A Strange Recursive Sequence

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Problem: Consider

0 01 01011 0101101011011 0101101011011010110101101101011011 ...

where each string is formed from the previous string by substituting 01 for 0 and 011 for 1 simultaneously at each occurance. Notice that each string is an initial substring of the previous string so that we may consider them all as initial substrings of an infinite string. The puzzle then is, given n, determine if the nth digit is 0 or 1 without having to construct all the previous digits. That is, give a non-recursive formula for the nth digit.

Solution: Let G equal the limit string generated by the above process and define the string F by

$$
\begin{aligned}
F[0] &= 0, \\
F[n] &= \begin{cases} 1 & \text{if } n = \lfloor \phi m \rfloor \text{ for some positive integer } m; \\ 0 & \text{if } n = \lfloor \phi^2 m \rfloor \text{ for some positive integer } m. \end{cases}
\end{aligned}
$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$ and $\phi = (1+\sqrt{5})/2$; I claim that $F = G$.

I will try to motivate my solution. Let $g(0) = 0$ and define $g(n+1)$ to be the string that results from replacing 0 in $g(n)$ with 01 and 1 with 011; furthermore, let $s(n)$ and $t(n)$ be the number of 0s and 1s in $g(n)$, respectively. Note that we have the following recursive formulas:

$$
\begin{aligned}
s(n+1) &= s(n) + t(n); \\
t(n+1) &= s(n) + 2t(n).
\end{aligned}
$$

I claim that $s(n) = \mathrm{Fib}(2n-1)$ and $t(n) = \mathrm{Fib}(2n)$, where $\mathrm{Fib}(m)$ is the mth Fibonacci number (defined by $\mathrm{Fib}(-1) = 1$, $\mathrm{Fib}(0) = 0$, $\mathrm{Fib}(n+1) = \mathrm{Fib}(n) + \mathrm{Fib}(n-1)$ for $n \geq 0$); this is easily established by induction. Now noting that $\lim_{n\to\infty} \mathrm{Fib}(2n)/\mathrm{Fib}(2n-1) = \phi$, we see that if the density of the 0s and 1s exists, they must be be $1/\phi^2$ and $1/\phi$, respectively. What is the simplest generating sequence which has this property? Answer: the one given above.

Proof: We start with **Beatty's Theorem:** If a and b are positive irrational numbers such that $1/a + 1/b = 1$, then every positive integer has a representation of the form $\lfloor am \rfloor$ or $\lfloor bm \rfloor$ (m a positive integer), and this representation is unique.

This shows that F is well-defined. I now claim that

Lemma: If $S(n)$ and $T(n)$ represent the number of 0s and 1s in the initial string of F of length n, then $S(n) = \lceil n/\phi^2 \rceil$ and $T(n) = \lfloor n/\phi \rfloor$ (where $\lceil x \rceil$ is the smallest integer $\geq x$).

Proof: Using the identity $\phi^2 = \phi + 1$ we see that $S(n) + T(n) = n$, hence for a given n either $S(n) = S(n-1) + 1$ or $T(n) = T(n-1) + 1$. Now note that if $F[n-1] = 1 \implies n - 1 = \lfloor \phi m \rfloor$ for some positive integer m and since

$$\begin{aligned} & \phi m - 1 < \lfloor \phi m \rfloor < \phi m \\ \implies & m - 1/\phi < (n-1)/\phi < m \\ \implies & T(n) = T(n-1) + 1. \end{aligned}$$

To finish, note that if $F[n-1] = 0 \implies n - 1 = \lfloor \phi^2 m \rfloor$ for some positive integer m, and since

$$\begin{aligned} & \phi^2 m - 1 < \lfloor \phi^2 m \rfloor < \phi^2 m \\ \implies & m - 1/\phi^2 < (n-1)/\phi^2 < m \\ \implies & S(n) = S(n-1) + 1. \end{aligned}$$

□

I will now show that F is invariant under the operation of replacing 0 with 01 and 1 with 011; it will then follow that $F = G$. Note that this is equivalent to showing that $F[2S(n) + 3T(n)] = 0$, $F[2S(n) + 3T(n) + 1] = 1$, and that if $n = [\phi m]$ for some positive integer m, then $F[2S(n) + 3T(n) + 2] = 1$. One could waste hours trying to prove some fiendish identities; watch how I sidestep this trap. For the first part, note that by the above lemma

$$\begin{aligned} & F[2S(n) + 3T(n)] = F[2\lceil n/\phi^2 \rceil + 3 \lfloor n/\phi \rfloor] \\ = \; & F[2n + \lfloor n/\phi \rfloor] = F[2n + \lfloor n\phi - n \rfloor] \\ = \; & F[\lfloor \phi n + n \rfloor] = F[\lfloor \phi^2 n \rfloor] \end{aligned}$$

$\implies F[2S(n)+3T(n)] = 0.$

For the second, it is easy to see that since $\phi^2 > 2$, if $F[m] = 0 \implies F[m] = 1$ hence the first part implies the second part. Finally, note that if $n = \lfloor \phi m \rfloor$ for some positive integer m, then

$$F[2S(n)+3T(n)+3] = F[2S(n+1)+3T(n+1)] = 0,$$

hence by the same reasoning as above $F[2S(n)+3T(n)+2] = 1$. $\square$