## **ON CATALAN'S CONJECTURE**

Preda Mihailescu (University of Paderborn)

Je ne fais pourtant de mal à personne En suivant les chemins qui n'mènent pas à Rome Mais les braves gens n'aiment pas que L'on suive une autre route qu'eux. George Brassens

 $x^{u} - y^{v} = 1 \implies (x, y; u, v) = (3, 2; 2, 3)$ 

Proof of Catalan's conjecture

The intriguing pair  $3^2 - 2^3 = 1 = 9 - 8$  had been shown in the 13th century to be the only pair of consecutive powers of 3 and 2; this was the Spanish-Jewish astronomer Ben Gershon. In 1844, Catalan conjectured they also are the only consecutive proper powers of integers (different from 0,1). He was right! Considering the equation  $x^p - y^q = 1$  with prime exponents p, q the following landmarks had been achieved by 2000: the case q = 2 (V. Lebesgue) and p = 2 (Ko Chao); Cassels showed that the "I cases of Catalan" had no solutions. Using Baker's theory, Tijdeman showed that the equation might have at most a finite number of solutions and eventually upper bounds on the exponents were improved to  $p < 7.10^{11}$  and  $q < 7.10^{16}$  (when p < q). Lower bounds were gained by verifying algebraic conditions stemming mainly from Inkeri. The lower bound  $p, q > 10^6$  was reached in 1999 and in 2000, using a new result, "the double Wieferich conditions", of the lecturer,  $p, q > 3.10^8$  was proved. If  $\mathbb{Q}(\zeta)$  is the *p*-th cyclotomic extension and (x, y; p, q) is a solution to Catalan, let  $\alpha = \frac{x-\zeta}{1-\zeta} \in \mathbb{Z}[\zeta]$ ; by Cassels' work, it is known that  $N(\alpha) = \nu^q$  and thus the ideal  $\mathfrak{a} = (\alpha, \nu) \subset \mathbb{Z}[\zeta]$  has  $\mathfrak{a}^q = (\alpha)$ . q-primary algebraic numbers of  $\mathbb{Q}(\zeta)$  being essentially such ones which are q-adic q-th powers, I prove that for every  $\theta \in \mathbb{F}_q[\operatorname{Gal}(\mathbb{Q}(\zeta)^+/\mathbb{Q})]$  which annihilates the q-primary cyclotomic units (mod q-th powers, certainly),  $(x - \zeta)^{\theta} = \nu^{q}$  holds. The algebraic integer  $\nu$  can then be found by power series expansion: in the reals not only the binomial series converges to  $\nu$ , but its sum also commutes with Galois action. This strong properties, together with generous lower bounds for x, which were given by Hyyrö, help prove that  $\nu$  can only be an algebraic integer if  $\theta = 0 \mod q$ . This again implies that all cyclotomic units should be q-primary, if a new solution to Catalan's equation should exist. However, if p > q one easily shows that all cyclotomic units cannot be q-primary, a contradiction which confirms Catalan's intuition. QED

KUWAIT TOUNDATION LECTURE 30 28 APRIL 2003 Preda MiHAiLESCU Je m fais pourtant de mal à personne / En suivant les climins qui n'ménent pas à Rome/Mair les braves gens n'aiment pas que/L'on suive une autre route qu'enx (q. Brassens)  $X' - Y' = 1 \Rightarrow (X, Y; u, V) = (3, 2; 2, 3)$ Proof of Cotolan's conjecture The intraguing pair 32-23=1=9-8 had been shown in the 13th century to be the only pair of consecutive powers of 3 and 2; this was the spenish-jewich astronomer Ben gershon. In 1844, Catalan conjectured they also are the only consecutive proper powers of integers (diffethey also are the only contraction of powers of inlegers (diffe-nont from 0,1). He was right! Considering the equation  $X^{-}y^{-}y^{-}=1$ with prime exponents P.9, the following landmarks had been achieven by 2000: the case q=2-V. Lebessone and p=2 (Ko (dao); Casuls dowed by 2000: the case q=2-V. Lebessone and p=2 (Ko (dao); Casuls dowed that the "I cases of Catalan" had no solutions. Using Baber's theory, Tijdeman shourd that the equation might have at most a finite number is also and that the equation might have at most a finite number Tijdeman stourd nor the equation hope bounds on the exponents were in-of colutions and eventhally upper bounds on the exponents were im-proved to p<7.10' and q<7.10' (shen p<q). Lover bounds were proved to p<7.10' and q<7.10 (shen p<q). Lover bounds were gained by verifying algebraic conditions, stemming mainly from Inkers. 2 Using The lower bound pig > 10° was reached in 1309 and in 2000, Using a new result, "He double wief end condition of the lectures, P,9>3.10" was proved. If Q(3) is the p-th cyclotomic extension and (x,y; P,9)'s a solution to latolen, let  $d = \frac{x-3}{x-3} \in \mathbb{Z}[3]$ ; by (ageds' work it is known that  $N(d) = v^2$  and  $H_m$  the idear  $O_1 = (d, v) \subset \mathbb{Z}[3]$  has:  $O_2^2 = (d)$ . q - primeryalgebraic numbers of Q(3) being essentially sad once which are gradic algebraic numeries of a constraining sold once which are gradic gril powers, I prove that for every O E FE Gal Q(s) / O] which annihis latic the g-primery cyclotomic units (mod g-th powers, certainly), (x-s) = V9 holds. The algebraic integer v can then be found by (x-s) = V9 holds. The algebraic integer v can then be found by to v, but its sum also commutes with galois action. This strong. To v, but its sum also commutes with galois action. This strong. proportion, together with generous lower bounds for x, which were given by Hyyro, help prove that v can only be an algebraic integer if by Hypro, he'p this spain implies that all cyclotomic units should B=0 modq. This spain implies that all cyclotomic units should be q-primary, if a new solution to Catalan's equation should toost. However, if p>q one easily shows that all a clotomic units cannot be However, if p>q one easily shows that all a clotomic units cannot be q-primary, a contradiction which confirms (atolan's intuition GED