RECURRENCE FOR THE ATKINSON-STEENWIJK INTEGRALS FOR RESISTORS IN THE INFINITE TRIANGULAR LATTICE

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ABSTRACT. The integrals $R_{n,n}$ obtained by Atkinson and van Steenwijk for the resistance between points of an infinite set of unit resistors on the triangular lattice obey P-finite recurrences. The main cause of these are similarities uncovered by partial integrations of their integral representations with algebraic kernels. All $R_{n,p}$ resistances to points with integer coordinates n and p relative to an origin in the lattice can be derived recursively.

1. Integral of Resistance in Infinite Triangular Lattice

The coordinates in the triangular lattice may be represented as integer pairs (n,p) where n is the number of steps into the $(1,0)$ direction of the Cartesian coordinates and p the number of steps into the $(-1/2,\sqrt{3}/2)$ direction of the Cartesian lattice. If all edges of the infinite lattice are equipped with resistors of a unit Ohm, the resistance between the (arbitrary, fixed) origin of the lattice to another lattice point at (n,p) is [2]

Definition 1.

$$R_{n,p} \equiv \frac{1}{\pi}\int_0^{\pi/2} \frac{dy}{\sinh x \cos y}\left[1-e^{-|n-p|x}\cos(n+p)y\right] \tag{1}$$

where

$$x \equiv \operatorname{arccosh}(\frac{2}{\cos y}-\cos y). \tag{2}$$

Some published values are [2]

$$R_{0,0}=0;\quad R_{1,0}=R_{0,1}=R_{1,1}=\frac{1}{3}. \tag{3}$$

$$R_{2,0}=R_{0,2}=R_{2,2}=\frac{8}{3}-\frac{4\sqrt{3}}{\pi}; \tag{4}$$

$$R_{1,2}=R_{2,1}=-\frac{2}{3}+\frac{2\sqrt{3}}{\pi}; \tag{5}$$

$$R_{1,3}=R_{3,1}=R_{2,3}=R_{3,2}=-5+\frac{10\sqrt{3}}{\pi}; \tag{6}$$

The aim of this paper is to provide a recursive algorithm to derive these expressions for arbitrary n and p.

The two principal integer parameters are:

Date: August 19, 2022.
2010 *Mathematics Subject Classification.* Primary 26A36; Secondary 28A12.

Definition 2.

$$\underline{n} \equiv |n-p|; \quad \bar{n} \equiv n+p. \tag{7}$$

2. Recurrences for $R_{n,n}$

2.1. **Chebyshev Connection.** In this section and we will consider the integral values 'on the diagonal' where $\underline{n}=0$, i.e.,

$$\pi R_{n,n} = \int_0^{\pi/2} \frac{dy}{\sinh x \cos y}[1-\cos(2ny)], \tag{8}$$

i.e., the calculation of the numbers

Definition 3. *(Integral on the ray $\underline{n}=0$)*

$$I_{\bar{n}} \equiv \int_0^{\pi/2} \frac{dy}{\sinh x \cos y}\,[1-\cos(\bar{n}y)] = I_{-\bar{n}} \tag{9}$$

for integer $\bar{n}$.

$$I_0 = 0. \tag{10}$$

The Fourier term in the integral kernel has a Bernstein-Polynomial expansion of the form [5, 1.331]

$$\cos(\bar{n}y) = \sum_{j=0}^{\lfloor \bar{n}/2 \rfloor} \binom{\bar{n}}{2j} (-1)^j \cos^{\bar{n}-2j} y \sin^{2j} y, \quad \bar{n}=0,1,2,3,\ldots \tag{11}$$

The substitution $\cos y = u$, $du/dy = -\sin y = -\sqrt{1-u^2}$ yields

$$I_{\bar{n}} = \int_1^0 \frac{-du}{u\sqrt{1-u^2}\sinh x}\left[1-\sum_{j=0}^{\lfloor \bar{n}/2 \rfloor}\binom{\bar{n}}{2j}(-1)^j u^{\bar{n}-2j}(1-u^2)^j\right] \tag{12}$$

The factor in the denominator is

$$\frac{1}{\sinh x} = \frac{1}{\sinh \operatorname{arccosh}(2/u-u)} = \frac{1}{\sinh \operatorname{arccosh} t} = \frac{1}{\sinh \ln[t+\sqrt{t^2-1}]} = \frac{1}{\sqrt{t^2-1}}$$

at $t \equiv 2/u - u \ge 1$, therefore

$$\begin{aligned} I_{\bar{n}} &= \int_0^1 \frac{du}{u\sqrt{1-u^2}}\frac{1}{\sqrt{(2/u-u)^2-1}}\left[1-\sum_{j=0}^{\bar{n}/2}\binom{\bar{n}}{2j}(-1)^j u^{\bar{n}-2j}(1-u^2)^j\right] \\ &= \int_0^1 \frac{du}{(1-u)(1+u)\sqrt{(2-u)(2+u)}}\left[1-\sum_{j=0}^{\bar{n}/2}\binom{\bar{n}}{2j}(-1)^j u^{\bar{n}-2j}(1-u^2)^j\right] \\ &= \int_0^1 \frac{du}{(1-u)(1+u)\sqrt{(2-u)(2+u)}} C_{\bar{n}}(u). \end{aligned} \tag{13}$$

The polynomials $C_{\bar{n}}$ are essentially the Chebyshev Polynomials and illustrated in Table 1:

Definition 4. *(complementary Chebyshev Polynomials)*

$$C_{\bar{n}}(u) \equiv 1-\cos(\bar{n}y) = C_{-\bar{n}}(u) = 1 - T_{\bar{n}}(u) \tag{14}$$

are polynomials of order $\bar{n}$.

$\bar n$	$C_{\bar n}$
0	0
1	$1-u = 1-u$
2	$2-2u^2 = 4(1-u) - 2(1-u)^2$
3	$1+3u-4u^3 = 9(1-u) - 12(1-u)^2 + 4(1-u)^3$
4	$8u^2-8u^4 = 16(1-u) - 40(1-u)^2 + 32(1-u)^3 - 8(1-u)^4$
5	$1-5u+20u^3-16u^5 = 25(1-u) - 100(1-u)^2 + 140(u-1)^3 - 80(u-1)^4 + 16(u-1)^5$
6	$2-18u^2+48u^4-32u^6$

TABLE 1. The polynomials $C_{\bar n}$ for small $\bar n$—see e.g. [1, Table 22.3][6, 18.5.14]—and associated expansion coefficients $c_{\bar n,i}$ for their expansions around $u = 1$.

The standard recurrence for the Chebyshev polynomials [1, 22.7.4] leads immediately to the recurrence

$$C_{\bar n}(u) = 2(1-u) + 2C_{\bar n-1}(u) - C_{\bar n-2}(u) - 2(1-u)C_{\bar n-1}(u). \tag{15}$$

Noticing that $C_{-1} = C_1$, all values of $C_{\bar n\geq 2}$ can be bootstrapped from the smaller expansions. In terms of the expansion coefficients

$$C_{\bar n}(u) \equiv \sum_{j=1}^{\bar n} c_{\bar n,j}(1-u)^j \tag{16}$$

this implies $c_{\bar n,j} = c_{-\bar n,j}$, $c_{0,j} = 0$, $c_{1,j} = \delta_{1,|j|}$ and

$$c_{\bar n,j} = 2\delta_{j,1} + 2c_{\bar n-1,j} - 2c_{\bar n-1,j-1} - c_{\bar n-2,j}. \tag{17}$$

Remark 1. *The unsigned coefficients $(-)^{j+1}c_{\bar n,j}$ are coefficients of Morgan-Voyce polynomials* [4, A211957][9, 8]. *The bivariate generating function is*

$$\sum_{\bar n\geq 0, j\geq 0} c_{\bar n,j} t^{\bar n} z^j = \frac{tz(1+t)}{[(1-t)^2 + 2tz](1-t)}. \tag{18}$$

A sum rule is

$$\sum_{j\geq 0} c_{\bar n,j} = 1 - T_{\bar n}(0) = \begin{cases} 1, & \bar n \text{ odd;} \\ 0, & 4 \mid \bar n; \\ 2, & 4 \nmid \bar n,\ \bar n \text{ even.} \end{cases} \tag{19}$$

A special value is—with $C_1(u) = 1-u$—reduced via [5, 2.281,2.261]

$$I_1 = \int_0^1 du \frac{1}{(1+u)\sqrt{(2+u)(2-u)}} = \int_{1/2}^1 dt \frac{1}{\sqrt{-1+2t+3t^2}} = \int_{1/2}^1 dt \frac{1}{\sqrt{(t+1)(3t-1)}} = \frac{1}{\sqrt{3}} \ln[1+\sqrt{3}/2] \approx 0.3601572\ldots \tag{20}$$

For $\bar{n} > 1$ the Taylor expansion (16) is inserted into (13):

$$
\begin{aligned}
(21)\quad I_{\bar n} &= \sum_{j=1}^{\bar n} c_{\bar n,j}\int_0^1 \frac{du}{(1-u)(1+u)\sqrt{(2+u)(2-u)}}(1-u)^j \\
&= \sum_{j=0}^{\bar n-1} c_{\bar n,j+1}\int_0^1 \frac{du}{(1+u)\sqrt{(2+u)(2-u)}}(1-u)^j \\
&= c_{\bar n,1}I_1 + \sum_{j=1}^{\bar n-1} c_{\bar n,j+1}(-)^j\int_0^1 \frac{du}{(1+u)\sqrt{(2+u)(2-u)}}(u-1)^j \\
&= c_{\bar n,1}I_1 + \sum_{j=1}^{\bar n-1}(-)^j c_{\bar n,j+1}\int_0^1 \frac{du}{(1+u)\sqrt{(2+u)(2-u)}}(u+1-2)^j \\
&= c_{\bar n,1}I_1 + \sum_{j=1}^{\bar n-1}(-)^j c_{\bar n,j+1}\sum_{k=0}^{j}\binom{j}{k}(-2)^{j-k}\int_0^1 \frac{du}{(1+u)\sqrt{(2+u)(2-u)}}(1+u)^k \\
&= c_{\bar n,1}I_1+\sum_{j=1}^{\bar n-1}(-)^j c_{\bar n,j+1}\left[(-2)^j I_1 + \sum_{k=1}^{j}\binom{j}{k}(-2)^{j-k}\int_0^1 \frac{du}{\sqrt{(2+u)(2-u)}}(1+u)^{k-1}\right] \\
&= \epsilon_{\bar n} I_1 + \sum_{j=1}^{\bar n-1} 2^j c_{\bar n,j+1}\left[\sum_{k=0}^{j-1}\binom{j}{k+1}(-2)^{-k-1}\int_0^1 \frac{du}{\sqrt{(2+u)(2-u)}}(1+u)^k\right]
\end{aligned}
$$

where $\epsilon_n \equiv n \pmod 2$ is 1 if n is odd, and 0 if n is even.

Definition 5.

$$(22)\qquad J_k \equiv \int_0^1 \frac{du}{\sqrt{2-u}\sqrt{2+u}}(1+u)^k, \quad k \ge 0,$$

such that [5, 2.261]

$$(23)\qquad J_0 = \pi/6 \approx 0.523599; \quad J_1 = \pi/6 + 2 - \sqrt{3} \approx 0.791548.$$

2.2. **Partial Integration.** By repeated partial integration the values for larger k can be derived via

$$(24)\qquad kJ_k = -2^{k-1}\sqrt{3} + 2 + (2k-1)J_{k-1} + 3(k-1)J_{k-2}.$$

Remark 2. *By telescoping the recurrence* (24) *can be written* $\sqrt{3}$*-free:*

$$(25)\qquad kJ_k + (-4k+3)J_{k-1} + (k-3)J_{k-2} + 6(k-2)J_{k-3} + 2 = 0.$$

Remark 3. *To keep the irrational terms separated in a computer algebra system, one may split* $J_k = \alpha_k\sqrt{3} + \sigma_k + \tau_k\pi$ *into three sequences* α_k, σ_k *and* τ_k *of rational numbers:*

$$(26)\qquad k\sigma_k + (1-2k)\sigma_{k-1} + 3(1-k)\sigma_{k-2} - 2 = 0.; \quad \sigma_0 = 0, \sigma_1 = 2; \sigma_2 = 4$$

$$(27)\qquad k\alpha_k + (-4k+3)\alpha_{k-1} + (k-3)\alpha_{k-2} + 6(k-2)\alpha_{k-3} = 0; \quad \alpha_0 = 0, \alpha_1 = -1, \alpha_2 = -5/2.$$

with generating function

$$(28)\qquad \sum_{k\ge 0}\alpha_k z^k = \frac{1}{\sqrt{(1-3z)(1+z}}\left[\frac{1}{\sqrt{3}}\arctan\frac{1-5z}{\sqrt{3}\sqrt{(1-3z)(1+z)}} - \frac{\pi}{6\sqrt{3}}\right].$$

$\bar n \backslash k$	0	1	2	3	4	5	6	
0								
1								
2	1							
3	2	1						
4	4	4	1					
5	6	11	6	1				
6	9	24	22	8	1			
7	12	46	62	37	10	1		
8	16	80	148	128	56	12	1	
9	20	130	314	367	230	79	14	1

TABLE 2. Table of $\hat c_{\bar n,k}$ for small indices.

$$k\tau_k + (-2k+1)\tau_{k-1} + 3(1-k)\tau_{k-2} = 0. \quad \tau_0 = \tau_1 = 1/6; \tau_2 = 1/2 \tag{29}$$

Interchanging the two summations in (21):

$$\begin{aligned} I_{\bar n} &= \epsilon_{\bar n} I_1 + \sum_{k=0}^{\bar n-2} \frac{J_k}{(-2)^{k+1}} \sum_{j=k+1}^{\bar n-1} 2^j c_{\bar n,j+1} \binom{j}{k+1} \\ &= \epsilon_{\bar n} I_1 + \sum_{k=0}^{\bar n-2} \frac{J_k}{(-2)^{k+1}} \sum_{j=0}^{\bar n-k-2} 2^{j+k+1} c_{\bar n,j+k+2} \binom{j+k+1}{k+1} \\ &= \epsilon_{\bar n} I_1 + \sum_{k=0}^{\bar n-2} (-)^{k+1} J_k \sum_{j=0}^{\bar n-k-2} 2^j c_{\bar n,j+k+2} \binom{j+k+1}{j}. \end{aligned} \tag{30}$$

The relevant coefficients are therefore

$$\begin{aligned} \hat c_{\bar n,k} &\equiv \frac{1}{2^{k+1}} (-1)^{\bar n+1} \sum_{j=0}^{\bar n-k-2} 2^j c_{\bar n,j+k+2} \binom{j+k+1}{j} \\ &= (-1)^{\bar n+1} \sum_{v=0}^{\bar n-1} (-)^v \binom{k+1+v}{2k+2} \end{aligned} \tag{31}$$

of Table 2, which is essentially one of Barry's Riordan arrays [4, A158454][3].

$$I_{\bar n} = \epsilon_{\bar n} I_1 + (-)^{\bar n+1} \sum_{k=0}^{\bar n-2} (-2)^{k+1} J_k \hat c_{\bar n,k}. \tag{32}$$

2.3. **Algorithm for $n = p$.** To compute $R_{n,n}$ one needs $I_{\bar n}$ for even $\bar n$, which are computed as follows: For $\bar n = 0$ and $\bar n = 1$ insert (10) and (20). For $\bar n > 1$ compute (32) where $J_{0,1}$ are the constants (23), other J_k recursively derived with (24), and the integer coefficients $c_{\bar n,j}$ recursively addressed with (17) or computed via (31).

Example 1.

$$I_0 = 0; \tag{33}$$

$$I_2 = \frac{1}{3}\pi; \tag{34}$$

$$I_4 = \frac{8}{3}\pi - 4\sqrt{3}; \tag{35}$$

$$I_6 = 27\pi - 48\sqrt{3}; \tag{36}$$

$$I_8 = \frac{928}{3}\pi - 560\sqrt{3}; \tag{37}$$

$$I_{10} = \frac{11249}{3}\pi - 6800\sqrt{3}; \tag{38}$$

$$I_{12} = 46872\pi - \frac{425076}{5}\sqrt{3}. \tag{39}$$

Conjecture 1.

$$I_{2n} = \beta_n\pi/3 - \gamma_n\sqrt{3} \tag{40}$$

where the sequences β_n and γ_n can be recursively computed by P-finite recurrences

$$(n-1)\beta_n - (15n-22)\beta_{n-1} + (15n-23)\beta_{n-2} - (n-2)\beta_{n-3} = 0 \tag{41}$$

and

$$(n-1)\gamma_n - (15n-22)\gamma_{n-1} + (15n-23)\gamma_{n-2} - (n-2)\gamma_{n-3} - 4 = 0 \tag{42}$$

starting at $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = 8$, $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = 4$.

Remark 4. *The first order homogeneous separable differential equation of the generating function derived from (41) can be solved as*

$$\beta(z) \equiv \sum_{n\ge 0} \beta_n z^n = \frac{z}{(1-z)\sqrt{1-14z+z^2}}. \tag{43}$$

The first differences $\beta_n - \beta_{n-1} = 1, 7, 73, 847, \ldots$ are Legendre Polynomials $P_n(7)$, [4, A084768]. Likewise the first order inhomogeneous linear differential equation derived for the generating function of (42) can be solved:

$$\gamma(z) \equiv \sum_{n\ge 0} \gamma_n z^n = \frac{2}{\sqrt{3}} \frac{z}{(1-z)\sqrt{1-14z+z^2}} \left[\arctan \frac{(1+z)\sqrt{3}}{\sqrt{1-14z+z^2}} - \frac{\pi}{3}\right]. \tag{44}$$

The merger of these two generating functions is

$$I(z) \equiv \sum_{n\ge 0} I_{2n} z^n = \frac{z}{(1-z)\sqrt{1-14z+z^2}} \left[\pi - 2\arctan \frac{(1+z)\sqrt{3}}{\sqrt{1-14z+z^2}}\right]. \tag{45}$$

Conjecture 2.

$$I_{2n+1} = I_1 + \phi_n - \eta_n\sqrt{3} \tag{46}$$

with P-finite recurrences

$$(-2n+1)\phi_n + 2(14n-13)\phi_{n-1} - 28\phi_{n-2} + 2(-14n+29)\phi_{n-3} + (2n-5)\phi_{n-4} = 0, \tag{47}$$

starting $\phi_0 = 0$, $\phi_1 = 8$, $\phi_2 = 232/3$, $\phi_3 = 12784/15$, and

$$(-2n+1)\eta_n + 32(n-1)\eta_{n-1} + 30(-2n+3)\eta_{n-2} + 32(n-2)\eta_{n-3} + (-2n+5)\eta_{n-4} = 0 \tag{48}$$

starting $\eta_0 = 0$, $\eta_1 = 4$, $\eta_2 = 44$, $\eta_3 = 2456/5$.

3. The Recurrence for $n \neq p$

The cases for $n \neq p$ are reduced to the values for $n = p$ by the symmetry properties of the grid. R is invariant applying elements of the cyclic group of order 6 of rotations by multiples of 60°:

$$R_{n,p} = R_{n-p,n} = R_{-p,n-p} = R_{-n,-p} = R_{-n+p,-n} = R_{p,-n+p}. \tag{49}$$

Any pair of indices is reduced by one of these to the region $n \geq 0$ and $p \geq 0$. The additional invariance

$$R_{n,p} = R_{p,n} \tag{50}$$

with respect to the sign of the difference of the two coordinates represents a mirror line in the lattice. These symmetries combined represent a dihedral group of order 12, see p6m in [7]. A combination of (50) and the first relation of (49) yields

$$R_{n,p} = R_{n,n-p} \tag{51}$$

which may be used to fold the cases $p > n/2$ to the 30° wedge of the 'irreducible' Brioullin zone for p6mm [10].

For a general point in that wedge of the lattice the unnumbered equation prior to [2, (13)] decreases the indices recursively until one or both become zero or both become equal, where $R_{n,0} = R_{n,n} = I_{2n}/\pi$ derived in Section 2.3 take over:

$$R_{n,p} = 6R_{n-1,p-1} - R_{n-1,p} - R_{n,p-1} - R_{n-2,p-1} - R_{n-1,p-2} - R_{n-2,p-2}, \quad n > 0, p \geq 2. \tag{52}$$

For $p = 1$ this equation includes terms with negative second indices on the right hand side; the second relation in (49) plus that swap (50) yield $R_{n,-1} = R_{n+1,1}$ to lift these, so for $p = 1$ (52) is effectively

$$R_{n,1} = 3R_{n-1,0} - R_{n-1,1} - \frac{1}{2}(R_{n,0} + R_{n-2,0}). \tag{53}$$

4. Summary

We have shown how the resistor values $R_{n,p}$ of the infinite triangular lattice can be computed recursively with standard techniques of integration.

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