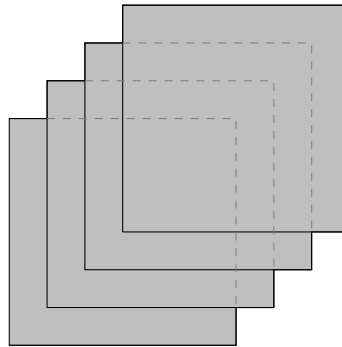1. Let $n \geq 2$ be an integer. Thibaud the Tiger lays n 2×2 overlapping squares out on a table, such that that the centers of the squares are equally spaced along the line $y = x$ from $(0, 0)$ to $(1, 1)$ (including the two endpoints). For example, for $n = 4$ the resulting figure is shown below, and it covers a total area of $\frac{23}{3}$.

   Find, with proof, the minimum n such that the figure covers an area of at least $\sqrt{63}$.

2. Let x and y be relatively prime integers. Show that $x^2 + xy + y^2$ and $x^2 + 3xy + y^2$ are relatively prime.

3. Find all sequences $a_1, a_2, a_3, \ldots$ of real numbers such that for all positive integers $m, n \geq 1$, we have

$$a_{m+n} = a_m + a_n - mn \text{ and}$$
$$a_{mn} = m^2 a_n + n^2 a_m + 2a_m a_n.$$

4. Alpha and Beta are playing a game on a 10×100 grid of squares. At each turn, they can fold the grid along any of the interior horizontal or vertical gridlines, which creates a smaller (folded) grid of squares (on the first move, they can choose one of 9 horizontal or 99 vertical gridlines). The person who makes the last fold wins. If both players play optimally and Alpha starts, determine with proof who wins.

5. 2022 lily pads are arranged in a circle. Each lily pad starts with height 1. A frog starts on one of the lily pads, and jumps around clockwise as follows: if the frog is on a lily pad of height k, the lily pad grows by 1 (becoming $k + 1$), and then the frog jumps k lily pads clockwise (i.e., jumping over $(k - 1)$). The frog continues doing this as long as it pleases.

   After n jumps, let $D(n)$ be the difference between the tallest lily pad and the shortest lily pad. Find, with proof, the maximum possible value of $D(n)$, or prove that $D(n)$ is unbounded.

6. An $m \times n$ grid of squares (with m rows and n columns) has some of its squares colored blue. The grid is called *fish-friendly* if a fish can swim from the left edge of the grid to the right edge of the grid only moving through blue squares. In other words, there is a sequence of blue squares, each horizontally or vertically adjacent to the previous square, starting in the first column and ending in the last column.

   Prove that the number of fish-friendly 42×49 grids is at least 2^{2022}.