# TRIANGULAR NUMBERS WHICH ARE SUMS OF TWO TRIANGULAR NUMBERS

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Professor Ernest Eckert, who taught at USCA in the early 90's, has written several papers on Pythagorean triangles and triangular numbers have played an important role in his research. He especially became interested in triangular numbers which are also the sum of two triangular numbers. When $t(z) = \frac{1}{2}z(z+1)$, positive integer solutions to the basic equation $t(z) = t(x) + t(y)$ produce special numbers $z$ that he and I now call *c-numbers* ( EIS entry A012132 )[1]. Positive numbers that are not *c-numbers* we call *e-numbers* (EIS entry A027861 ). An important open question is whether there is an infinity of *e-numbers*. This paper is at most a prelude to answering that question; here it is our modest goal to simply gain new information about *c-numbers*. However, we do show in Section 5 that there are arbitrarily large gaps in the series of *e-numbers*. This, together with figure 5 at the end of this paper, shows that the series of *e-numbers* shares properties with the series of prime numbers.

In Section 1 we show that for a fixed positive integer $z$, the solutions $(x, y)$ to the basic equation lie on a special circle. By examining this circle we determine if there are positive integer solutions $(x, y)$ and hence we establish geometric conditions to decide when a number is, or is not, a *c-number*. In Section 2 we show positive integral solutions $(x, y)$ also lie on parabolas. From this we develop families of sequences of *c-numbers*. Section 4 expands on Eckert's treatment of the surface $t(z) = t(x) + t(y)$, which it turns out, is a hyperboloid closely connected to the hyperboloid $x^2 + y^2 = z^2 + 1$. Other sections will be devoted to related topics stemming from Sections 1, 2 and 4. The material in this paper is easily accessible requiring little more than a knowledge of analytic geometry and beginning number theory.

**Note:** All referenced figures occur at the end of this document.

## 1. The circles $\mathcal{C}[n]$

A *triangular number* $t(n)$ is the sum $t(n) = \sum_{i=1}^{n} i$ , or equivalently, is defined as $t(n) = \frac{1}{2}n(n+1)$. This latter definition has the advantage of being defined for all real numbers $n$, an advantage we are about to exploit. We call a positive integer $n$ a *c-number* if $t(n) = t(a) + t(b)$ for positive

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[1]EIS is The Encyclopedia of Integer Sequences [4]

integers $a$ and $b$. For example, $t(3) = t(2) + t(2)$ and therefore 3 is a *c-number*. In fact, all triangular numbers are *c-numbers* because if $m = t(n)$, then $t(m) = t(m-1) + m = t(m-1) + t(n)$. On the other hand, one can check by inspection that $4, 5$ and 7 are not *c-numbers*. The equation $t(8) = t(6) + t(5)$ makes 8 the first non-triangular *c-number*. We let $C$ be the set of all *c-numbers* and we make the following definition:

**Definition 1.1.** *The equation $t(z) = t(x) + t(y)$ can be rewritten as*

$$(x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 = z^2 + z + \frac{1}{2}$$

*and this equation defines a circle with center at $(-\frac{1}{2}, -\frac{1}{2})$ passing through $(0, z)$ and $(z, 0)$ which we call $\mathcal{C}[z]$. The circle $\mathcal{C}[z]$ has the polar equation*

$$P_z(\theta) = (-\frac{1}{2} + R\cos(\theta), -\frac{1}{2} + R\sin(\theta))$$

*where $R = R_z = \sqrt{z^2 + z + \frac{1}{2}}$.*

In particular, $\mathcal{C}[n]$ exists for all positive integers $n$ (Figure 1 shows $\mathcal{C}[10]$); our interest in this section is to determine properties of the circle which distinguish between *c-numbers* and non *c-numbers*. Non *c-numbers* we call *e-numbers* and we set $E$ to be the set of all *e-numbers*. We have already shown that $E \neq \emptyset$. By definition, $C \cup E$ consists of all positive integers.

Ernest Eckert [3], among others, has conjectured that $E$ is infinite. Sierpinski [10] presents a proof that this is so; however his proof depends upon there being an infinite number of primes of the form $n^2 + (n+1)^2$. [2]

We are only interested in the upper right quarter of $\mathcal{C}[n]$ and this quarter-circle can be broken down further into the upper eighth-circle consisting of the points $P_n(\theta)$ for $0 < \theta \leq \pi/4$ which we name $\mathrm{ure}[n]$ (note that we include the point $P_n(\frac{\pi}{4})$) [3]. The sector $\mathrm{ure}[n]$ can be seen on $\mathcal{C}[10]$ in Figure 1 between the two dashed lines. We denote by $\Lambda$ the fundamental point-lattice of all pairs of integers. Most importantly, we are only interested in the $\Lambda$ points on these circle parts. Hence, we make the following definition:

**Definition 1.2.** *Let $\Lambda_n$ be $\Lambda$ minus the point $(n, 0)$. We use $\mathcal{U}$ to select out exactly the points in* ure *with positive integer coordinates:*

$$\mathcal{U}[n] = \mathrm{ure}[n] \cap \Lambda_n.$$

The importance of $\mathcal{U}[n]$ is that its points $(a, b)$ are exactly the positive integer solutions to the equation $t(n) = t(a) + t(b)$ with $a \geq b$. Figure 1 shows the one and only point $(9, 4)$ on $\mathcal{U}[10]$. Figure 2 shows $\mathcal{U}[n]$ for the first 17 *c-numbers*

$$3, 6, 8, 10, 11, 13, 15, 16, 18, 20, 21, 23, 26, 27, 28, 31, 33.$$

[2] this would follow from an hypothesis of Schinzel [8], [9], and [11].

[3] $P_n(\frac{\pi}{4})$ rarely has integer coordinates. $3, 20, 119, 696, 4059$, and $23660$ are the only such $n$ between 1 and 100,000. See EIS entry A001652

The next theorem follows directly from Definitions 1.1 and 1.2.

**Theorem 1.1.** *If $a$ and $b$ are positive integers less than $n$, then*

$$t(n) = t(a) + t(b) \Leftrightarrow (a, b) \in \mathcal{C}[n].$$

*Hence the integer $n \in C$ if and only if $\mathcal{U}[n] \neq \emptyset$.*

*Proof.* The only comment needed here is that if $a$ and $b$ are positive integers and $(a, b) \in \mathcal{C}[n]$, then either $(a, b)$ or $(b, a)$ belongs to $\mathcal{U}[n]$. □

We need the following lemma for the next corollary as well as in Section 3:

**Lemma 1.1.** *For each positive integer $n$, $n^2 + (n+1)^2$ and all of its prime factors are of the form $4k+1$. Hence, $n^2+(n+1)^2$ is composite if and only if there exist integers $p$ and $q$, $0 \leq p < n < q$, such that $n^2+(n+1)^2 = p^2+q^2$.*

*Proof.* The lemma follows mainly from two theorems. The first theorem is: Every prime of the form $l = 4k + 1$ can be written as the sum $p^2 + q^2 = l$, $0 < p < q$ in one and only one way; [6] page 56. The second theorem is: If an odd prime divides the sum of the squares of two relatively prime integers, then it must be of the form $4k + 1$; [11] page 378.

Let $m = n^2 + (n + 1)^2$. Note that $m = 2n(n + 1) + 1$ and thus is of the form $4k + 1$. All prime factors of $m$ must be odd; hence, because $n$ and $n + 1$ are relatively prime, all prime factors of $m$ must be of the form $4k+1$. To complete the proof, we must show that when $m$ is composite then it can be represented by a sum of two squares in more than one way. Let $m = p \cdot q$. The formula 4.3 ([6] page 55 ) shows that products of sums of two squares are also a sum of two squares. Thus let $p = a^2 + b^2$ and $q = c^2 + d^2$. Therefore, either $q \cdot p = (ac + bd)^2 + (bc - ad)^2$ is a representation different from $p \cdot q$ or $p \cdot q = (ac + bd)^2$. Either leads to the desired result. □

We can now give a geometric proof of a relevant theorem of Sierpinski [10].

**Corollary 1.1.** *Sierpinski's Theorem. $n \in C \Leftrightarrow n^2 + (n+1)^2$ is composite.*

*Proof.* We note first that

$$P_n(\theta) = (a, b) \Rightarrow P_n(\theta + \pi/2) = (-b - 1, a).$$

We position the square inscribed in $\mathcal{C}[n]$ so that one corner is at the point $(n, 0)$. See Figure 1. By our formula above, one side of the square has end points at $(n, 0)$ and $(-1, n)$; hence this square has area $n^2 + (n + 1)^2$.

Suppose $n \in C$. Then there exists a point $(a, b) \in \mathcal{U}[n]$. Hence we can reposition the square so that one side has end points at $(a, b)$ and $(-b-1, a)$. Thus, the area of the square is now $(a + b + 1)^2 + (a - b)^2$. That is,

$$n^2 + (n + 1)^2 = (a + b + 1)^2 + (a - b)^2$$

and because $0 \leq a - b < a < n$, we have that the value $n^2 + (n + 1)^2$ can be represented as the sum of two squares in two different ways. According to Lemma 1.1 this shows $n^2 + (n + 1)^2$ is composite.

Conversely, if $n^2 + (n+1)^2$ is composite then again by Lemma 1.1 there exist integers $p$ and $q$ such that $n^2+(n+1)^2 = p^2+q^2$ where $0 \leq p < n < q$. For any $(a,b) \in \mathcal{C}[n]$, we have also that $(-b-1, a) \in \mathcal{C}[n]$ and therefore

$$(a+b+1)^2 + (a-b)^2 = n^2 + (n+1)^2 = p^2 + q^2.$$

We solve the pair of equations $a+b+1 = p,\ a-b = q$ to obtain $a = -\frac{1}{2} + \frac{p+q}{2}$ and $b = -\frac{1}{2} + \frac{q-p}{2}$ and because both $q-p$ and $p+q$ are odd positive integers ($p$ and $q$ must have opposite parity), $a$ and $b$ are positive integers with $a > b$. Hence, $(a,b) \in \mathcal{U}[n]$ and thus $n \in C$. $\square$

## 2. The parabolas $\mathcal{P}$

If one makes a graph of a number of the concentric circles parts ure$[n]$ imposed on a $\Lambda$ lattice graph (See Figure 2. ), one will see the lattice points $\mathcal{U}[n]$, for a few $n \geq 3$, and one will see that some of the lattice points seem to lie on parabolas. In fact we will see that all of the lattice points lie on parabolas as well as on circles (See Figure 3 .[4]). The lowest parabolic sequence of points on the graph, one will note, lie on the circles $\mathcal{C}[t(n)]$. This is the sequence of $\Lambda$ points $(t(n)-1, n)$, for $n \geq 2$. The points are solutions to the equation $t(t(n)) = t(t(n)-1) + t(n)$; this, as we pointed out earlier, is the equation which proves all triangular numbers are *c-numbers*.

**Lemma 2.1.** *Let $u_k = (2k+1)^2$ and $v_k = 8k$. Consider the family of parabolas $f_k(x) = \frac{1}{2}(-1 + \sqrt{u_k + v_k x})$ Then for all $k > 0$ and all $x > 0$, $t(x+k) = t(x) + t(f_k(x))$.*

*Proof.* The lemma is proved by solving the following equation for y:

$$t(x+k) = t(x) + t(y).$$

We set $f_k(x)$ equal to the positive solution $y = \frac{1}{2}(-1+\sqrt{1+4k+4k^2+8kx})$. $\square$

It is not difficult to see that $f_1(t(n)-1) = n$. Thus the points $(t(n)-1, n)$ lie on the parabola $f_1$. The plot in Figure 3 shows the truth of the next lemma:

**Lemma 2.2.** *The curve*

$$f_1(x) = \frac{-1+\sqrt{9+8x}}{2}, \quad 0 \leq x$$

*is the lower bound for the set* $\bigcup_{3 \leq n} \mathcal{U}[n]$

*Proof.* Let $(a,b) \in \mathcal{C}[n]$ and suppose $b < \frac{-1+\sqrt{9+8a}}{2}$. Then $(2b+1)^2 < 9+8a$. From $(2b+1)^2 = 8t(a)+1$, we conclude $t(a) < a+1$. Hence, $t(n) = t(a)+t(b) < t(a)+a+1 = t(a+1)$. Thus we have $t(a) \leq t(n) < t(a+1)$ which forces $t(a) = t(n)$ and therefore forces $b = 0$. This proves the lemma. $\square$

[4] One also sees two lines of points; more about this in Section 5

We note that according to Lemma 2.1, if $(a, b)$ is a lattice point on the parabola $f_k$, then $a + k$ is a *c-number*. This leads us to the next definition:

**Definition 2.1.** *We denote by $\mathcal{P}[k]$ the set of lattice points on the parabola with equation $y = f_k(x)$. If $(a, b)$ is a point on $\mathcal{P}[k]$, we say $\mathcal{P}[k]$ implicitly defines the c-number $a + k$. We let $\mathcal{N}[k]$ be the set of* c-numbers *which are implicitly defined by $\mathcal{P}[k]$.*

**Lemma 2.3.** *For $k \geq 1$, $\mathcal{P}[k]$ is an infinite set. This follows from:*

$$\mathcal{P}[k] = \{(\frac{1}{k}(t(n) - t(k)), n) : k \mid t(n-k)\}.$$

*Proof.* The point $(x, f_k(x))$ is on $\mathcal{P}[k]$ if and only if both $x$ and $f_k(x)$ are positive integers. $f_k(x)$ is the integer $n$ if and only if $u_k + v_k x$ is the square of $2n + 1$. Solving $u_k + v_k x = (2n + 1)^2$ for $x$, we have

$$x = \frac{n^2 + n - k^2 - k}{2k} = \frac{1}{k}(t(n) - t(k)).$$

From $t(n - k) = t(n) - t(k) + k^2 - nk$, we have $k \mid (t(n) - t(k)) \Leftrightarrow k \mid t(n - k)$. $\square$

Note: Lemma 2.5 also proves that $\mathcal{P}[k]$ is infinite.

We define $\mathcal{S} = \bigcup_{n \geq 3} \mathcal{U}[n]$. Thus, by definition the family of finite sets $\mathcal{U}[n]$, $n \geq 3$, is a partition of $\mathcal{S}$. We show this true for $\mathcal{P}[k]$ as well:

**Theorem 2.1.** *The family of the infinite sets $\mathcal{P}[k]$, $k \geq 1$, is a partition of $\mathcal{S}$. Thus, $C = \bigcup_{k>0} \mathcal{N}[k]$ where $C$ is the set of all* c-numbers*.*

*Proof.* The equation

$$f_{k+1}(x) - f_k(x) = \frac{4(k + x + 1)}{\sqrt{u_{k+1} + v_{k+1} x} + \sqrt{u_k + v_k x}}$$

shows that for positive $x$, $f_k(x) < f_{k+1}(x)$, and hence that the family of sets $\mathcal{P}[k]$ is disjoint.

Suppose $(i, j) \in \mathcal{S}$. Then for some $n$, $t(n) = t(i) + t(j)$. Set $k = n - i$. Hence, $t(n) = t(i + k) = t(i) + t(f_k(i))$ and thus $f_k(i) = j$ and therefore $(i, j) \in \mathcal{P}[k]$. $\square$

In Figure 3 one can see that on $\mathcal{P}[k]$, for $k =$ 2,3,4,or 5, the points appear to be in groups of two. But in Figure 4, which shows $\mathcal{P}[6]$, the points are in groups of four. These groupings, we will show, depend upon the prime factorization of $2k$.

The set $C$ of all *c-numbers* does not have easily discernible patterns; however we will show that $C$ is made up of numerous subsequences of simple composition which we find by studying the sets $\mathcal{P}[k]$. In order to understand this and the grouping mention in the preceding paragraph, we must explore in detail the condition

$$k \mid (t(n) - t(k)) \Leftrightarrow k \mid t(n - k) \Leftrightarrow 2k \mid (n - k)(n - k + 1).$$

Thus, given a positive integer $m$, we need a general method of finding all $i$ such that $m|i(i+1)$. This leads us to the following definitions, examples, and theorems:

**Definition 2.2.** set$[m]$ *is the set of prime power factors of* $m$ *and* number$[m]$ *is the number of elements of* set$[m]$. *That is, if* $m = \prod_{i=1}^{n} p_i^{\alpha_i}$ *is the prime factorization of the positive integer* $m$*, we write* set$[m] = \{p_i^{\alpha_i} : 1 \leq i \leq n\}$ *and* number$[m] = n$.

For example, if $m = 100800$, set$[m] = \{9, 64, 25, 7\}$ and number$[m] = 4$.

If $p$ and $q$ are relatively prime positive integers it is well know that there are an infinite number of pairs $(a, b)$ such that $ap-bq = 1$. This set consists exactly of the pairs $(a_0+iq, b_0+ip)$ where $(a_0, b_0)$ is an arbitrary solution to $ap - bq = 1$ while $i$ varies over all integers. We want to select from this set the positive integer pair nearest the origin. We do this in the next lemma:

**Lemma 2.4.** *If* $p$ *and* $q$ *are relatively prime positive integers, then there exists exactly one pair of positive integers* $(a, b)$ *such that* $ap-bq = 1$*,* $a < q$*, and* $b < p$*. Hence, there also exists exactly one pair of positive integers* $(c, d)$ *such that* $cq - dp = 1$*,* $c < p$*, and* $d < q$*. In fact,* $c = p - b$ *and* $d = q - a$*. Finally, we note for future reference, that* $bq + dp = pq - 1$.

*Proof.* First we show there exists at least one such pair $(a, b)$. If $p$ and $q$ are relatively prime, we choose a pair of positive integers $(a_0, b_0)$ such that $a_0p - b_0q = 1$. Let $i_0$ be the least integer $i$ such that both members of the pair $(a_0+iq, b_0+ip)$ are positive and set $(a, b) = (a_0+i_0q, b_0+i_0p)$. Neither $a = q$ nor $b = p$ is possible because, for example, if $p = b$ then $(a - q)p = 1$ which is not possible. If $a > q$ then $q(p - b) < 1$; hence, $b > p$. But now, if we set $i_1 = i_0 - 1$, both members of the pair $(a_0 + i_1q, b_0 + i_1p)$ are positive contradicting the definition of $i_0$. Thus, $0 < a < q$ and $0 < b < p$.

Now note that if $\alpha p - \beta q = 1$, there is an integer $i$ such that $\alpha = a + iq$ and $\beta = b + ip$. If $i > 0$ then $\alpha > q$ and $\beta > p$. If $i < 0$, then both $\alpha$ and $\beta$ are negative. This proves there is no more than one such pair $(a, b)$. Finally, if we set $c = p - b$ and $d = q - a$, we have $cq - dp = 1$, $c < p$, and $d < q$.

The last statement of the lemma follows from the computation $bq + dp = bq + (q - a)p = bq + pq - ap = pq - 1$. □

Suppose we wish to find all non-negative integers $i$ such that $12|i(i+1)$. By inspection, we see that the first four $i$ are $0, 3, 8$, and $11$. A little thought tells us that all other $i$ are obtained by adding multiples of 12 to the first four numbers. In fact, we can see these numbers in rows and columns as follows:

$$\begin{pmatrix} 0 & 3 & 8 & 11 \\ 12 & 15 & 20 & 23 \\ 24 & 27 & 32 & 35 \\ 36 & 39 & 44 & 47 \\ 48 & 51 & 56 & 59 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

See, also, EIS entry A108752:

In other words, a matrix which we shall call S[12], of four columns and an infinity of rows. The entire set is known once we find the first row or in fact any one row. Rather than by inspection we can find the first row as follows: set[12] $= \{3, 4\}$; hence find positive integers $a, b, c$, and $d$ satisfying the previous lemma with $p = 3$ and $q = 4$. We see that $a = 3, b = 2, c = 1$, and $d = 1$ gives us $3 \cdot 3 - 2 \cdot 4 = 1$ and $1 \cdot 4 - 1 \cdot 3 = 1$. Thus it is the lesser products $bq$ and $dp$, in this case, 8 and 3, that lie in the first row. The first and last numbers in the first row will always be 0 and $pq - 1$; in this case, 0 and 11. In this example, the elements of the first row come in pairs $(\alpha, \beta)$ such that $\alpha + \beta = pq - 1$ and we will see this is true in general. Except for the basic pair $(0, pq - 1)$ all other pairs result from Lemma 2.4 from the lesser products $(bq, dp)$.

For any positive integer $m$, where set[$m$] has two elements, finding all solutions to $m|i(i+1)$ goes exactly as the preceeding example. When set[$m$] has only one element, the situation is quite simple. The matrix of solutions to $m|i(i + 1)$ will have two infinite columns headed by 0 and $m - 1$.

These two examples show in essence, given a positive integer $m$, how we solve the problem of finding all $i$ such that $m|i(i+1)$. The extra complication in the general case will be when set[$m$] has more than two elements.

**Theorem 2.2.** *Suppose* number[$m$] $= n$. *We define* $\mathrm{S}[m] = \{i : m \mid i \cdot (i + 1)\}$. $\mathrm{S}[m]$ *can be written as the matrix of* $2^n$ *columns and an infinite number of rows, with entries* $\mathrm{S}[m](i, j)$, *as follows:* $\mathrm{S}[m](1, 1) = 0$, $\mathrm{S}[m](1, 2^n) = m - 1$, *and* $\mathrm{S}[m](k + 1, i) = \mathrm{S}[m](1, i) + k \cdot m$. *Hence, the matrix is completely determined by its first row. The method of determining the remaining integers* $\mathrm{S}[m](1, i)$ *in the first row is explained in the proof.*

***Note:*** *The sequence of all first rows of* S[$m$] *for* $m > 1$ *is EIS entry A108760.*

*Proof.* Let FR[$m$] denote the first row of S[$m$]. The elements of FR[$m$] come in non-ordered pairs (as exemplified in the example above) $\pi(i) = \{i, m - 1 - i\}$ where $0 \le i < m$. We will show these pairs are in one-to-one correspondence with the partitions of set[$m$] into two sets provided we include the set $\{\emptyset, \mathrm{set}[m]\}$ as one partition (which, strictly speaking is not a partition) that corresponds to the pair $\pi(0)$. The Stirling number $2^{n-1} - 1$ is the number of ways we can partition set[$m$] into two non-empty subsets and therefore the lemma is proven once we demonstrate the one-to-one correspondence.

Each $i$ in FR[$m$] determines a partition of set[$m$]; namely

$$A_1 = \{q \in \mathrm{set}[m] : q|i\},\ A_2 = \{q \in \mathrm{set}[m] : q|(i + 1)\}.$$

It is not difficult to show that $m - 1 - i$ determines the same partition of set[$m$]. Hence, $\pi(i)$ corresponds uniquely to the partition $\{A_1, A_2\}$.

Conversely, let the two sets $A$ and $B$ be a partition of set[$m$] and let $p = \prod A$ (the product of all integers in $A$) and $q = \prod B$. Then according to lemma 2.4, provided neither set in the partition is empty, there exist four positive integers integers $a, b, c,$ and $d$ so that $bq + dp = m - 1$, and such that $ap = bq + 1$ and $cq = dp + 1$; hence, the pair $\pi(bq) = \pi(dp) = \{bq, dp\}$ belongs to $\mathrm{FR}[m]$. Thus the partition consisting of the two sets $A$ and $B$ corresponds uniquely to $\pi(bq)$. $\square$

From lemma 2.3, we know that

$$\mathcal{P}[k] = \{(\frac{1}{k}(t(n) - t(k)), n) : k \mid t(n-k)\}.$$

Hence, we have the following lemma and definitions:

**Definition 2.3.**

$$\xi(k, n) = \frac{1}{k}(t(n) - t(k))$$

**Lemma 2.5.**

$$\mathcal{P}[k] = \{(\xi(k, n), n) : n - k \in \mathrm{S}[2k]\}.$$

*Proof.* The lemma follows from

$$k \mid t(n-k) \Leftrightarrow 2k \mid (n-k)(n-k+1).$$

$\square$

The condition $n - k \in \mathrm{S}[2k]$ means that for some $i \geq 0$, $n = m + k + 2ki$ where $m$ is an element in the first row of $\mathrm{S}[2k]$. This gives rise to the functions $a$ and $b$ defined in the next definition:

**Definition 2.4.** *For each pair of integers $k$ and $m$, if $0 < k$, $0 < m < 2k$, and if $2k|m(m+1)$, we define*

$$b(k, m, i) = k + m + 2ki \text{ and } a(k, m, i) = \xi(k, b(k, m, i))$$

The previous definition shows that the matrix $\mathrm{S}[2k]$ generates three more matrices of the same dimensions (the same number of columns and an infinite number of rows) which we will call $\mathrm{R}[k]$, $\mathrm{T}[k]$, and $\mathrm{W}[k]$:

**Definition 2.5.** *Let $m_j$ be the $j$th element in the first row of $S[2k]$. Then $\mathrm{R}[k]$ is the matrix whose $j$th column is the infinite sequence $b(k, m_j, i)$, $\mathrm{T}[k]$ is the matrix whose $j$th column is the infinite sequence $a(k, m_j, i)$, and $\mathrm{W}[k]$ is the matrix whose $j$th column is the infinite sequence $a(k, m_j, i) + k$.*

**Theorem 2.3.** *If $A$ and $B$ are matrices of the same dimension, we define $A \star B$ to be the set of all pairs $(A[i, j], B[i, j])$ for appropriate $i$ and $j$. Thus, we have:*

$$\begin{aligned} &\mathcal{P}[k] = T[k] \star R[k] - \{(0, k)\} \\ &\textit{and} \\ &\mathcal{N}[k] = W[k] - \{k\}. \end{aligned}$$

*Hence, $C = \bigcup_{k>0}(W[k] - \{k\})$ where $C$ is the set of all $c$-*numbers.

*Note: The two sets $\{(0, k)\}$ and $\{k\}$ are removed because they give rise to the false $c$-number $n$ satisfying the equation $t(n) = t(0) + t(n)$.*

*Proof.* The theorem follows from Theorem 2.1, Lemma 2.5, and Definition 2.5. □

Example:

$$
\begin{array}{cccc}
\mathrm{S}[12] & \mathrm{R}[6] & \mathrm{T}[6] & \mathrm{W}[6] \\
| & | & | & | \\
\begin{pmatrix} 0 & 3 & 8 & 11 \\ 12 & 15 & 20 & 23 \\ 24 & 27 & 32 & 35 \\ 36 & 39 & 44 & 47 \\ 48 & 51 & 56 & 59 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} &
\begin{pmatrix} 6 & 9 & 14 & 17 \\ 18 & 21 & 26 & 29 \\ 30 & 33 & 38 & 41 \\ 42 & 45 & 50 & 53 \\ 54 & 57 & 62 & 65 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} &
\begin{pmatrix} 0 & 4 & 14 & 22 \\ 25 & 35 & 55 & 69 \\ 74 & 90 & 120 & 140 \\ 147 & 169 & 209 & 235 \\ 244 & 272 & 322 & 354 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} &
\begin{pmatrix} 6 & 10 & 20 & 28 \\ 31 & 41 & 61 & 75 \\ 80 & 96 & 126 & 146 \\ 153 & 175 & 215 & 241 \\ 250 & 278 & 328 & 360 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}
\end{array}
$$

In general, these matrices serve several important purposes. For example, the matrices above give reason why the plot of $\mathcal{P}[6]$ shows points in groups of four [5]. They also show that for each integer $k > 0$, if number$[2k] = n$, $\mathcal{P}[k]$ is partitioned by the $2^n$ sequences of pairs $(a(k, m_j, i), b(k, m_j, i))$, for $1 \leq j \leq 2^n$. In addition, $\mathcal{N}[k]$ is partitioned by the family of sequences $a(k, m_j, i) + k$, for $1 \leq j \leq 2^n$.

They automatically provide formulas for simple sequences of points on $\mathcal{P}[k]$; for example we see in $\mathcal{P}[6]$ (Figure 4.) the point sequence

$$\{(25, 18), (74, 30), (147, 42), (244, 54), \cdots\}$$

What is important about this sequence is that the right coordinates of these points is the arithmetic sequence $b(k, m_j, i)$ with common difference $d_k = 2k$ (in this case $d_6 = 12$) while the left coordinates are the image of the right coordinates under the function $\xi$. This provides a certain order and structure to $\mathcal{P}[k]$ that otherwise, on first viewing in Figure 2 at least, seems a random jumble of points.

In the next corollary and theorem, we develop a formula for *c-numbers* which essentially stands alone making no references to the constructs in this paper such as $\mathcal{C}[k]$, $\mathcal{N}[k]$, and $\mathcal{P}[k]$. However it does indirectly call on the construct $\mathrm{S}[m]$ but in such a way that one could use the formula without knowledge of $\mathrm{S}[m]$.

**Corollary 2.1.** *For every c-*number *$n$ there exists a positive integer $k$ and non-negative integers $m$ and $i$ such that*

$$t(n) = t(n - k) + t(k + m + 2ki).$$

*Proof.* If $n \in C$, then there exists positive integers $a$ and $b$ so that $t(n) = t(a) + t(b)$. Hence, by Theorem 2.1 there is a $k$ such that $(a, b) \in \mathcal{P}[k]$. Thus, by Theorem 2.3 there exists an integer $m$, $0 \leq m < 2k$, such that $2k|m(m+1)$ and there exists a non-negative integer $i$ such that $a = a(k, m, i)$ and $b = b(k, m, i)$. Because $n = k + a(k, m, i)$ and $b(k, m, i) = k + m + 2ki$, the corollary is proved. □

[5] By the same reasoning, the points on $\mathcal{P}[k]$, for $k = 3$ *and* 5, also occur in groups of four though this is not readily apparent in Figure 3.

The corollary above essentially reiterates that every *c-number* can be found in at least one matrix $W[k]$. The next theorem shows that every *c-number* $n$ is in fact in the first row of some matrix $W[k]$ where $k < n$.

**Definition 2.6.** *We define the function $w(k,m)$ for positive integer values of $k$ and $m$ follows:*

$$w(k,m) = \begin{cases} k + m + \frac{m(m+1)}{2k} & \text{if } 2k|m(m+1) \text{ and } 0 < m < 2k; \\ 0 & \text{otherwise.} \end{cases}$$

*The value 0 in the second case is arbitrary; we include the second case simply to have $w$ defined as an integer for all integer values of $k$, $m$. Note that $k$ plus the non-zero outputs of $w(k,m)$ make up the first row of $W[k]$.*

**Theorem 2.4.** *When $w(k,m) \neq 0$, $w(k,m) \in C$. More importantly, if $n$ is a $c$-*number *then there exists a pair of positive integers $k < n$ and $m$ such that $w(k,m) = n$. Thus, $w$ maps $NxN$ onto $C \cup \{0\}$.*

*Proof.* For the first assertion, note that if $w(k,m) \neq 0$ then $w(k,m)$ is in $W[k] - \{k\}$ and hence is a *c-number*.

To prove the second assertion, we start with the identity

$$n^2 - a^2 - b^2 = 2(n-a)(n-b) - (a+b-n)^2$$

for all numbers $a$, $b$, and $n$. If $t(n) = t(a) + t(b)$ then we also have the equation

$$n^2 - a^2 - b^2 = a + b - n.$$

Therefore combining these two equations we get the identity

$$(id) : (a+b-n)^2 + (a+b-n) = 2(n-a)(n-b).$$

These calculations were suggested by Professor Eckert. From Corollary 2.1, we know there exists $k$, $m$, and $i$ such that $k = n - b$ and $m + 2ki = a - k = a + b - n$. Assuming $b \leq a$ we show that $i = 0$.

We only need show that $m = a + b - n$ satisfies the first condition of Definition 2.6. First note that the identity (id) guarantees that $2k|m(m+1)$. Since $m = n^2 - a^2 - b^2$, we have $m > 0$. To prove $m < 2k = 2(n-b)$ we show $3(n-b) > a$. This can be done analytically by returning to the parametric equations of $\mathcal{C}[n]$ given in Definition 1.1:

$$a = a(n,\theta) = -0.5 + R_n \cos\theta \ ; \ b = b(n,\theta) = -0.5 + R_n \sin\theta$$

where $R_n = \sqrt{n^2 + n + 0.5}$. We then define

$$f(n,\theta) = 3(n-b) - a = 0.5 - R_n \cos\theta + 3(0.5 + n - R_n \sin\theta).$$

One can check that $f(n,\theta)$ is concave upward for $0 \leq \theta \leq \pi/2$, that $f(n,0) = 3n+2-R_n > 0$, that $f(n,\theta) = 0$ at $\theta \approx 0.97$, and that $f(n,\theta)$ has a minimum at $\theta \approx 1.24$. We conclude from this that $f(n,\theta) > 0$ when $0 \leq \theta \leq \pi/4$ which proves $m < 2k$ when $b \leq a$. $\square$

Summing up and refining is the purpose of the next corollary:

**Corollary 2.2.** *Every* $c$-number $n$ *equals* $w(k,m)$ *where* $\frac{n+2}{5} \leq k < n$. *Furthermore, if* $t(n) = t(a) + t(b)$ *and* $b \leq a$*, then* $n = w(n-b, a+b-n)$. *Conversley, if* $n = w(k,m)$ *and* $n > 0$*, then* $t(n) = t(m+k) + t(n-k)$.

*Proof.* $w(k, 2k-1) = 5k-2$ is the largest value of $w(k,m)$ for a given $k$. Thus, we derive the lower bound $\frac{n+2}{5}$ for $k$. □

**Theorem 2.5.** *The set* $E$ *can be obtained by a sieve similar to the "sieve of Eratosthenes". For each* $k > 0$*, let* $w[k]$ *be the set of non-zero values of* $w(k,m)$ *as* $0 < m < 2k$*. Recursively set:*

$$sieve(0) = N, \ sieve(k) = sieve(k-1) - w[k].$$

*Then* $E = lim_{k\to\infty} sieve(k)$.

One can use the sieve above to generate initial segments of the set $E$ but better methods exist. For example, we can use the Mathematica predicate "$PrimeQ[n]$", which is true when $n$ is a prime, and then make use of Sierpinski's Theorem (corollary 1.1). This will of course not prove that $E$ is infinite but we can use these segments to produce visual evidence in favor of this proposition. As an example, see Figure 5. Here we have the distribution function $\pi(n)$, which is the number of primes not larger than n and we have the distribution function pi$(n)$, which is the number of *e-numbers* not larger than n. The figure shows the graphs of $\pi(n)/n$ in yellow and of pi$(n)/n$ in red for $2 \leq n \leq 15000$. In addition it shows the graph of $\frac{1.5}{Log(x)}$ in green. These graphs are certainly consistent with $E$ being infinite.

## 3. Gauss's Circle Problem

Gauss studied the problem of how many lattice points lie on a circle; this problem can be solved for $\mathcal{C}[n]$. Also we want to count the number of points in $\mathcal{U}[n]$. Corollary 1.1 can be generalized to the following theorem:

**Theorem 3.1.** *For a positive integer* $n$*, if* $\tau(n)$ *is the number of ways that* $n^2+(n+1)^2$ *can be represented as a sum of the squares of two integers, then there are* $\tau(n)$ $\Lambda$ *points lying on* $\mathcal{C}[n]$. *Furthermore, for each representation*

$$p^2 + q^2 = n^2 + (n+1)^2$$

*the* $\Lambda$ *point* $(a,b)$ *lies on* $\mathcal{C}[n]$ *where* $a = -\frac{1}{2} + \frac{p+q}{2}$ *and* $b = -\frac{1}{2} + \frac{q-p}{2}$.

We make use of Gaussian integer theory. We use the formula $\tau$ in Theorem 8, Chapter 13.6, [11] to compute the number of representations. We have this corollary to Theorem 3.1:

**Corollary 3.1.** *If the prime factorization of* $n^2 + (n+1)^2 = \prod p_j^{\alpha_j}$*, there are exactly* $\tau(n) = 4\prod(\alpha_j + 1)$ $\Lambda$ *points lying on* $\mathcal{C}[n]$.

The computation of the formula $\tau$ depends upon the fact that, according to Lemma 1.1, each prime $p_j$ in the factorization of $n^2+(n+1)^2$ is a $4k+1$ prime. Each such prime is uniquely represented as a sum of two squares $a_j^2+b_j^2$; therefore the representations of $p_j^{\alpha_j}$ as a sum of two squares can be computed from the products

$$\prod (a_j+ib_j)^{\lambda_j}(a_j-ib_j)^{\alpha_j-\lambda_j}, \qquad 0\le \lambda_j\le \alpha_j$$

where $z_j=a_j+ib_j$ and $z_j'=a_j-ib_j$ are conjugate complex prime factors of $p_j$. The fact that $0\le\lambda_j\le\alpha_j$, gives rise to $\alpha_j+1$ Gaussian integers for each $j$. We must also count the four associates $z,-z,iz,-iz$ of each $z$ produced by the product above; hence the 4 in the formula $\tau$.

For a given $z$ produced by the formula, the associates of $z$ and $z'$ yield eight different Gaussian integers provided $z\ne z'$. Otherwise the associates of $z$ and $z'$ yield four different Gaussian integers. For the purpose of counting points in $\mathcal{U}[n]$, we want a formula $T$ which counts only the representations $n^2+(n+1)^2=x^2+y^2$ with $0\le x<y$. The four, or eight, associates of $z$ and $z'$ produce exactly one such Gaussian integer $x+iy$ per $z$. Hence, the following lemma:

**Lemma 3.1.** *The number of representations of* $n^2+(n+1)^2=\prod p_j^{\alpha_j}$ *as a sum of two squares* $x^2+y^2$ *with* $0\le x<y$ *is given by the formula*

$$T(n)=\begin{cases}\frac{\prod(\alpha_i+1)}{2} & \text{if } \prod(\alpha_i+1) \text{ is even;}\\ \frac{1+\prod(\alpha_i+1)}{2} & \text{if } \prod(\alpha_i+1) \text{ is odd.}\end{cases}$$

*Therefore, the number of points in* $\mathcal{U}[n]$ *is* $T(n)-1$. *If there are no representations other than* $x=n$ *and* $y=n+1$ *(i.e.* $T(n)=1$*), then* $n$ *is an* $e$-number*.*

*Proof.* Directly one can see that

$$T(n)=\begin{cases}\frac{\tau(n)}{8} & \text{if } \prod(\alpha_i+1) \text{ is even;}\\ \frac{\tau(n)-4}{8}+1 & \text{if } \prod(\alpha_i+1) \text{ is odd.}\end{cases}$$

We only need note that $T$ counts the Gaussian integer $n+i(n+1)$ which gives rise to the representation $n^2+(n+1)^2$ which in turn gives rise to the point $(n,0)$ on $\mathcal{C}[n]$. □

**Note:** In a paper by Ono, Robins, and Wahl [7], the authors develop the function $\delta_k(n)$ that counts the number of representations of $n$ as a sum of k triangular numbers. We are interested in $Q(n)=\delta_2(t(n))$ that counts sums $t(x)+t(y)$ twice because order of summands matter, provided $x\ne y$, and also counts twice the sum $t(n) + t(0)$. Hence it is not difficult to see that

$$T(n) - 1 = \begin{cases} \frac{Q(n)-3}{2} + 1 & \text{if } Q(n) \text{ is odd;} \\ \frac{Q(n)-2}{2} & \text{otherwise.} \end{cases}$$

The authors' results concerning $Q$ and our results concerning $T$ also follow from the authors' next lemma, called Proposition 2 in [7]:

**Lemma 3.2.** *If $q(n)$ counts the number of representations of $t(n)$ as a sum of two odd squares, then*

$$Q(n) = q(8t(n) + 2).$$

## 4. Eckert's Hyperboloid

The equation $t(z) = t(x) + t(y)$ defines the surface we will call "Eckert's Hyperboloid". This is of course the same equation we use in Section 1 to define the circle $\mathcal{C}[z]$ for a fixed real number $z$ and in particular the circle $\mathcal{C}[n]$ for a fixed integer $n$. The difference in this section is that we obtain the surface by allowing $z$ to vary over all real numbers. Professor Eckert rewrites the equation $t(z) = t(x)+t(y)$ in the form $(2x+1)^2+(2y+1)^2 = (2z+1)^2+1$ and from this we see that the surface defined by the equation $t(z) = t(x) + t(y)$ is mapped into the hyperboloid H with equation $X^2 + Y^2 = Z^2 + 1$ by the function $\tau(v) = 2v + (1, 1, 1)$. More precisely we have the following definition:

**Definition 4.1.** *In this context we define the lattice $\Lambda$ to be the set of triples with positive integer coordinates and we define* EH *(Eckert's hyperboloid) to be the set of $\Lambda$ triples on the surface $t(z) = t(x) + t(y)$. The translator $\tau$, defined above, and its inverse, takes us between $EH$ and the $\Lambda$ points on the surface* H *with odd coordinates. In Section* 1 *notation we can define*

$$\mathrm{EH} = \{(a, b, n) : (a, b) \in \mathcal{U}[n] \text{ or } (b, a) \in \mathcal{U}[n], n \in N_+\}$$

*where $N_+$ is the set of positive integers.*

A Pythagorean triangle ($PT$) is a triple of positive integers $(a, b, c)$ such that $a^2 + b^2 = c^2$. A primitive Pythagorean triangle ($PPT$) is a $PT$ $(a, b, c)$ such that the integers $a$, $b$ and $c$ are relatively prime. Eckert in [3], shows three linear operators, that when applied repeatedly to $PPT$s, generate $PPT$s. Eckert cites, among others, F. J. M. Barning [1] and A. Hall [5] for proofs that these operators can generate the complete $PPT$ tree. These operators are described in the next definition.

**Definition 4.2.** *The operators* D*,* A *, and* U*.*

$$\mathrm{D} = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix} \quad \mathrm{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \quad \mathrm{U} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

*Also we define the Lorentz norm*

$$\| v \| = v_1^2 + v_2^2 - v_3^2.$$

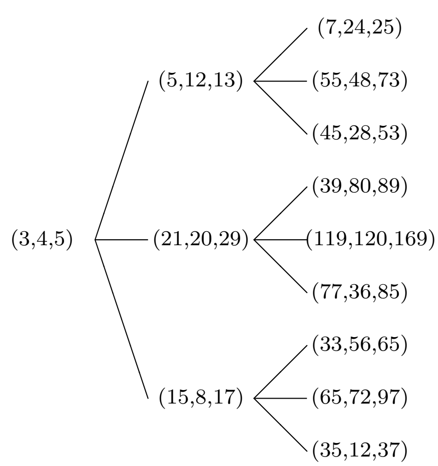
*The important connection between the operators and the norm is this: If $O$ is any one of the three operators* D*,* A*, or* U *then for any vector $v$*

$$\| O \cdot v \| = \| v \| .$$

*Hence, as pointed out in* [3]*, the Lorentz norm is invariant under under* D*,* A*, and* U*.*

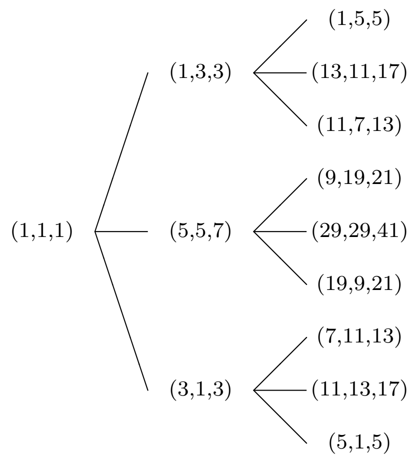
**Note** In order to keep everything on the same line, we will write vectors horizontally as $1 \times 3$ matrices and when multiplying a square matrix $A$ by a vector $v$ we will write $Av$ horizontally in place of the correct form $(Av^T)^T$.

The invariance of the Lorentz norm guarantees that the operators map $PPT$s to $PPT$s. The following partial tree appears in [3] and in [5]:




The operators D, A , and U are so named for "down", "across", and "up" with respect to the tree above. Thus U$(3, 4, 5)$ goes "up" to $(5, 12, 13)$ and so forth.

Our main interest here is in how Eckert applies the operators to the hyperboloid H. He makes use of the fact that the invariance of the Lorentz norm guarantees that repeated applications of the operators to a vector on H produces vectors on H. More specifically, he shows that the operators map $\tau$(EH) into $\tau$(EH). The following partial tree also appears in [3]:




In order to keep track of what the operators are doing, we define the path function. An example of how the function works is this: if $s$ is the string of letters "*uad*", then $\text{path}(s, v) = \mathrm{U}(\mathrm{A}(\mathrm{D}\, v))$. Hence, a path represents a composition of some combination of the three operators. The definition for path is recursive:

**Definition 4.3.**

$$\begin{aligned} \text{path}(``", v) &= v \\ \text{path}(``u", v) &= \mathrm{U}\, v \\ \text{path}(``a", v) &= \mathrm{A}\, v \\ \text{path}(``d", v) &= \mathrm{D}\, v \\ \text{path}(s, v) &= \text{path}(\mathit{head}(s)), \text{path}(\mathit{rest}(s), v)) \end{aligned}$$

*where if $s$ is a string of letters, $head(s)$ is the leftmost letter of $s$ and $rest(s)$ is all of $s$ except the leftmost letter. For example, $head$("aeiou") = "a" and $rest$("aeiou") = "eiou".*

Because paths are compositions of non-singular linear operators, paths themselves are non-singular linear operators. Hence we have the next lemma:

**Lemma 4.1.** *The function* path *is one-to-one with respect to the vector variable. That is,* $\text{path}(t, v_1) = \text{path}(t, v_2) \Rightarrow v_1 = v_2$.

The surface H is a hyperboloid of one sheet and is therefore a doubly ruled surface which means it can be swept out by a moving line in space called a ruling. In fact it can be swept out by two such rulings. Details can be found in Eric W. Weisstein's World of Mathematics [12]. Citing from this work, these lines are of the form $\beta(u) + v\delta(u)$ where $\beta$ is called the base curve and $\delta$ is called the director curve. See figure 6 at the end of this paper; this is a graphic of H in terms of its rulings. For H we will call these two rulings $R_1$

and $R_2$ and their equations are as follows:

$$\begin{aligned}
\beta(\theta) &= (\cos\theta, \sin\theta, 0)\\
\delta(\theta) &= (\cos\theta, \sin\theta, 1)\\
R_1(\theta, z) &= \beta(\theta) + z\delta(\theta + \pi/2)\\
R_2(\theta, z) &= \beta(\theta) + z\delta(\theta - \pi/2).
\end{aligned}$$

The rulings shown in figure 6 are at angles $\theta$ spaced 5 degrees apart.

**Lemma 4.2.** *Let* $v = (p, q, r) \in \mathrm{H}$. *Then the equations*

$$R_1(\theta_1, r) = v \ , \ R_2(\theta_2, r) = v$$

*have solutions*

$$\begin{aligned}
\cos\theta_1 &= \frac{p + rq}{r^2 + 1} \ , \ \sin\theta_1 = \frac{q - pr}{r^2 + 1}\\
\cos\theta_2 &= \frac{p - rq}{r^2 + 1} \ , \ \sin\theta_2 = \frac{q + pr}{r^2 + 1}
\end{aligned}$$

*Hence,* $\theta_1$ *is in the fourth quadrant and* $\theta_2$ *is in the second quadrant.*

*Proof.* If $(p, q, r) = R_1(\theta_1, r)$, then

$$p = \cos\theta_1 - r\sin\theta_1 \text{ and } q = \sin\theta_1 + r\cos\theta_1$$

while if $(p, q, r) = R_2(\theta_2, r)$, then

$$p = \cos\theta_2 + r\sin\theta_2 \text{ and } q = \sin\theta_2 - r\cos\theta_2$$

The proof follows directly from these equalities. □

There is an important and interesting relationship between the operators U, A and D, points on $\tau(\mathrm{EH})$, and $PPT$s. Eckert shows in [3] that if the point $P = (p, q, r) \in \mathrm{H}$, then the two vectors $P_1 = (pr - q, qr + p, r^2 + 1)$ and $P_2 = (pr + q, qr - p, r^2 + 1)$ are $PT$s in the direction of the rulings $R_1$ and $R_2$. This fact is confirmed by the previous lemma. Furthermore, we can see that $P_1$ is in the direction of $R_1$ and that $P_2$ is in the direction of $R_2$.

What we will show is that $P$, and the the two $PPT$s corresponding to $P_1$ and $P_2$ are all produced by the same path. But before continuing we need to point out the following fact which we put in the form of a lemma:

**Lemma 4.3.** *Although the vectors* $(1, 0, 1)$ *and* $(0, 1, 1)$ *are not strictly speaking* $PPT$*s, they can be used to generate all* $PPT$*s by applying the operators* U, A *and* D.

*Proof.* In point of fact, applying these three operators as written to $(1, 0, 1)$ we obtain respectively $(3, 4, 5)$, $(3, 4, 5)$, and $(1, 0, 1)$. Similarly applying these three operators as written to $(0, 1, 1)$ we obtain respectively $(0, 1, 1)$, $(4, 3, 5)$, and $(4, 3, 5)$. From this, one can see that in order to actually produce all $PPT$s as triples (order of coordinates matter) then we must start with the vectors $(1, 0, 1)$ and $(0, 1, 1)$. □

**Example:**

$P = \text{path}(\text{“}d\text{”}, (1,1,1)) = (3,1,3)$

and in addition using the same paths we generate two PPT's

$\Delta_1 = \text{path}(\text{“}d\text{”}, (1,0,1)) = (1,0,1)$

$\Delta_2 = \text{path}(\text{“}d\text{”}, (0,1,1)) = (4,3,5).$

and we note that

$$P_1 = (8,6,10) = 2 \cdot \Delta_2 \text{ and } P_2 = (10,0,10) = 10 \cdot \Delta_1$$

Furthermore, Lemma 4.2 shows that

$$R_1(\theta_1, 3) = P, \quad \text{where } \theta_1 = -\arccos(\tfrac{3}{5})$$
$$R_2(\theta_2, 3) = P, \quad \text{where } \theta_2 = \arccos(\tfrac{0}{1}) = \tfrac{\pi}{2}$$

Figure 7 at the end of this paper provides some geometrical insight into this example. In order to prove our main theorem, Theorem 4.1, that summarizes the information of this example, we need the next lemma. The lemma and its proof is due to Professor Eckert:

**Lemma 4.4.** *If $P$ and $Q$ are distinct points on $\Lambda \cap \mathrm{H}$ then $P$ and $Q$ lie on the same ruling iff $P - Q$ is an integer multiple of a $PPT$.*

*Proof.* If $P$ and $Q$ are both lattice points on the same ruling of H, say $R_1(\theta, z)$, then $P - Q = z\delta(\theta + \pi/2)$ which is a $PT$ and hence an integer multiple of a $PPT$.

The rulings of H lie on tangent planes to H. Suppose $P = (p, q, r)$. A tangent plane to H passing through $P$ has equation

$$z - r = \frac{p}{r}(x - p) + \frac{q}{r}(y - q)$$

which, because $p^2 + q^2 = r^2 + 1$, simplifies to

$$px + qy - rz = 1.$$

Letting $Q = (x, y, z)$, if $P - Q = n(a, b, c)$, $n \in N$, where $(a, b, c)$ is a $PPT$, then

$$(p - x)^2 + (q - y)^2 - (r - z)^2 = n^2(a^2 + b^2 - c^2) = 0$$

and because $x^2 + y^2 = z^2 + 1$, this simplifies to

$$px + qy - rz = 1$$

meaning $Q$ is both on H and a tangent plane to H passing through $P$. Therefore, $P$ and $Q$ are on the same ruling. □

**Theorem 4.1.** *If $P = \text{path}(t, (1,1,1))$ then director vectors for the two rulings $R_1$ and $R_2$ through $P$ are the $PPT$ vectors $\Delta_1$ and $\Delta_2$ given by the path formulas*

$$\Delta_1 = \text{path}(t, (1,0,1)) \text{ and } \Delta_2 = \text{path}(t, (0,1,1)).$$

*Conversely, if the director curve $\delta$ of a ruling $R$ is a multiple of a $PPT$, then there exists at least one point of $\Lambda \cap \mathrm{H}$ on $R$.*

*Proof.* The proof follows from two simple equations and lemma 4.4. The equations are

$$P = \text{path}(t, (1, 1, 1)) = \text{path}(t, (1, 0, 1)) + \text{path}(t, (0, 1, 0))$$

$$P = \text{path}(t, (1, 1, 1)) = \text{path}(t, (0, 1, 1)) + \text{path}(t, (1, 0, 0))$$

By setting $Q_1 = \text{path}(t, (0, 1, 0))$ and $Q_2 = \text{path}(t, (1, 0, 0))$ we know that $P$ and $Q_i$, $i = 1, 2$, are on the same rulings and that $P - Q_i = \Delta_i$ are the director vectors of the respective rulings. By lemma 4.1, we know $\Delta_1 \neq \Delta_2$ and thus they are director vectors for two distinct rulings.

**Note:** Whether $\Delta_i$, $i = 1$ or 2, is the director vector for $R_1$ or $R_2$ seems to depend on whether the path $t$ has an even number of $a$'s or an odd number of $a$'s.

For the converse, suppose the ruling is $R_1$ and that $\delta(\theta) = z(a, b, c)$ for some real number $z$. Then we calculate that $\cos\theta = b/c$ and $\sin\theta = -a/c$ where $(a, b, c)$ is the $PPT$ in question. Hence

$$R_1(\theta, r) = (b/c, -a/c, 0) + r(a/c, b/c, 1) = \left(\frac{ra + b}{c}, \frac{rb - a}{c}, r\right).$$

We want to show there are positive integers $m$ and $n$ such that for some $r$, $ra + b = mc$ and $rb - a = nc$. We have the following chain of equivalences:

$$(ra + b = mc) \wedge (rb - a = nc) \Leftrightarrow mbc - b^2 = nac + a^2 \Leftrightarrow mb - na = c.$$

We know there exists positive integers $i$ and $j$ so that $ib - ja = 1$. Thus, $(ci)b - (cj)a = c$ and therefore we set $m = ci$ and $n = cj$ and finally we choose $r$ so that:

$$r = \frac{mc - b}{a} = \frac{nc + a}{b}.$$

With this value for $r$ we obtain

$$R_1(\theta, r) = \left(\frac{ra + b}{c}, \frac{rb - a}{c}, r\right) = (m, n, r).$$

It remains to show that $r$ is an integer but this follows from

$$r^2 = m^2 + n^2 - 1.$$

A similar proof holds if the ruling is $R_2$. □

**Corollary 4.1.** *If $R$ is a ruling whose director curve $\delta$ is a multiple of the $PPT$ $\Delta$ and if $P$ is the point of $\Lambda \cap \mathrm{H}$ on the ruling $R$ of minimum distance from the origin , then*

$$R \cap \Lambda \cap \mathrm{H} = \{P + n\Delta : n \in N\}.$$

*Furthermore, if $P$ is the point of $\tau(\mathrm{EH})$ on the ruling $R$ of minimum distance from the origin , then*

$$R \cap \tau(\mathrm{EH}) = \{P + 2n\Delta : n \in N\}$$

*where $N$ is the set of non-negative integers.*

*Proof.* From theorem 4.1 we know there is at least one lattice point on $R$. Thus there is one that is nearest the origin; call it $P$. Hence, by lemma 4.4, $R \cap \Lambda \cap \mathrm{H} = \{P + n\Delta : n \in N\}$. The second part of the corollary follows from the simple fact that if $P \in \tau(\mathrm{EH})$ then so to does $P + 2n\Delta$. □

## 5. Special sequences

In Figure 3, amongst the parabolas, one can see two lines and with just a little effort (tracking down the coordinates of a few points) one can see the lines have equations $3x - 4y + 2 = 0$ and $3x - 4y - 3 = 0$. The two lines are related to two special sequences of *c-numbers* $\alpha_n = 3 + 5n$ and $\beta_n = 6 + 5n$ defined for all $n \geq 0$. These sequences were pointed out by Professor Eckert who first read about them in Dickson's *History of the Theory of Numbers* [2]. The proof that the sequences produce *c-numbers* is given by the following two identities:

$$t(3 + 5n) = t(2 + 4n) + t(2 + 3n) \text{ and } t(6 + 5n) = t(5 + 4n) + t(3 + 3n).$$

Setting $x_n = 2 + 4n$ and $y_n = 2 + 3n$, we obtain the linear identity $3x_n - 4y_n + 2 = 0$. Similarly, with $x_n = 5 + 4n$ and $y_n = 3 + 3n$, we obtain the identity $3x_n - 4y_n - 3 = 0$. This begins to explain the two lines in Figure 3. These lines contain positive integer solution points $(a, b)$ to the equation $t(c) = t(a) + t(b)$ where $c = \alpha_n$ for the first equation and $c = \beta_n$ for the second equation. Again, one can see this by tracking down the coordinates of a few points on the lines in Figure 3. However, there is a better explanation for this connection. The sequences above are special cases of a more general scheme. For this we must return to section 4. From corollary 4.1 we know that given a ruling $R$ whose director curve $\delta$ is a multiple of a $PPT$ $\Delta$, then

$$R \cap \tau(\mathrm{EH}) = \{P + 2n\Delta : n \in N\}$$

where $P$ is the point of $R \cap \tau(\mathrm{EH})$ nearest the origin. This leads to the following theorem:

**Theorem 5.1.** *Given a $PPT$ $\Delta = (a, b, c)$, there exists a point $(x, y, z) \in$ EH such that*

$$t(z + cn) = t(x + an) + t(y + bn).$$

*Therefore, we have an infinite sequence of what we might call $c$-*number *triples $(x + an, y + bn, z + cn)$ and specifically an infinite sequence of $c$-*numbers *$z + cn$ defined for all $n \geq 0$.*

*Proof.* Given a $PPT$ $\Delta = (a, b, c)$, there is a path $t$ such that $\Delta = \mathrm{path}(t, v)$ where $v$ is either the vector $(1, 0, 1)$ or $(0, 1, 1)$. According, then, to theorem 4.1, there exists a ruling $R$ whose director curve $\delta$ is a multiple of $\Delta$. And thus by corollary 4.1 we have, as mentioned above, a point $(p, q, r) \in R \cap \tau(\mathrm{EH})$ such that

$$R \cap \tau(\mathrm{EH}) = \{(p + 2na, q + 2nb, r + 2nc) : n \in N\}.$$

Applying the inverse of the translator function $\tau$ to the vector

$$(p + 2na, q + 2nb, r + 2nc)$$

we obtain

$$(x + na, y + nb, z + na) = \tau^{-1}(p + 2na, q + 2nb, r + 2nc)$$

with $(x, y, z) = \tau^{-1}(p, q, r) \in \mathrm{EH}$. $\square$

The following is a corollary to lemma 4.2 and theorem 4.1.

**Corollary 5.1.** *Theorem 5.1 guarantees that given a PPT* $\Delta = (a, b, c)$, *there exists a point* $(x, y, z) \in \mathrm{EH}$ *such that*

$$t(z + cn) = t(x + an) + t(y + bn).$$

*There are in fact two such points* $(x_i, y_i, z_i)$, $i = 1, 2$, *which are found as follows: define* $\theta_1 = -\arccos(b/c)$ *and* $\theta_2 = \arccos(-b/c)$. *Then, we choose* $P_i$ *to be the point of* EH *on* $R_i(\theta_i)$ *that is nearest the origin and we set* $(x_i, y_i, z_i) = \tau^{-1}(P_i)$, $i = 1, 2$. *By* $R_i(\theta_i)$ *we mean the set*

$$\{R_i(\theta_i, z) : z \text{ a real number}\}.$$

From this corollary we can generate an infinite number of *c-number* sequences. For example, let $\Delta = (5, 12, 13)$. We set $\theta_1 = -\arccos(12/13)$ and $\theta_2 = \arccos(-12/13)$. Then one can see by calculation (using Mathematica perhaps) that

$$R_1(\theta_1) \cap \tau(\mathrm{EH}) = \{(9, 19, 21), (19, 43, 47), \cdots\}$$
$$R_2(\theta_2) \cap \tau(\mathrm{EH}) = \{(11, 29, 31), (21, 53, 57), \cdots\}$$

and that $(4, 9, 10) = \tau^{-1}(9, 19, 21)$ and $(5, 14, 15) = \tau^{-1}(11, 29, 31)$. This gives rise to the two sequences of *c-number* triples

$$(4 + 5n, 9 + 12n, 10 + 13n)$$
$$(5 + 5n, 14 + 12n, 15 + 13n)$$

and the two *c-number* sequences

$$10 + 13n \text{ and } 15 + 13n \text{ for } n \geq 0.$$

The special sequences we have described above are generated by applying $\tau^{-1}$ to subsets of rulings on H. This observation is the motivation for corollary 5.3 below.

**Corollary 5.2.** *Each point* $(x, y, z)$ *on the surface* EH *defines two distinct special sequences in* EH *of the form* $(x + na, y + nb, z + nc)$.

*Proof.* Let $(p, q, r) = \tau(x, y, z)$. We return to the constructs

$$P_1 = (pr - q, qr + p, r^2 + 1) \text{ and } P_2 = (pr + q, qr - p, r^2 + 1)$$

used in the proof of lemma 4.2. $P_i$, for $i = 1, 2$, are $PT$s in the direction of the rulings $R_i$ and hence are integer multiples of two $PPT$s $\Delta_i$. If $\Delta_i = (a_i, b_i, c_i)$, then the two special sequences are:

$$(x + na_i, y + nb_i, z + nc_i)$$

for $i = 1, 2$. $\square$

**Corollary 5.3.** *The special sequences of* $c$-number *triples guaranteed by the theorem and corollary above lie on special rulings of the hyperboloid defined by the equation $t(z) = t(x) + t(y)$ or equivalently by the equation*

$$(2x + 1)^2 + (2y + 1)^2 = (2z + 1)^2 + 1.$$

*The special rulings are exactly those that are inverse images, under $\tau$, of rulings of* H *whose director curve $\delta$ is a multiple of a $PPT$.*

We include figures 8 and 9 showing the two hyperboloids H and the surface $t(z) = t(x) + t(y)$ side-by-side. Figure 8 shows rulings of the two hyperboloids. We also include figure 10 showing the lattice points $(x, y, z)$ on the surface $t(z) = t(x) + t(y)$ with $x \geq y$ and $z \leq 500$. One can see a number of the rulings. Recall here, we are only looking at lattice points on the rulings, not at all points on the ruling. These lattice points are packed in a density determined by the size of the $PPT$ that defines them; this according to theorem 5.1. The most densely packed are the rulings whose $PPT = (3, 4, 5)$.

*******************

We end this section with an application of special sequences. There are a number of reasons to think of the *e-numbers* as being, in some sense, similar to the set of prime numbers. The distribution function whose graph is in figure 5 is one reason. The next theorem is offered as another reason. The set $E$, like the set of prime numbers, has arbitrarily large gaps. We will prove this using special sequences that generate arbitrarily large chains of successive *c-numbers*. Before stating the theorem we make the following definition:

**Definition 5.1.** *Special sequences of* $c$-numbers *arise from corollary 5.2; each is determined by a* $c$-number *$z$ and a hypotenuse $c$ of a $PPT$ and each has the form $z + nc$. We designate a special sequence with the symbol $ss[z, c]$ and write $ss[z, c](n) = z + nc$.*

**Theorem 5.2.** *The set of* $e$-numbers *$E$ has gaps of arbitrary length.*

*Proof.* By induction, we show that for each integer $n \geq 2$, there is a set $S$ of special sequences $ss[z_k, c_k]$, and a set of positive integers $m_k$ such that

$$ss[z_k, c_k](m_k) = ss[z_{k+1}, c_{k+1}](m_{k+1}) + 1 \text{ for } k = 1, 2, \cdots, n - 1.$$

The proof proceeds this way: for $n = 2$, given two special sequences of *c-numbers* $ss[z_1, c_1]$ and $ss[z_2, c_2]$ so that $c_1$ and $c_2$ are relatively prime, it is guaranteed there exist positive integers $m_1$ and $m_2$ such that

$$ss[z_1, c_1](m_1) = ss[z_2, c_2](m_2) + 1$$

because this equation reduces to

$$z_1 - z_2 - 1 = m_2c_2 - m_1c_1.$$

Now assume that we have a set $S$ of special sequences $ss[z_k, c_k]$ and a set of positive integers $m_k$ such that

$$ss[z_i, c_i](m_k) = ss[z_{i+1}, c_{i+1}](m_{k+1}) + 1 \text{ for } k = 1, 2, \cdots, n-1$$

and such that the set of $c_k$ are mutually relatively prime. First note that if we define $P = \prod c_k$ and $P_k = P/c_k$ then we have

$$ss[z_i, c_i](m_k + iP_k) = ss[z_{i+1}, c_{i+1}](m_{k+1} + iP_{k+1}) + 1 \text{ for } k = 1, 2, \cdots, n-1$$

and for all non-negative integers $i$.

Suppose we now have a special sequence $ss[z, c]$ so that $c$ is relatively prime to each $c_k$. Then as in the case $n = 2$, there is a pair of integers $(i, j)$ such that $ss[z_n, c_n](j) = ss[z, c](i) + 1$. In fact there is an infinite family of such pairs of the form $(j+rc, i+rc_n)$ for all non-negative integers $r$. Because $P_n$ and $c$ are relatively prime, there exist positive integers $r_0$ and $k_0$ so that $m_n + k_0P_n = j + r_0c$. We redefine $m_k = m_k + k_0P_k$ for $k = 1, 2, \cdots, n$ and we define $m_{n+1} = i + r_0c_n$. With this definition, note that $m_n$ now equals $j + r_0c$; hence setting $z_{n+1} = z$ and $c_{n+1} = c$ we have

$$ss[z_k, c_k](m_k) = ss[z_{k+1}, c_{k+1}](m_{k+1}) + 1 \text{ for } k = 1, 2, \cdots, n$$

which proves the theorem. □

We end this section with some examples of special sequences. In fact, we list all special sequences possible to the depth of level three in the $PPT$ tree:

Special sequences from level one, the root $(3, 4, 5)$ of the $PPT$ tree.

$$t(5n + 3) = t(4n + 2) + t(3n + 2) \tag{1}$$

$$t(5n + 1) = t(4n + 1) + t(3n + 0) \tag{2}$$

Special sequences from the second level of the $PPT$ tree.

$$t(13n + 10) = t(12n + 9) + t(5n + 4) \tag{3}$$

$$t(13n + 2) = t(12n + 2) + t(5n + 0) \tag{4}$$

$$t(29n + 8) = t(20n + 5) + t(21n + 6) \tag{5}$$

$$t(29n + 20) = t(20n + 14) + t(21n + 14) \tag{6}$$

$$t(17n + 10) = t(8n + 4) + t(15n + 9) \tag{7}$$

$$t(17n + 6) = t(8n + 3) + t(15n + 5) \tag{8}$$

Special sequences from the third level of the $PPT$ tree.

$$t(169n+119) = t(120n+84) + t(119n+84) \tag{9}$$
$$t(169n+49) = t(120n+35) + t(119n+34) \tag{10}$$
$$t(97n+37) = t(72n+27) + t(65n+25) \tag{11}$$
$$t(97n+59) = t(72n+44) + t(65n+39) \tag{12}$$
$$t(73n+13) = t(48n+8) + t(55n+10) \tag{13}$$
$$t(73n+59) = t(48n+39) + t(55n+44) \tag{14}$$
$$t(85n+23) = t(36n+9) + t(77n+21) \tag{15}$$
$$t(85n+61) = t(36n+26) + t(77n+55) \tag{16}$$
$$t(37n+21) = t(12n+6) + t(35n+20) \tag{17}$$
$$t(37n+15) = t(12n+5) + t(35n+14) \tag{18}$$
$$t(53n+41) = t(28n+21) + t(45n+35) \tag{19}$$
$$t(53n+11) = t(28n+6) + t(45n+9) \tag{20}$$
$$t(89n+27) = t(80n+24) + t(39n+12) \tag{21}$$
$$t(89n+61) = t(80n+55) + t(39n+26) \tag{22}$$
$$t(65n+41) = t(56n+35) + t(33n+21) \tag{23}$$
$$t(65n+23) = t(56n+20) + t(33n+11) \tag{24}$$
$$t(25n+21) = t(24n+20) + t(7n+6) \tag{25}$$
$$t(25n+3) = t(24n+3) + t(7n+0) \tag{26}$$

## 6. Distribution function for the set $E$

We are going to write recursive definitions for the distribution functions for both the set of primes and for the the set $E$ of *e-numbers*. A distribution function $d(n)$ for a set $S$ counts the number of numbers of $S$ less than or equal to $n$. The distribution function for the primes is usually denoted by $\pi$ and we will denote the distribution function for $E$ by pi. Besides the recursive definition for $\pi$ we give below, the Prime Number Theorem gives an asymptotic approximation for $\pi$ of the form

$$\pi(n) \sim \frac{n}{ln(n)}.$$

Similarly, we will show that pi has, as well as a recursive definition, a companion definition; this time in a closed form formula.

First, the recursive definitions:

$$\pi(2) = 1$$

$$\pi(n) = \begin{cases} \pi(n-1)+1 & \text{if } n \text{ is prime;} \\ \pi(n-1) & \text{otherwise.} \end{cases}$$

$$\mathrm{pi}(2) = 1$$
$$\mathrm{pi}(n) = \begin{cases} \mathrm{pi}(n-1)+1 & \text{if } n^2+(1+n)^2 \text{ is prime;} \\ \mathrm{pi}(n-1) & \text{otherwise.} \end{cases}$$

We have proven that $n$ is a $c$-number iff $\mathcal{U}[n] \neq \emptyset$. This is because if the natural number pair $(a,b) \in \mathcal{U}[n]$, then $t(n) = t(a) + t(b)$. Hence, $n$ is an *e-number* iff $\mathcal{U}[n] = \emptyset$. We use this fact to define a closed form definition for pi. First some preliminary functions:

$$\mathrm{a}(b,n) = \sqrt{(n-b)(n+b+1)+0.25} - 0.5$$

The function $\mathrm{a}(b,n)$ is the left-coordinate of points on $\mathcal{C}[n]$ when $b$ is the right coordinate. That is: $(\mathrm{a}(b,n), b) \in \mathcal{C}[n]$. This formula comes directly from definition 1.1.

$$\lim(n) = -0.5 + \frac{r_n\sqrt{2}}{2}$$

$\lim(n)$ is the largest right-coordinate of the points in ure$[n]$.

$$\text{floor}(n) \text{ is the greatest integer function}$$
$$\text{ceiling}(n) \text{ is the least integer function}$$

The names floor and ceiling are used in Mathematica for the same functions.

$$\mathrm{p}(n) = \prod_{b=1}^{lim(n)} (\mathrm{a}(b,n) - \text{floor}(\mathrm{a}(b,n)))$$

If $\mathrm{a}(b,n)$ takes on an integer value then $\mathrm{p}(n) = 0$; otherwise $0 < \mathrm{p}(n) < 1$. Hence, ceiling(p($n$)) is either 0 if $n$ is a $c$-number or 1 if $n$ is an $e$-number. Finally

$$\mathrm{pi}(k) = \sum_{n=2}^{k} \text{ceiling}(\mathrm{p}(n))$$

This is our desired closed form formula for pi. Again, we refer the reader to Figure 5 for a graphical comparison of $\pi$ and pi.

## 7. Mathematica Programs

We are including a few Mathematica notebooks that one can down load to their computer; these will enable the reader to go beyond the figures in this paper. This of course assumes that one has Mathematica installed on their computer. If not, it may still be possible to read and execute the files using Mathreader from Wolfram Research; this utility can be down loaded for free at http://www.wolfram.com/products/mathreader.

Note: If the different links keep loading the same program, exit Mathematica between opening the programs.

The programs:

*Section*1 $\mathcal{U}[n]$. This program produces the contents of $\mathcal{U}[n]$ for any $n$. If $n$ is an *e-number*, the program outputs $\{\}$. Can be used to generate initial segments of $C$ and $E$.

*Section*2 S[$n$]. This program outputs each the four matrices S,R,T, and W as either a set of numbers, a set of rows of numbers, or as a proper matrix.

*Section*3 counting function $\tau$. This program implements the function $\tau(n)$ which counts the number of lattice points on $\mathcal{C}[n]$ and the function $T(n)$ which counts the number of points in $\mathcal{U}[n]$ plus 1 (because it counts the point $(n, 0)$). $T$ can also be used to generate initial segments of $C$ and $E$.

$\mathcal{C}[n]$,$\mathcal{U}[n]$ and $\mathcal{P}[k]$. This program overlays $\mathcal{C}[n]$, $\mathcal{U}[n]$, and $\mathcal{P}[k]$ for a range of $n$ and $k$; it was used to produce Figure 3.

$Sierpinski's\ Theorem$. This program makes use of Corollary 1.1 to generate initial segments of $E$ and to plot the distribution function for the segment. We think this is the most efficient way to produce such segments when using Mathematica. See the explanation at the end of Section 2. The program was used to make figure 5.

*Section*4 path. This program implements the function path$(s, v)$ of Section 4 as well as a findpath function.

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FIGURE 1. $\mathcal{C}[10]$

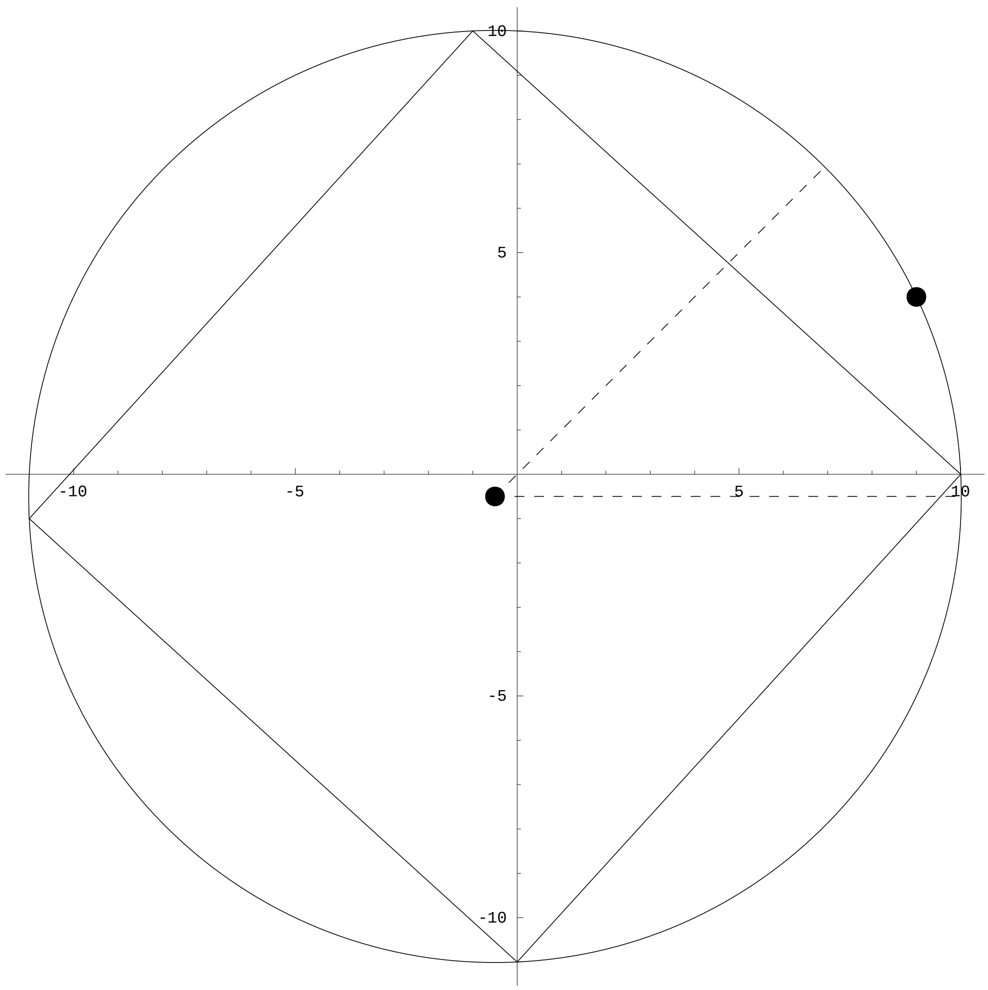

FIGURE 2. Multiple $\mathcal{U}[n]$

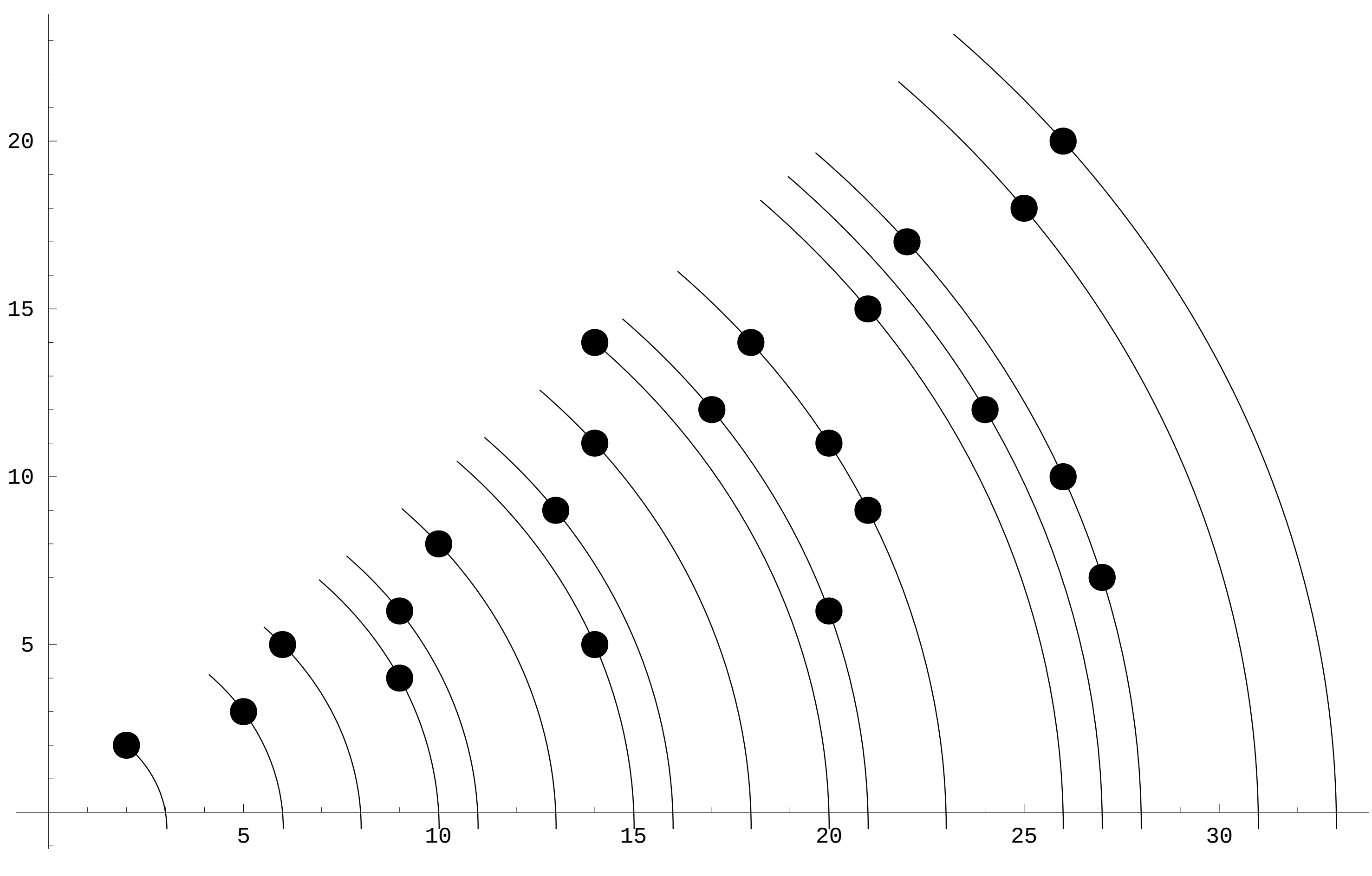

FIGURE 3. Parabolas superimposed upon $\mathcal{U}[n]$

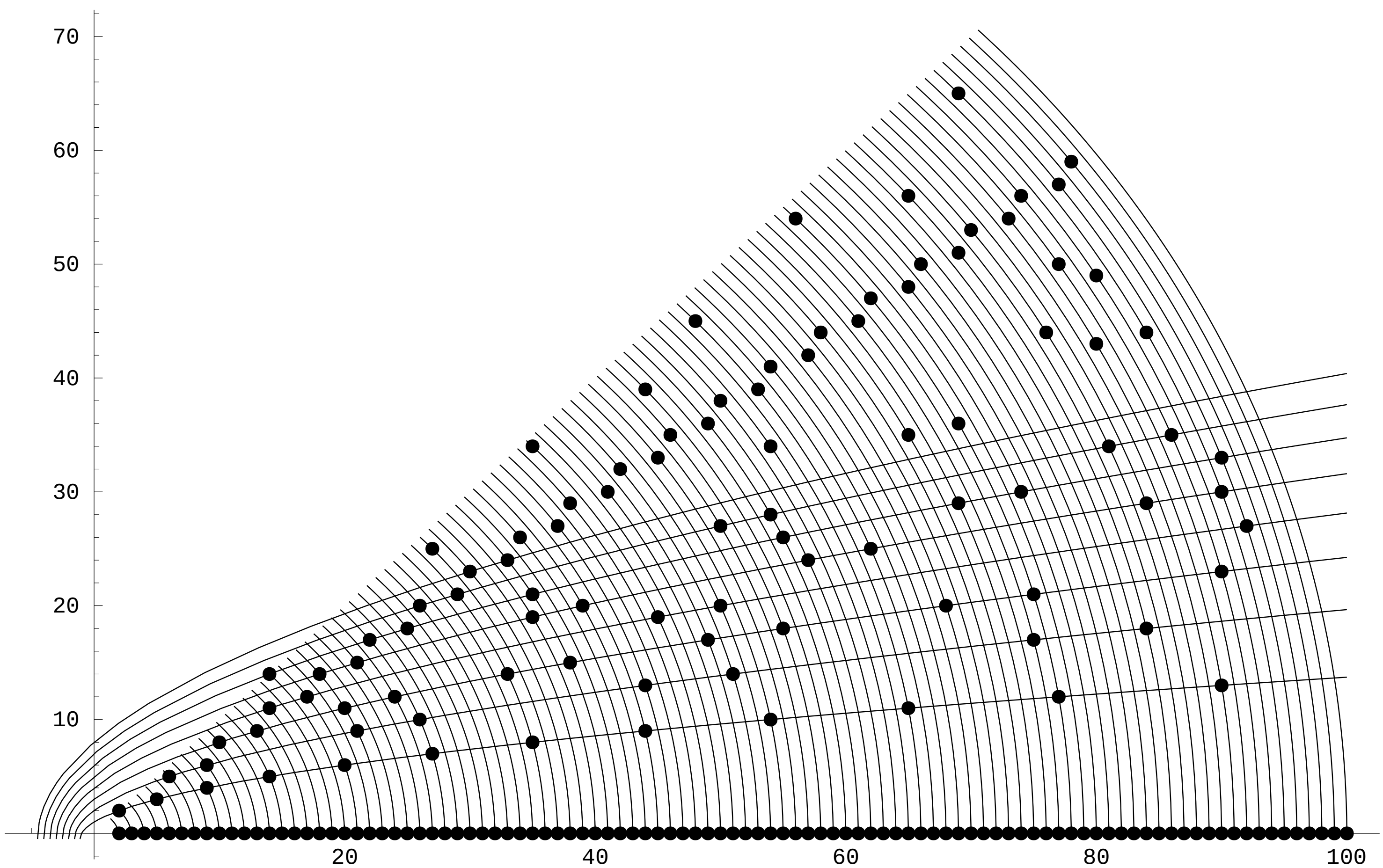

FIGURE 4. $\mathcal{P}[6]$

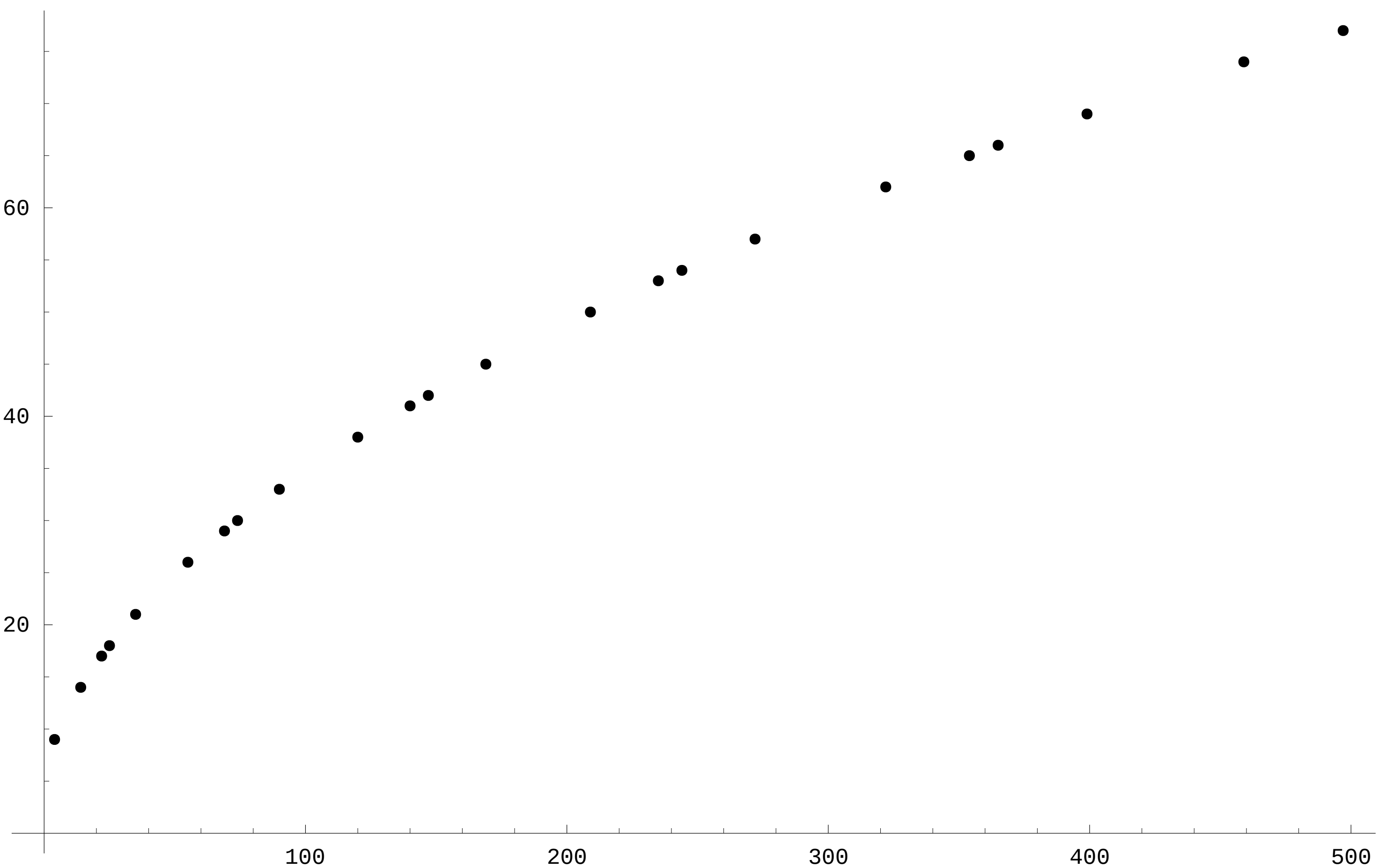

FIGURE 5. Distribution function for *e-numbers* in red and for primes in yellow

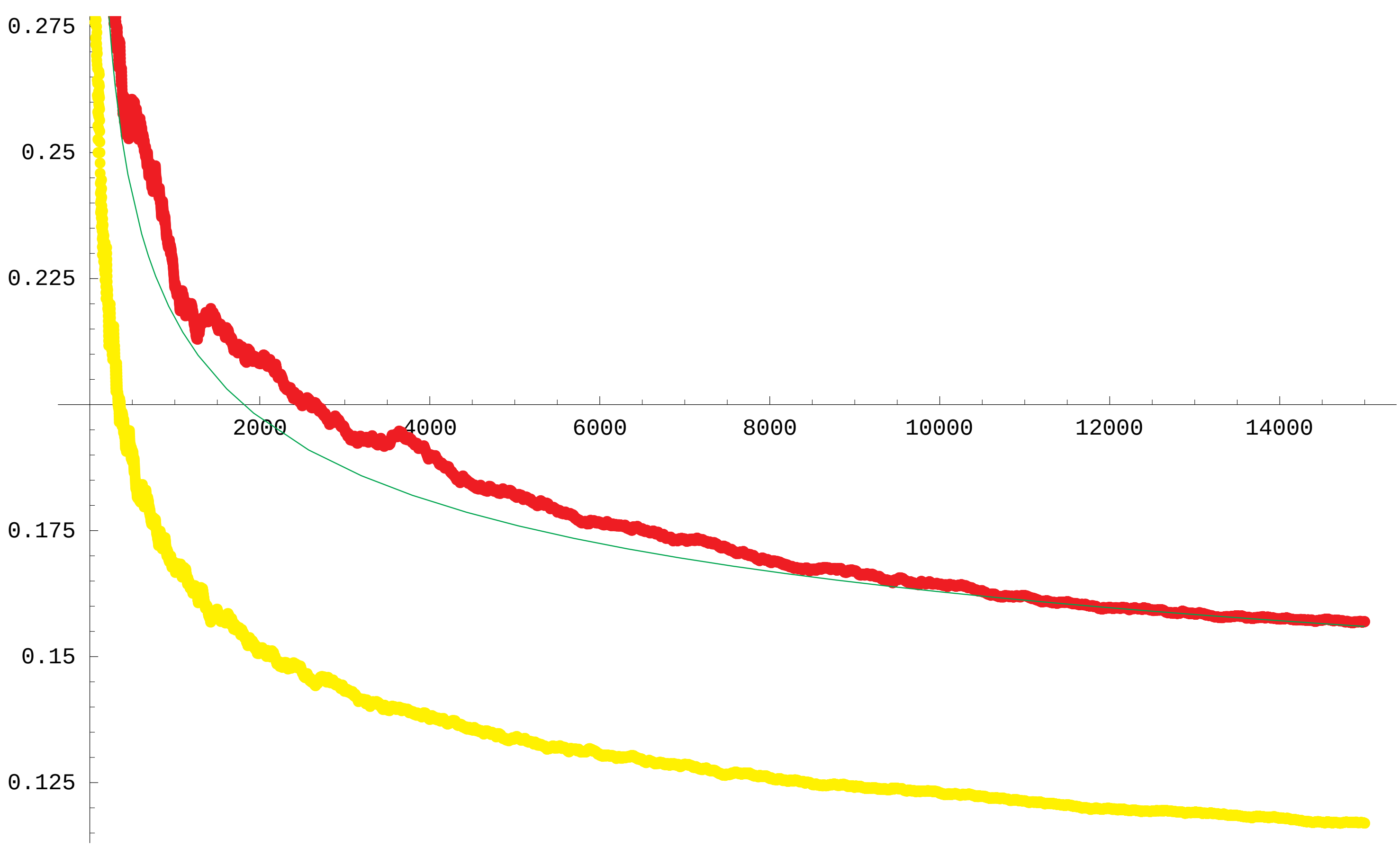

Figure 6. H swept out by rulings 5 degrees apart

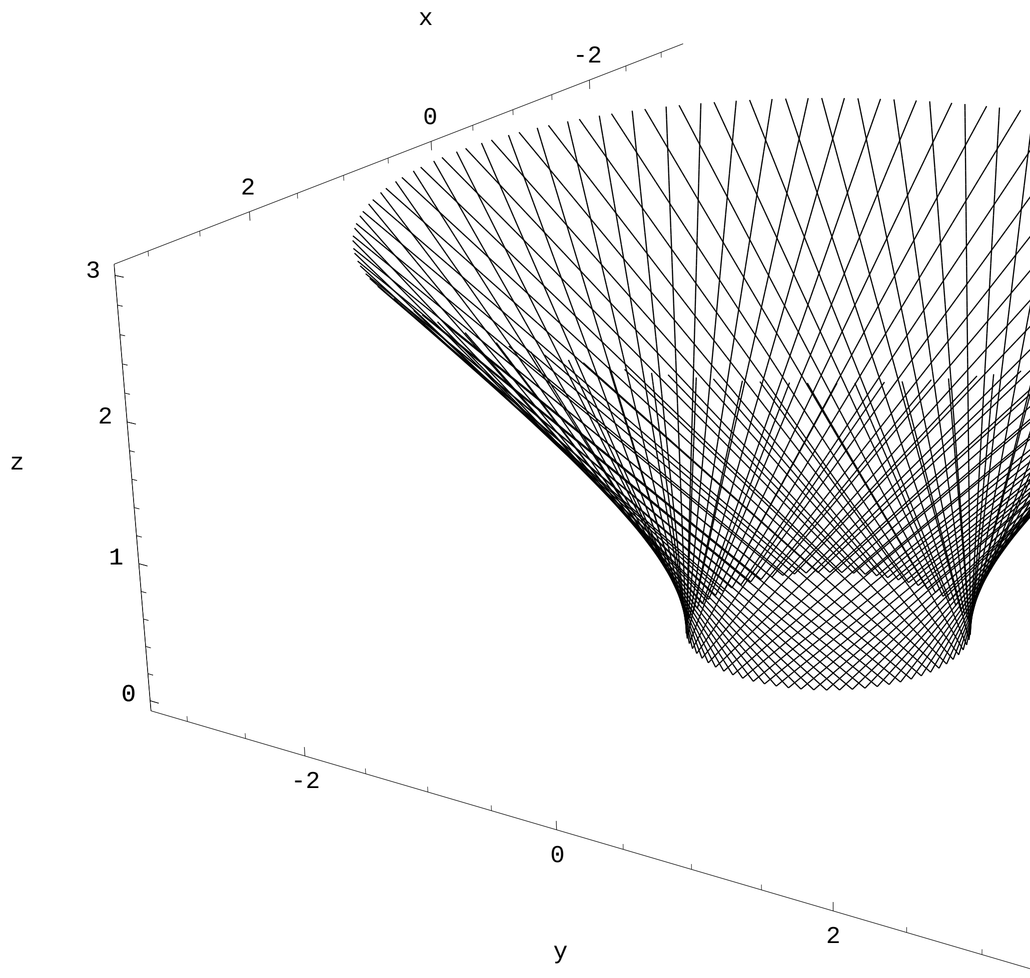

Figure 7. Rulings through $(3, 1, 3)$ on H

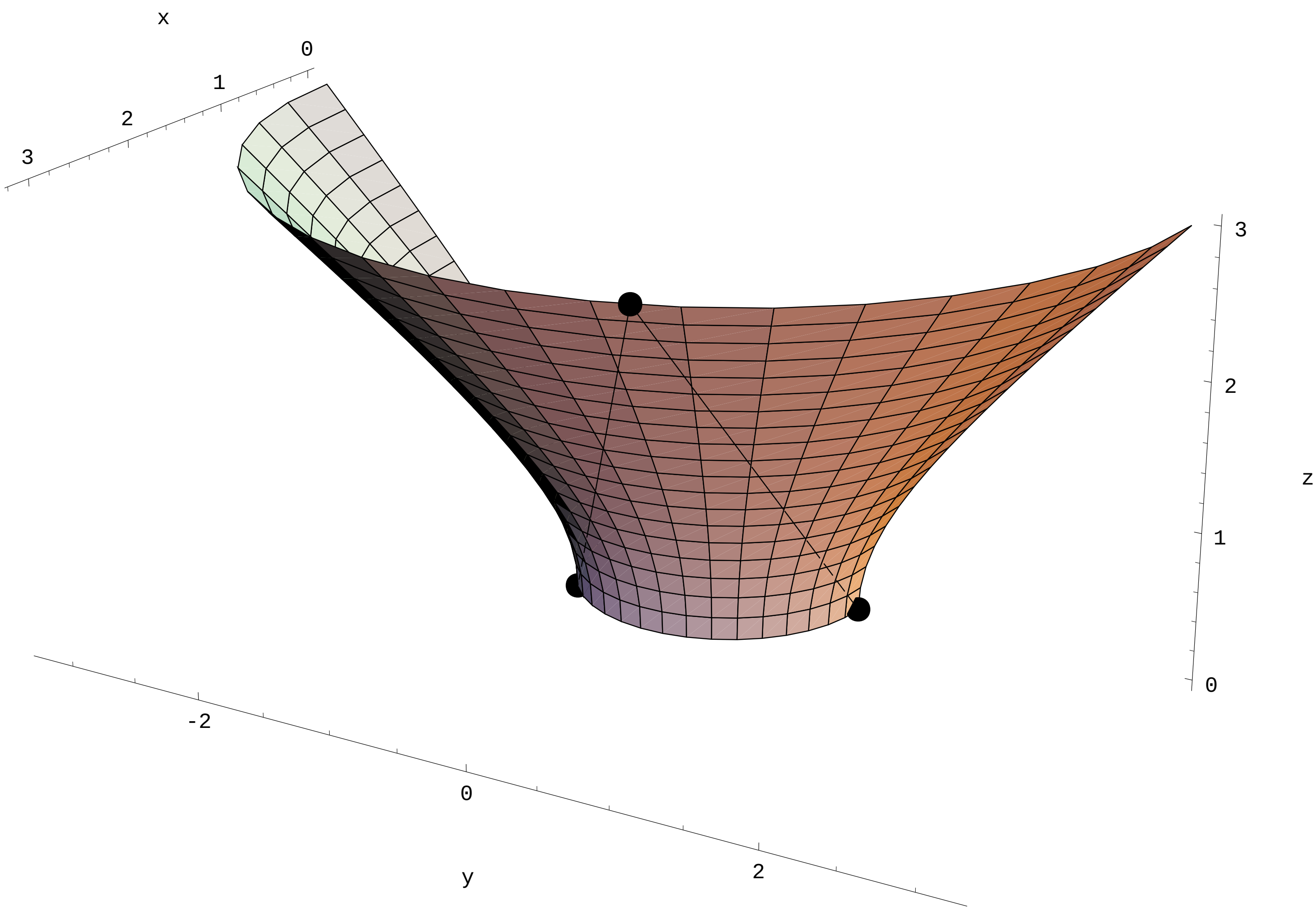

FIGURE 8. Rulings of the surface $t(z) = t(x) + t(y)$ inside of rulings of H

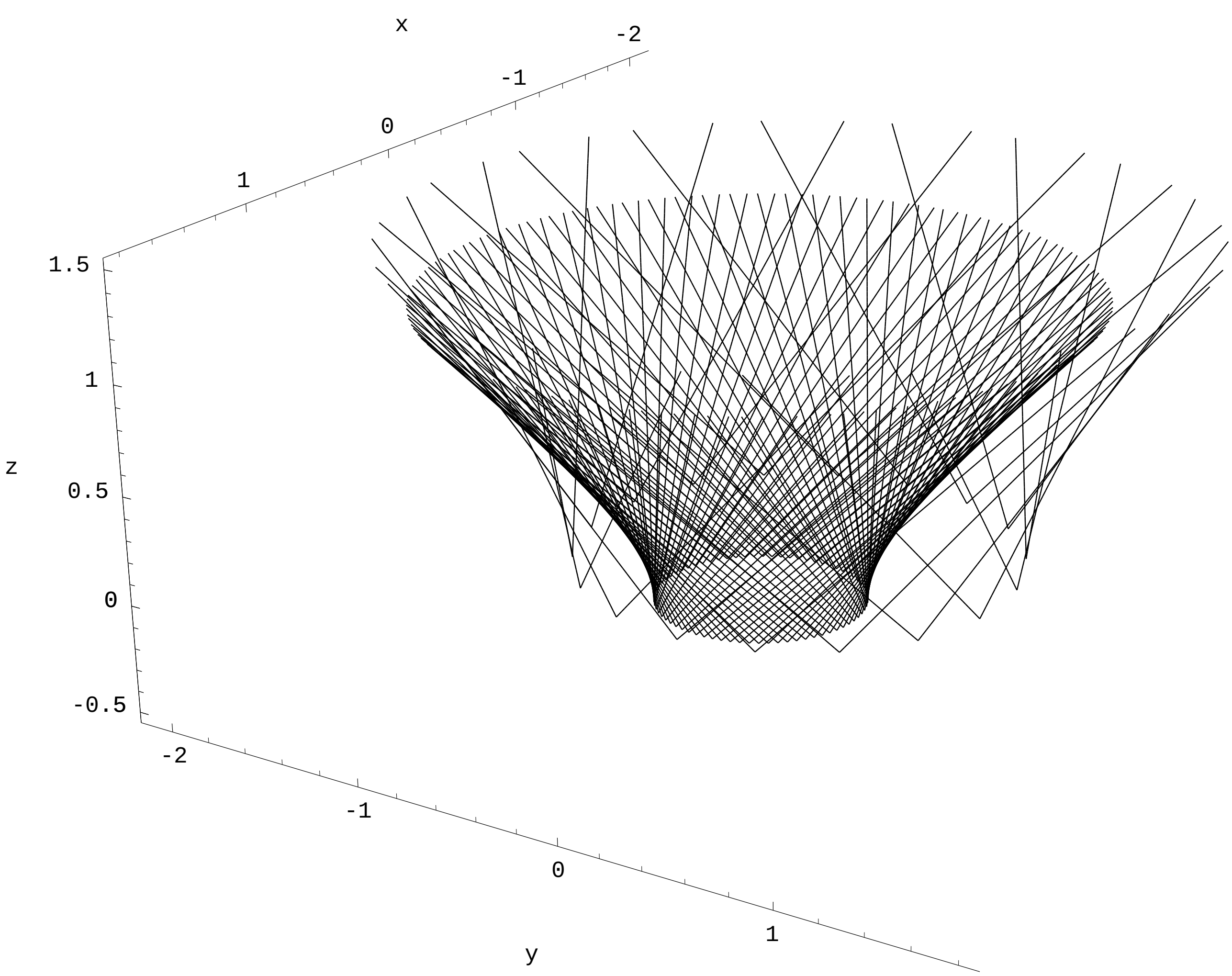

FIGURE 9. The surface $t(z) = t(x) + t(y)$ inside of H

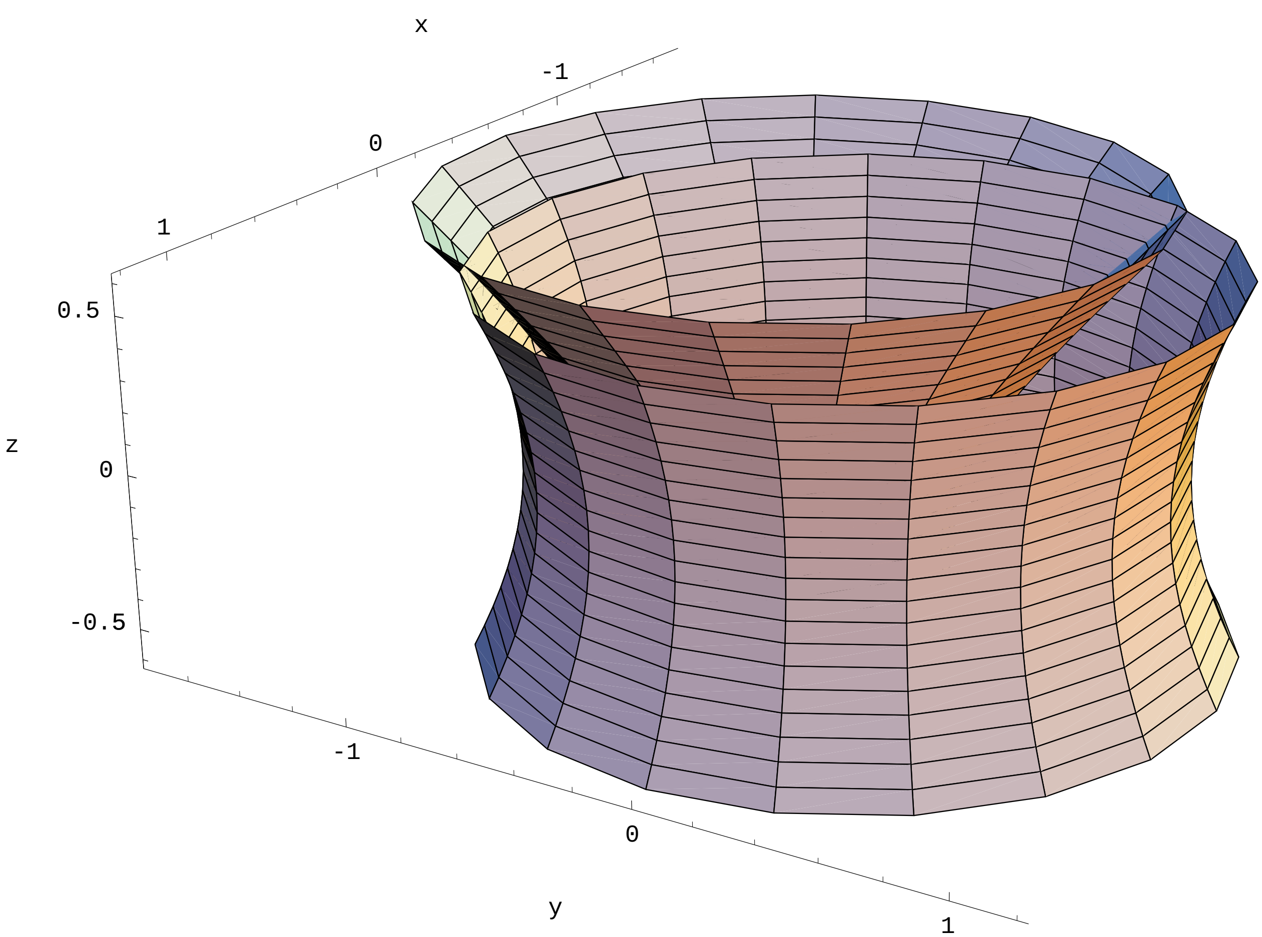

Figure 10. The lattice points on the surface $t(z) = t(x) + t(y),\ x \geq y,\ z \leq 500$