0a_1 \dots a_{n+1}$. The vertices (0-cells) of this cellular decomposition are in bijection with the parenthesizings, that is with the planar binary trees with $n + 2$ leaves. The 1-cells correspond to elementary homotopies between parenthesizings, the 2-cells to homotopies between homotopies, etc. This is the space which is called the *Stasheff complex*. It plays a prominent role in the recognition of spaces homotopy equivalent to loopspaces (that is almost topological groups).

In low dimension it is obvious that the Stasheff complex can be realized as a polytope (pentagon for $n = 2$). However it is was not until 1989 that a proof of the general case was published (cf. [Lee]). Different realizations, some more explicit than others, were obtained. See for instance Gelfand, Kapranov and Zelevinsky (cf. [GKZ]). Their realization uses the interpretation of the parenthesizings in terms of triangulations of polygons. See also the appendix of [Sta2] by Shnider and Stasheff. The realization described in this article is probably the most simple of all.

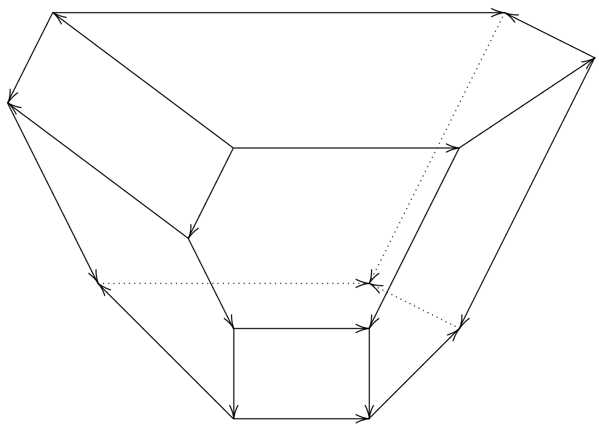
4. Combinatorics

The homotopic interpretation of the Stasheff complex given in the preceding section suggests the existence of a partial order structure on the set of vertices, that is on the set of planar binary trees with a given number of leaves.

4.1. **Order structure.** Given a partial order structure, denoted $<$, on a set E one says that $x < y$ is a *covering relation* (between x and y) if there is no element z, different from x and y, so that $x < z < y$. An order structure on a finite set is obviously determined by its covering relations. One defines a covering relation on the set Y_n as follows : for all $t, s \in Y_n$, t is said to be smaller than s, denoted $t < s$, if s is obtained from t by replacing, locally, a subtree of t of the form  by a subtree of the form  . This covering relation induces a partial order structure on Y_n often called the *Tamari order*. So one has $t < s$ for the partial order relation if and only if there exists a sequence

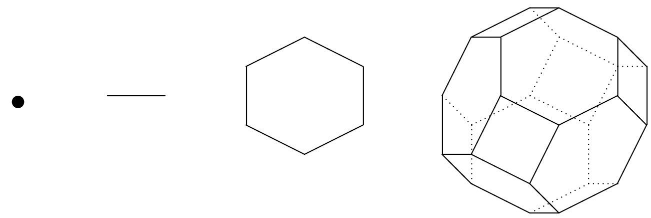
$$t = t_0 < t_1 < \cdots < t_{k-1} < t_k = s$$

where $t_i < t_{i+1}$ is a covering relation for all i. It happens that each covering relation corresponds to an edge of the associahedron. See above for the pentagon. In dimension 3 one gets:



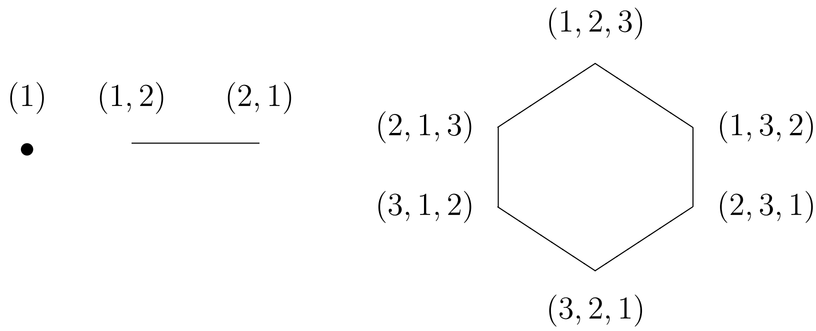
Recall that any poset (= partially ordered set) admits a geometric realization, which is a cellular complex. In the Tamari poset case this realization is precisely the Stasheff complex. The euclidean realization described in section 2 has the following property with respect to this order relation. There is a smallest vertex, called the North pole, which corresponds to the parenthesizing $(\cdot((a_0a_1)a_2)\cdots a_{n+1})$. There is a largest vertex, called the South pole, which corresponds to the parenthesizing $(a_0(a_1(\cdots(a_na_{n+1})\cdot)))$. If one orientates the polytope so that the North pole is on top, the partial ordering is given by the latitude of a vertex, as one runs North-South along the edges. See the picture in section 3 for $\mathcal{K}^2$.

There exists another polytope with analogous properties, it is called the permutohedron, and its vertices are in bijection with the permutations. Let S_n be the symmetric group made of the permutations σ of the set $\{1,\ldots,n\}$. Consider $M(\sigma)=(\sigma(1),\ldots,\sigma(n))$ as a point in $\mathbb{R}^n$. All the points $M(\sigma)$ lie in the hyperplane H_n since $1+2+\cdots+n=\frac{n(n+1)}{2}$. The convex hull of the points $M(\sigma)$ for $\sigma\in S_n$ forms an $(n-1)$-dimensional polytope denoted $\mathcal{P}^{n-1}$ and called the *permutohedron.*



$\mathcal{P}^0$ $\quad$ $\mathcal{P}^1$ $\quad$ $\mathcal{P}^2$ $\quad$ $\mathcal{P}^3$

In low dimension the coordinates of the vertices are:



$\mathcal{P}^0$ $\quad$ $\mathcal{P}^1$ $\quad$ $\mathcal{P}^2$

The set of vertices of $\mathcal{P}^{n-1}$, that is the symmetric group S_n, is equipped with a partial order such that the covering relations correspond to the edges of $\mathcal{P}^{n-1}$, this is the *weak Bruhat order.* It is defined as follows. Let s_i be the *transposition* which exchanges i and $i+1$. The group S_n is presented by the generators s_i, $i=1,\cdots,n-1$, and the relations

$$\begin{cases} {s_i}^2 = 1, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ s_i s_j = s_j s_i \quad \text{si } |i-j|\geq 2. \end{cases}$$

One can show that any permutation admits a unique minimal length writing. The covering relation $\sigma < \omega$ for the weak Bruhat order is determined by: $\omega = \sigma s_i$ for some integer i and the minimal length of ω is strictly greater than the minimal length of σ. Example:

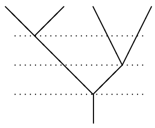
$$\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & \\
s_1 \arrow[d] & & s_2 \arrow[d] \\
s_1 s_2 \arrow[dr] & & s_2 s_1 \arrow[dl] \\
& s_1 s_2 s_1 = s_2 s_1 s_2 &
\end{tikzcd}$$

We now focus on the close relationship between the permutohedron and the associahedron. First, note that the polytope $\mathcal{K}^n$ contains the polytope $\mathcal{P}^n$, or, from a more "solid" point of view, $\mathcal{P}^n$ can be constructed out of $\mathcal{K}^n$ by truncation. Here is the trick. The faces of $\mathcal{P}^{n-1}$ lie in some affine hyperplanes, which can be described using "shuffles". A $(k, n-k)$-*shuffle* is a permutation $(\omega_1, \cdots, \omega_k; \omega_{k+1}, \cdots, \omega_n)$ of $(1, 2, \cdots, n)$ such that $\omega_1 < \cdots < \omega_k$ and $\omega_{k+1} < \cdots < \omega_n$. Consider the polynomial

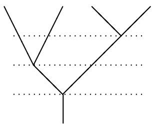
$$p_\omega(x_1, \ldots, x_n) := \\ (n-k)(x_{\omega_1} + \cdots + x_{\omega_k}) - k(x_{\omega_{k+1}} + \cdots + x_{\omega_n}) + \frac{1}{2} nk(n-k).$$

Denote by $\mathcal{H}_\omega$ the hyperplane of $\mathbb{R}^n$ whose equation is $p_\omega(x_1, \ldots, x_n) = 0$. The affine hyperplanes of H_n containing the faces of $\mathcal{P}^{n-1}$ are the spaces $\mathcal{H}_\omega \cap H_n$ associated to the shuffles ω. When the sequence of integers $(\omega_1, \ldots, \omega_k)$ of the shuffle ω is a sequence of consecutive integers, one observes that $\mathcal{H}_\omega \cap H_n$ is exactly the affine sub-space $F_{\omega_1,k} \cap H_n$ containing a face of $\mathcal{K}^{n-1}$. This assertion follows from the comparison of the equations of these hyperplanes.

One can also built the associahedron from the permutohedron by the following method. One first builds a set map $\phi : S_n \to Y_n$ as the composite $S_n \cong \tilde{Y}_n \to Y_n$ where $\tilde{Y}_n$ is the set of *leveled planar binary trees*. The difference between the elements of $\tilde{Y}_n$ and those of Y_n is that, in $\tilde{Y}_n$, two different vertices are located at different levels. So the two trees



and



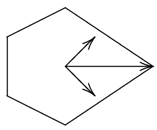
are distinct in $\tilde{Y}_n$ but they represent the same element in Y_n. It is easy to see that there is a bijection between $\tilde{Y}_n$ and S_n: to any leveled tree one associates the permutation $i \mapsto$ level of the ith vertex. The surjection $\tilde{Y}_n \to Y_n$ is simply given by forgetting the levels.

We are now ready to build the euclidean realization of $\mathcal{K}^{n-1}$ from the euclidean realization of $\mathcal{P}^{n-1}$. Let $t \in Y_n$ be a planar binary tree and let $\phi^{-1}(t)$ be the set of permutations σ in S_n having t as image under ϕ. Then one gets the following result.

4.2. **Proposition.** [Lo3]. *Let C the center of the permutohedron $\mathcal{P}^{n-1}$ whose coordinates are $C = (\frac{n+1}{2}, \dots, \frac{n+1}{2})$. One has the following equality of vectors:*

$$\overrightarrow{CM(t)} = \sum_{\sigma \in \phi^{-1}(t)} \overrightarrow{CM(\sigma)} \ .$$

Hence the point $M(t)$ is obtained from the points $M(\sigma)$ where σ is a preimage of t :

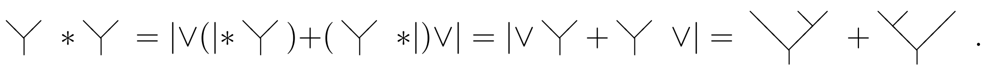


5. Algebra

On the vector space spanned by the planar binary trees there is associative structure defined inductively by the following formula: for any $t = t^l \vee t^r \in Y_p$ and $s = s^l \vee s^r \in Y_q$ on puts

$$t * s := t^l \vee (t^r * s) + (t * s^l) \vee s^r \ .$$

When t or s is equal to $|$ we put $t * | = t$ and $| * s = s$, and this starts the induction. So one has

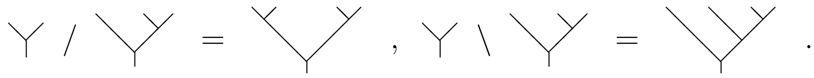


Denote by $K[Y_n]$ the vector space over K spanned by the elements of Y_n.

5.1. **Proposition.** *The operation $*$ is associative, and so $(\bigoplus_{n\geq 0} K[Y_n]), *)$ is a unital associative algebra over K.*

Proof. By induction ! □

Until now this structure involves only the planar binary trees, that is the vertices of the associahedron, and the notion of grafting. However we will see that it is related to the geometrical structure of the polytope, via the structure of poset. We have seen that, given two trees t and s, one can graft onto a new root. But one can also define an alternative grafting. Denote by t/s (read t over s) the tree obtained by identifying the root of t with the left leaf of s, and denote by $t\backslash s$ (read t under s) the tree obtained by identifying the root of s with the right leaf of t. So the number of internal vertices of t/s (as well as $t\backslash s$) is the sum of the number of internal vertices of t and of s. For instance:



For the Tamari partial order one always has

$$t/s < t\backslash s$$

or equality if one of the trees is $|$.

We can show the following result which relates the algebra product $*$ and the Tamari order:

5.2. **Theorem.** [LR2] *For any planar binary trees t and s one has*

$$t * s = \sum_{t/s \leq x \leq t\backslash s} x \ .$$

The sum is extended to all the trees which lie in between t/s and $t\backslash s$. This subset is called an *interval* in combinatorics.

In the definition of the product $*$ on $\bigoplus_n K[Y_n]$ one sees two distinct terms. Let us introduce the two operations $\prec$, called *left*, and $\succ$, called *right*:

$$t \prec s := t^l \vee (t^r * s) \quad , \quad t \succ s := (t * s^l) \vee s^r \ .$$

So we have

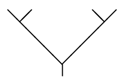
$$t * s := t \prec s + t \succ s$$

by definition, and one says that the associative operation $*$ *splits* into two operations. These new operations are not associative, however the associativity relation of $*$ splits into the following three relations:

$$\begin{cases} (x \prec y) \prec z &= x \prec (y * z), \\ (x \succ y) \prec z &= x \succ (y \prec z), \\ (x * y) \succ z &= x \succ (y \succ z). \end{cases}$$

A vector space A over K equipped with two binary operations $\prec$ and $\succ$ satisfying the three relations above (where $* = \prec + \succ$ of course) is called a *dendriform algebra*. One sees immediately that a dendriform algebra is an associative algebra for the product $*$.

It happens that $(\bigoplus_n K[Y_n], \prec, \succ)$ is a very special type of dendriform algebra. Indeed, it is the dendriform algebra analogue of the one variable polynomial algebra (in the theory of associative algebras). More precisely it is characterized by the following universal property: for any dendriform algebra A and any element $a \in A$ there exists a unique morphism of dendriform algebras $f : \bigoplus_n K[Y_n] \longrightarrow A$ which sends the tree  to a. One says that $(\bigoplus_n K[Y_n], \prec, \succ)$ is the free dendriform algebra on one generator (cf. [Lo1]).

5.3. **Dendriform series.** This result has many interesting properties. For instance it permits us to generalize the notion of formal power series $\sum_n a_n x^n$ by replacing the integers n by planar binary trees t. In this way one can work with series of the form $\sum_t a_t x^t$, that is, add them, multiply them, and even compose them. The crucial point, which is a consequence of the universal property, consists in giving a meaning to $(\sum_t a_t x^t)^s$ for any tree s. The process begins by writing s in terms of the generator, and then replacing this generator by the element we want to take the power of. For instance if $s =$  then, since

$s =$  $\succ$  $\prec$ , one has

$$(\sum_t a_t x^t)^s = (\sum_t a_t x^t) \succ (\sum_t a_t x^t) \prec (\sum_t a_t x^t).$$

For more details see [Lo2] and [BF].

5.4. **Zinbiel algebras.** One knows that the commutative algebras form an important class among the associative algebras. One defines a *commutative dendriform algebra* as being a dendriform algebra which verifies the commutativity relation $x \succ y = y \prec x$ for any x and y. As a consequence the associative product $*$ is commutative. Since any right product is also a left product, one can rewrite the three defining relations with, for instance, the left product. So the first relation becomes

$$(x \prec y) \prec z = x \prec (y \prec z) + x \prec (z \prec y). \qquad (Zb)$$

One easily verifies that the third relation gives also the relation (Zb) and that the second relation is a consequence of (Zb). So a commutative dendriform algebra is defined by a unique operation, that we have chosen to be $\prec$, satisfying only one relation, namely (Zb). This structure was already known in a different context under the name *Zinbiel algebra*, cf. [Lo1].

The main feature of Zinbiel algebras is the following: they are to commutative algebras what associative algebras are to Lie algebras:

$$\begin{array}{lcl} \text{Zinbiel algebras} & \longrightarrow & \text{commutative algebras} \\ (Z, x \prec y) & \mapsto & (Z, xy = x \prec y + y \prec x) \\ & & \\ \text{associative algebras} & \longrightarrow & \text{Lie algebras} \\ (A, xy) & \mapsto & (A, [x,y] = xy - yx). \end{array}$$

6. Inversion of power series

Let us consider the formal power series

$$f(x) = x + a_1x^2 + a_2x^3 + \cdots + a_nx^{n+1} + \cdots$$

and let

$$g(x) = x + b_1x^2 + b_2x^3 + \cdots + b_nx^{n+1} + \cdots$$

be its inverse for composition, that is $f(g(x)) = x$. The coefficient b_n is a polynomial in $a_i, 1 \leq i \leq n$. In low dimension one gets

$$\begin{array}{rcl} b_1 & = & -a_1 \\ b_2 & = & 2a_1^2 - a_2 \\ b_3 & = & -5a_1^3 + 5a_1a_2 - a_3 \\ b_4 & = & 14a_1^4 - 21a_1^2a_2 + 6a_1a_3 + 3a_2^2 - a_4 \end{array}$$

and more generally

$$b_n = \sum (-1)^{\sum n_i} \lambda(n_1, \ldots, n_k) a_1^{n_1} \cdots a_k^{n_k}$$

where the sum is extended to all the $k$tuples of integers $(n_1, \ldots, n_k)$ so that $n_1 + 2n_2 + \cdots + kn_k = n$. Here the coefficient $\lambda(n_1, \ldots, n_k)$ is the number of cells of the associahedron $\mathcal{K}^{n-1}$ which are isomorphic to the cartesian product $(\mathcal{K}^0)^{n_1} \times \cdots \times (\mathcal{K}^{k-1})^{n_k}$. The translation in terms of planar trees is the following. To any planar tree t one associates the $k$tuple $n(t) = (n_1(t), \ldots, n_k(t))$ where $n_i(t)$ is the number of vertices of t having one root and $i+1$ leaves. So for the corolla one gets $(0, \ldots, 0, 1)$. For a planar binary tree one gets (n) if there are $n+1$

leaves. Then the coefficient $\lambda(n_1, \ldots, n_k)$ is the number of planar trees t with $n+1$ leaves so that $n(t) = (n_1, \ldots, n_k)$.

For instance $\lambda(n)$ is the number of planar binary trees with $n+1$ leaves, that is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

There exists a short operadic proof of the above formula which explicitly involves the parenthesizings, but it would be interesting to find one which involves the topological structure of the associahedron.

7. Variations, questions, problems

7.1. **Families of polytopes.** The simplices, the hypercubes, the associahedrons (also called associahedra), the permutohedrons (also called permutohedra) form families of polytopes whose number of edges in dimension 2 is respectively 3, 4, 5, 6. How about $k \geq 7$? Let us remark that for $k = 2$ it is natural to consider the family of globular spaces (2 cells in each dimension, except in the maximal dimension where there is only one cell), even if they are not polytopes. See the figure "Polytopes" below.

7.2. **Other simple polytopes.** In the hyperplane H_n the permutohedron can be defined by inequalities, namely $p_\omega(x_1, \ldots, x_n) \geq 0$ where ω runs over the $(k, n-k)$-shuffles, cf. 4.1. We have seen that by restricting to some shuffles, we get the associahedron. Another exemple is given by restriction to ω of the form $(1, 2, \ldots, k; k+1, \ldots, n)$ or $(k+1, \ldots, n; 1, 2, \ldots, k)$. The polytope obtained that way is a hypercube (up to an affine transformation (cf. [Lo3]). It is still a simple polytope, that is a polytope of dimension $n-1$ such that any vertex comes with n edges. Can one characterize the families of shuffles which give rise to simple polytopes ?

7.3. **The trefoil knot.** There is a strange relation between the trefoil knot and the associahedron $\mathcal{K}^3$. Let us draw on $\mathcal{K}^3$ a path starting from the center of a square (which may be a rectangle in fact) goes to the center of an adjacent pentagon, then goes to the other adjacent square, and so on, until we are back to the starting point (the path goes once through all the pentagon centers and twice through the square centers). By replacing (judiciously) the intersection at the centre of the squares by an under-over crossing, we obtain the trefoil knot.

7.4. **Hopf algebras.** The dendriform algebra built on the planar binary trees in section 5 bears a richer structure: it is a Hopf algebra (cf. [LR1]). In fact it is the non-commutative version of the Connes-Kreimer Hopf algebra. It is related to the renormalization problem, cf. [BF].

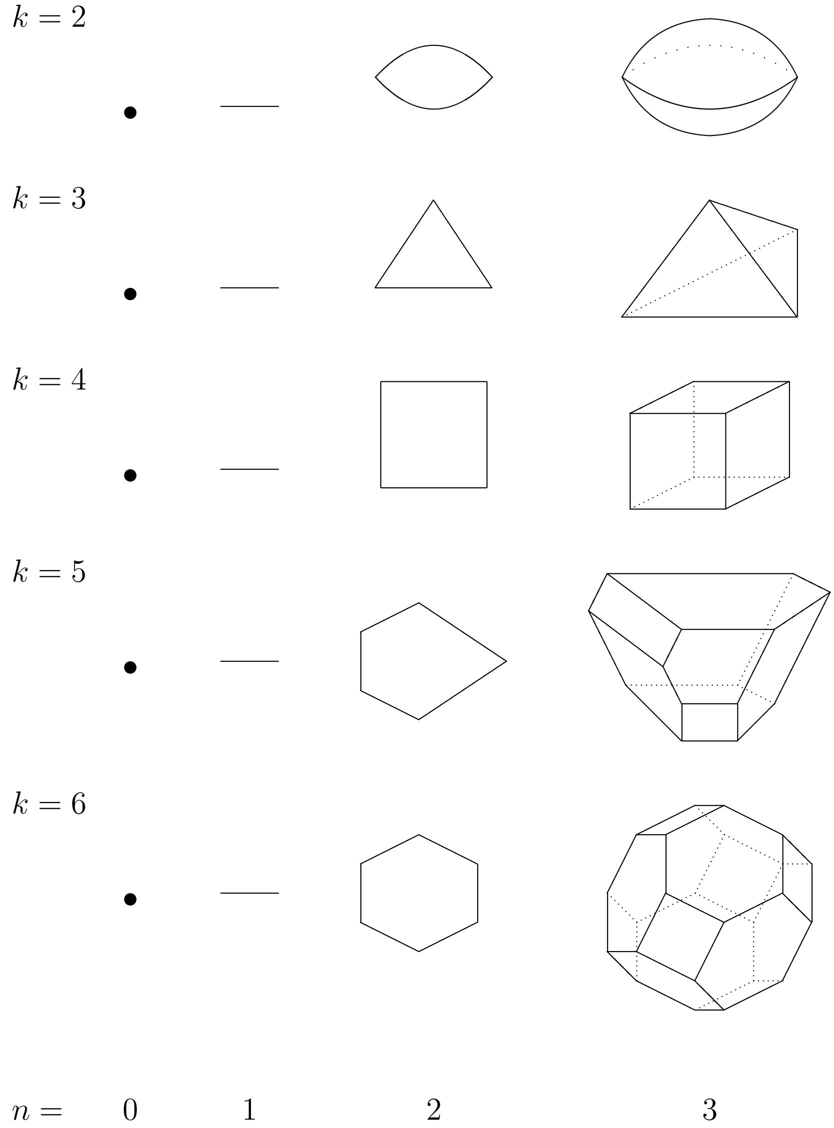
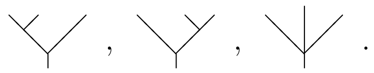


Figure 1. Polytopes

7.5. **Planar trees.** One can ask if the algebraic and combinatorial properties of the planar binary trees described in section 5 can be extended to all the planar trees (not just binary trees). First, note that the family of such trees are in bijective correspondence with the cells of the associahedron. We already remarked that the vertices are in one to one correspondence with the planar binary trees. At the other extreme, the corolla (tree with only one vertex) corresponds to the top cell.

For the trees with three leaves we now have three trees:



The first two correspond to the operations right and left respectively. So it is natural to introduce a third operation to take the corolla into account. Then we obtain a new algebra structure with 3 generating operations. We can show that they are related by 7 relations (one for each cell of the triangle). See [LR3] and [PR] for more details.

7.6. **Moduli spaces.** The real points $\overline{\mathcal{M}}_0^n(\mathbb{R})$ of the moduli space of Riemann spheres with n labeled punctures form a space which admits a tiling by associahedrons, see for instance [Dev].

7.7. **Ehrhart polynomial.** Here is a question about the associahedron. The Strasbourg mathematician Eugène Ehrhart has defined an interesting invariant of polytopes, which is now called the Ehrhart polynomial, cf. [E]. Problem: compute the Ehrhart polynomial of the associahedron. Some computations of the Ehrhart polynomial of the permutohedron can be found in [BP].

7.8. **Physics-Chemistry.** Mother Nature offers us numerous examples of molecules arranged as a tetrahedron or a cube. One can also find permutohedrons, called *Birkhoff cells* in this context. Does there exist a molecule arranged as an associahedron ?

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Solution of the exercice. The face of $\mathcal{K}^3$ which contains these four points is in the hyperplane $F_{2,2}$.

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