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THE DISTRIBUTION OF VALUES OF A CERTAIN CLASS OF ARITHMETIC FUNCTIONS AT CONSECUTIVE INTEGERS

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1. INTRODUCTION

The purpose of this paper is to study several aspects of the problem of the distribution of values of certain arithmetic functions, especially of the values at consecutive integers. Our main motivation is the function $a(n)$, which represents the number of non-isomorphic Abelian groups with n elements. This is a well-known multiplicative function which satisfies $a(p^\alpha) = P(\alpha)$, where henceforth p will denote primes and $P(k)$ will denote the number of (unrestricted) partitions of k. It is known that $a(n)$ possesses a positive mean value, and more precisely one has

$$(1.1) \qquad \sum_{n\le x} a(n) = \sum_{m=1}^{3} A_m x^{1/m} + R(x), \quad A_m = \prod_{k=1,k\ne m}^{\infty} \zeta(k/m).$$

It was proved long ago by P. Erdős and G. Szekeres [3] that $R(x) \ll x^{1/2}$, and after many subsequent improvements the best known bound is $R(x) \ll x^{97/381}\log^{35}x$, which is due to G. Kolesnik [17]. There are several interesting arithmetic aspects of the function $a(n)$ and similar functions. For example, $a(p) = 1$ for every prime p, so that $a(n) = a(s(n))$, where $s(n)$ is the squarefull part of n. Namely every integer $n \geq 1$ may be uniquely decomposed as $n = q(n)s(n)$, $(q(n), s(n) = 1)$, where $q(n)$ is squarefree and $s(n)$ is squarefull (s is squarefull if $p^2|s$ whenever $p|s$). Nonnegative, integer-valued arithmetic functions $f(n)$ such that $f(n) = f(s(n))$ for every $n \geq 1$ were called functions with squarefull kernel, or simply s-functions by A. Ivić and G. Tenenbaum [16]. It is to be noted that s-functions are not necessarily multiplicative: $\Omega(n) - \omega(n)$ is clearly an additive s-function, where as usual $\Omega(n)$ and $\omega(n)$ denote the number of all prime factors of n and the number of distinct prime factors of n, respectively. It was shown in [16] by a simple argument that the local density

$$(1.2) \qquad d_k = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x, f(n)=k} 1$$

of an s-function always exists for any fixed $k \geq 0$. In fact, it was proved in [16] that an asymptotic formula implying (1.2) with the uniform error term $O(x^{1/2}\log^2 x)$

exists. More precise results on local densities of $a(n)$ and related functions were proved by E. Krätzel [20] and A. Ivić [15]. The property that $a(n)$ is an s-function is essential in the proof of the asymptotic formulas

$$(1.3) \qquad \sum_{n\le x,\, a(n)=a(n+1)} 1 = Ax + O(x^{3/4}\log^4 x),$$

$$\sum_{n\le x} a(n)(\Omega(n+1) - \omega(n+1)) = Bx + O(x^{3/4+\varepsilon}),$$

where $A,B > 0$ are two constants which may be explicitly evaluated. These formulas are new, and in proving them we shall also make use of the order result

$$(1.4) \qquad a(n) \le \exp\left(\left(\frac{\log 5}{4} + \varepsilon\right)\frac{\log n}{\log\log n}\right) \quad (n \le n_0(\varepsilon)).$$

This was proved by E. Krätzel [19], and later sharpened by W. Schwarz and E. Wirsing [28], and J.-L. Nicolas [26]. The above mentioned properties of $a(n)$ are essentially the only ones needed in the proof of (1.3). In 2. we shall prove a more general result than (1.3) for suitable s-functions satisfying a mild growth condition analogous to (1.4). In order to avoid unnecessary technicalities we have not tried to obtain the most general possible form of our results. This is also true of other sections, where the results obtained for $a(n)$ may be readily generalized to many other multiplicative functions.

It will turn out, however, that the method used in proving Theorem 1 and Theorem 2 in 2. (the appropriate generalizations of (1.3)) may yield asymptotic formulas for other related arithmetic sums. This is discussed in 3. where several other applications of our method are discussed. These include the asymptotic formulas

$$(1.5) \qquad \sum_{n\le x} a(n)d(n+1) = C_1 x\log x + C_2 x + O(x^{8/9+\varepsilon})$$

and

$$(1.6) \qquad \sum_{n\le x} a(n)\omega(n+1) = D_1 x\log\log x + D_2 x + O(x/\log x)$$

with $C_1, D_1 > 0$, which seems to be new.

In [15] A. Ivić defined the functions $C(x)$, $D(x)$, $E(x)$ as follows: $C(x)$ denotes the number of distinct values taken by $a(n)$ for $n\le x$, $D(x)$ denotes the number of $n\le x$ such that $n=a(m)$ for some integer m, $E(x)=\sum_{n\le x} b(n)$, where $b(n)$ is the number of solutions in squarefull s of the equation $n=a(s)$ for a fixed n. The functions $C(x)$ and $D(x)$ determine to a great extent the distribution of values of $a(n)$, and it is obvious that

$$(1.7) \qquad C(x)\le D\left[\exp\left\{\left(\frac{\log 5}{4}+\varepsilon\right)\frac{\log x}{\log\log x}\right\}\right], \quad D(x)\le E(x).$$

It was proved in [15] that

$$(1.8) \qquad C(x)\le\exp((3^{-1/2}\,2\pi+\varepsilon)(\log x/\log\log x)^{1/2}) \quad (x\ge x_0(\varepsilon)),$$

and the significance of E(x) is that it may be fairly precisely evaluated. Thus it was shown in [15] that, as $x \to \infty$,

$$(1.9) \qquad E(x) = \exp((B+o(1))\log^{2/3}x), \quad B = \tfrac{3}{2}(6\zeta(3)\pi^{-2})^{1/3},$$

and this formula was refined by J. Herzog and W. Schwarz [12], [13]. Problems involving the estimation of C(x) and D(x) are discussed in 4. where explicit dependence of these functions on the structure of prime factors of the partition function P(k) is exhibited. It is very plausible to conjecture that

$$(1.10) \qquad C(x) = \exp(\log^{1/2+o(1)}x), \quad D(x) = \exp(\log^{2/3+o(1)}x), \quad (x \to \infty)$$

and these formulas are proved if a certain conjecture involving the function P(k) is assumed. Our construction of the lower bound for C(x) may be generalized to various other multiplicative functions besides a(n). Thus if $d_k(n)$ represents the number of ways n may be written as a product of $k \ge 2$ fixed factors ($d_2(n) = d(n)$) and C(x) is again the corresponding number of distinct values of $d_k(n)$ for $n \le x$, then our construction gives

$$(1.11) \qquad \exp(E_1(k)(\log x)^{1/2}/\log\log x) \ll C(x) \ll \ll \exp(E_2(\log x/\log\log x)^{1/2})$$

for some $E_1(k)$, $E_2 > 0$. This seems to be new for $k > 2$, but for $k = 2$ (i.e. for $d(n)$) this problem has been practically solved by P. Erdős and L. Mirsky [5], who proved

$$(1.12) \qquad C(x) = \exp((2\pi(2/3)^{1/2} + o(1))(\log x)^{1/2}/\log\log x) \quad (x \to \infty).$$

For some recent results concerning (1.12) and the related B-numbers defined in [5], see the works of M. Nair and P. Shiu [25], [29].

The aforementioned paper of Erdős and Mirsky considered problems concerned with long blocks of consecutive integers such that the values of $d(n)$ at these integers are all distinct. It was shown that there exist infinitely many n such that $d(n+1), d(n+2), \ldots, \ldots, d(n+k)$ are all distinct, where $k = [c(\log n)^{1/2}/\log\log n]$. It was also conjectured that $d(n) = d(n+1)$ has infinitely many solutions, and this difficult problem has been only recently solved in the affirmative by D.R. Heath-Brown [11]. Both of these problems for $a(n)$ are duscussed in 5. and naturally they can be considered for various other arithmetic functions. The intrinsic property that $a(p) = 1$ gives easily the existence of infinitely many n such that $a(n) = a(n+1) = a(n+2)$ holds. This follows from the old result of L. Mirsky [23] that

$$(1.13) \qquad \sum_{n \le x} \mu^2(n)\mu^2(n+1)\mu^2(n+2) = \prod_p (1 - 3p^{-2})x + O(x^{2/3+\varepsilon}),$$

since $a(n)=a(n+1)=a(n+2)=1$ if $\mu^2(n)\mu^2(n+1)\mu^2(n+2)=1$. Hence this problem for $a(n)$ is considerably easier than the corresponding problem for $d(n)$ (not much seems to be known about n for which $d(n)=d(n+1)=d(n+2)$). We shall prove that there are infinitely many n such that the values $a(n+1), \ldots, a(n+t)$ are all distinct for $t=[C(\log n/\log\log n)^{1/2}]$ $(C>0)$, and infinitely many n such that $a(n+1)=\ldots=a(n+k)$ for $k=[D\log n\log\log\log n/(\log\log n)^2]$ $(D>0)$. Quantitative forms of these results are contained in Theorem 6 and 7 of 5.

Finally we wish to thank A. Schinzel for kindly letting us use his unpublished result (Lemma 2) on prime factors of the partition function.

2. ASYMPTOTIC FORMULAS FOR CERTAIN SUMMATORY FUNCTIONS

In this section we shall prove a general result which contains (1.3). First we make the following

DEFINITION. Let $\mathcal{B}$ denote the class of nonnegative, integer-valued s-functions $b(n)$ (i.e. $b(n)=b(s(n))$ for $n\geq 1$) such that $b(n)\ll n^{\varepsilon}$ for any $\varepsilon>0$.

In view of (1.4) it follows that $a(n)$ (generated by $\zeta(s)\zeta(2s)\zeta(3s)\ldots$) belongs to $\mathcal{B}$, and many other multiplicative functions also belong to $\mathcal{B}$. Two further examples are $S(n)$ (the number of non-isomorphic semi-simple rings with n elements, generated by $\prod_{m,r>1} \zeta(mr^2s)$) and $d^{(e)}(n)$ (the number of exponential divisors of n, defined by the relation $d^{(e)}(p^\alpha) = d(\alpha)$). But as noted in 1. s-functions need not be multiplicative, and e.g. $\Omega(n) - \omega(n)$ belongs to B, since $\Omega(n) - \omega(n) \le \frac{\log n}{\log 2} - 1$ for $n \ge 1$), and other examples of additive functions from B may be readily found.

Our aim in this section is to study the asymptotic behaviour of the sums $\sum_{n\le x, f(n)=g(n+1)} 1$ and $\sum_{n\le x} f(n)g(n+1)$ when $f,g \in B$. The growth condition imposed on the functions of B is a mild one, and a condition of this sort is necessary if one wishes to avoid trivial results. One expects that both of the above sums are asymptotic to Cx for some $C \ge 0$. This, and much more, turns out to be true. Our results are contained in the following two theorems.

THEOREM 1. Let $f(n)$ and $g(n)$ be two functions belonging to B. If s_1, s_2 denote squarefull numbers, then we have

$$(2.1) \qquad \sum_{n\le x, f(n)=g(n+1)} 1 = Ax + O(x^{3/4}\log^4 x),$$

where

$$(2.2) \qquad A = A(f,g) = \frac{6}{\pi^2} \sum_{s_1,s_2=1,(s_1,s_2)=1,f(s_1)=g(s_2)}^{\infty} \frac{1}{s_1 s_2} \times$$

$$\times \prod_{p|s_1s_2} (1-\frac{1}{p}) \prod_{p\nmid s_1s_2} (1-\frac{2}{p^2}).$$

THEOREM 2. Let $f(n)$ and $g(n)$ be two functions belonging to $\mathbf{B}$. If s_1, s_2 denote squarefull numbers, then we have

$$(2.3)\qquad \sum_{n\le x} f(n)g(n+1) = Cx + O(x^{3/4+\varepsilon}),$$

where

$$(2.4)\qquad C = C(f,g) = \frac{6}{\pi^2} \sum_{s_1,s_2=1,(s_1,s_2)=1}^{\infty} \frac{f(s_1)g(s_2)}{s_1s_2}$$

$$\times \prod_{p|s_1s_2} (1-\frac{1}{p}) \prod_{p\nmid s_1s_2} (1-\frac{2}{p^2}).$$

Note that $A > 0$ if $f(s_1) = g(s_2)$ has at least one solution in squarefull s_1, s_2 such that $(s_1,s_2) = 1$, while $C > 0$ if $f(s_1), g(s_2) > 0$ for at least one pair s_1, s_2 such that $(s_1,s_2) = 1$. It may be remarked that Theorem 1 and Theorem 2 have relatively wide applicability, since $f(n)$, $g(n) \in \mathbf{B}$ implies both $f(n)g(n) \in \mathbf{B}$ and $f^k(n) \in \mathbf{B}$ for any integer $k \ge 1$. The proofs of both theorems depend on a certain asymptotic formula involving squarefree numbers. This is

LEMMA 1. Let a,b be two given natural numbers such that $(a,b) = 1$ and let

$$(2.5)\qquad S = \sum_{ka-lb=1,\, lb\le x,\, (k,a)=(l,b)=1} \mu^2(k)\mu^2(l),$$

where k,l denote natural numbers. Then uniformly for $1 \le b \le x^{\frac12}$

$$(2.6)\qquad S = \frac{6x}{\pi^2 ab} \prod_{p|ab} \left(1-\frac{1}{p}\right) \prod_{p\nmid ab} \left(1-\frac{2}{p^2}\right) + O\left(x^{\frac34}(ab)^{-\frac12} 2^{\omega(ab)} + \left(\frac{x}{b}\right)^{\frac12} + \left(\frac{x}{a}\right)^{\frac12} \frac{1}{b}\right).$$

PROOF OF LEMMA 1. Let H be a parameter which satisfies $1 \le H \le (x/b)^{\frac12}$. Observe that the number of solutions in natural numbers u and v of the equation $uu_0 - vv_0 = 1$, $vv_0 \le x$, $(u_0,v_0) = 1$, $u_0,v_0 \in \mathbf{N}$ is uniformly $x/(u_0v_0) + O(1)$. Using then $\mu^2(l) = \sum_{mc^2=l} \mu(c)$ we have, for S defined by (2.5),

$$S = \sum_{\substack{c^2m\le x/b,\, (c,ab)=1,\, (m,ab)=1,\\ mbc^2=ka-1,\, (k,a)=1}} \mu(c)\mu^2(k)$$

$$= \sum_{c\le (x/b)^{\frac12} (c,ab)=1} \mu(c) \sum_{\substack{k\le (x+1)/a,\, (k,abc^2)=1,\\ ka=mbc^2+1,\, (m,b)=1}} \mu^2(k)$$

$$= \sum_{c\le H,\, (c,ab)=1} \mu(c) \sum_{\substack{k\le (x+1)/a,\, (k,abc^2)=1,\\ ka=mbc^2+1,\, (m,b)=1}} \mu^2(k) +$$

$$+ O\left(\sum_{H\le c\le (x/b)^{\frac12}} \sum_{k\le (x+1)/a,\, k\equiv a'(\mathrm{mod}\ bc^2)} 1 \right)$$

$$(a' = a'(b,c))$$

$$= \sum_{c\le H,(c,ab)=1} \mu(c) \sum_{d\le((x+1)/a)^{\frac12},(d,abc)=1} \mu(d)$$

$$\times \sum_{ad^2r=mbc^2+1,mbc^2\le x,(m,b)=1,(r,a)=1} 1$$

$$+ O(x/(abH) + (x/b)^{\frac12})$$

$$= \sum_{c\le H,(c,ab)=1} \mu(c) \sum_{d\le((x+1)/a)^{\frac12},(d,abc)=1} \mu(d) \sum_{e|a} \mu(e)$$

$$\times \sum_{f|b} \mu(f) \sum_{\substack{ad^2er_1=fm_1bc^2+1,fm_1bc^2\le x \\ (r_1,bcf)=(m_1,ade)=1}} 1$$

$$+ O(x/(abH) + (x/b)^{\frac12}) \qquad (r = r_1e,\ m = m_1f)$$

$$= \sum_{c\le H,(c,ab)=1} \mu(c) \sum_{d\le((x+1)/a)^{\frac12},(d,abc)=1} \mu(d)$$

$$\times \left(\prod_{p|ab} (1-\frac{1}{p}) \frac{x}{abc^2d^2} + O(2^{\omega(ab)})\right)$$

$$+ O(x/(abH) + (x/b)^{\frac12})$$

$$= \frac{x}{ab} \prod_{p|ab} (1-\frac{1}{p}) \sum_{c=1,(c,ab)=1}^{\infty} \mu(c)c^{-2} \sum_{d=1,(d,abc)=1}^{\infty} \mu(d)d^{-2}$$

$$+ O(\frac{x}{abH}) + O((\frac{x}{b})^{\frac12}) + O(H(\frac{x}{a})^{\frac12}2^{\omega(ab)}) + O((\frac{x}{a})^{\frac12} \frac{1}{b}).$$

Now we choose $H = x^{\frac{1}{4}}b^{-\frac{1}{2}}$, so that the condition $H \leq (x/b)^{\frac{1}{2}}$ is obvious, and $H \geq 1$ for $b \leq x^{\frac{1}{2}}$, and the above error terms are

$$\ll \; x^{\frac{3}{4}}(ab)^{-\frac{1}{2}}2^{\omega(ab)} + (x/b)^{\frac{1}{2}} + (x/a)^{\frac{1}{2}}b^{-1}.$$

It remains to evaluate the constant which appears in the above main term. This is equal to

$$(ab)^{-1} \prod_{p|ab} (1-p^{-1}) \sum_{c=1,(c,ab)=1}^{\infty} \mu(c)c^{-2}$$

$$\times \sum_{d=1,(d,abc)=1}^{\infty} \mu(d)d^{-2}$$

$$= (ab)^{-1} \prod_{p|ab} (1-p^{-1}) \prod_{p\nmid ab} (1-p^{-2})6\pi^{-2}$$

$$\times \sum_{c=1,(c,ab)=1}^{\infty} \mu(c)c^{-2} \prod_{p|c} (1-p^{-2})^{-1}$$

$$= 6(\pi^2 ab)^{-1} \prod_{p|ab} (1-p^{-1}) \prod_{p\nmid ab} (1-p^{-2})(1-p^{-2}(1-p^{-2})^{-1})$$

$$= 6(\pi^2 ab)^{-1} \prod_{p|ab} (1-p^{-1}) \prod_{p\nmid ab} (1-2p^{-2}),$$

so that the proof of Lemma 1 is complete. For $x^{\frac{1}{2}} < b \leq x$ in most cases the trivial bound

(2.7) $$S \ll \frac{x}{ab} + 1$$

suffices. (2.7) follows from

$$S \le \sum_{\substack{l\le x/b,\, lb\equiv -1 \pmod a, \\ (a,b)=1}} 1 \le \sum_{l\le x/b,\, l\equiv l' \pmod a} 1 = \frac{x}{ab} + O(1),$$

where $l' = l'(b)$.

PROOF OF THEOREM 1 AND THEOREM 2. Having at our disposal Lemma 1 it will be a relatively simple matter to prove both Theorem 1 and Theorem 2. Let $s(n)$ be the squarefull part of n, let q and s denote squarefull and squarefree numbers respectively, and let H be a parameter which satisfies $1 \le H \le x^{\frac{1}{2}}$. Then we have

$$\sum_{n\le x,\, f(n)=g(n+1)} 1 = \sum_{n\le x,\, f(n)=g(n+1),\, s(n)\le H} 1 + \sum_{n\le x,\, f(n)=g(n+1),\, s(n)>H} 1$$

$$= \sum_{n\le x,\, f(n)=g(n+1),\, s(n)\le H} 1 + O\Big(\sum_{H\le s\le x}\ \sum_{q\le x/s} 1\Big)$$

$$= \sum_{n\le x,\, f(n)=g(n+1),\, s(n)\le H} 1 + O(xH^{-\frac{1}{2}})$$

$$= \sum_{n\le x,\, f(n)=g(n+1),\, s(n)\le H,\, s(n+1)\le H} 1 + O(xH^{-\frac{1}{2}})$$

$$= \sum_{\substack{s_1,s_2\le H,\, (s_1,s_2)=1 \\ f(s_1)=g(s_2)}}\ \sum_{\substack{q_2s_2-q_1s_1=1, \\ q_1s_1\le x,\, (q_1,s_1)=(q_2,s_2)=1}} 1 + O(xH^{-\frac{1}{2}}).$$

In the last expression the inner sum may be estimated by Lemma 1. Hence (2.6) gives $(s_1 \le H \le x^{\frac{1}{2}})$

$$\sum_{n\le x,\, f(n)=g(n+1)} 1 = \frac{6x}{\pi^2} \sum_{\substack{s_1,s_2\le H,\,(s_1,s_2)=1\\ f(s_1)=g(s_2)}} \frac{1}{s_1 s_2}$$

$$\prod_{p|s_1s_2} \left(1-\frac{1}{p}\right) \prod_{p|s_1s_2} \left(1-\frac{2}{p^2}\right) \tag{2.8}$$

$$+ O(xH^{-\frac12}) + O\left(\sum_{s_1,s_2\le H} x^{\frac12} s_1^{-\frac12}\right) +$$

$$+ O\left(\sum_{s_1,s_2\le H} x^{\frac34} 2^{\omega(s_1)} s_1^{-\frac12} 2^{\omega(s_2)} s_2^{-\frac12}\right).$$

But if s denotes squarefull numbers, then by easy elementary arguments we have, as $x\to\infty$,

$$\sum_{s\le x} 1 \sim \frac{\zeta(3/2)}{\zeta(3)} x^{\frac12}, \qquad \sum_{s\le x} 2^{\omega(s)} s^{-\frac12} << \log^2 x, \qquad \sum_{s>x} s^{-1} \ll x^{-\frac12}. \tag{2.9}$$

Using (2.9) in (2.8) we obtain

$$\sum_{n\le x,\, f(n)=g(n+1)} 1 = Ax + O(xH^{-\frac12}) + O(x^{\frac12}H^{\frac12}\log x) + O(x^{\frac34}\log^4 x). \tag{2.10}$$

If we choose $H = x^{\frac12}$, then (2.10) immediately gives (2.1).

The proof of Theorem 2 is completely analogous. We have

$$\sum_{n\le x} f(n)g(n+1) = \sum_{n\le x,\, s(n)\le H,\, s(n+1)\le H} f(n)g(n+1) + \tag{2.11}$$

$$+ O(x^{1+\varepsilon}H^{-\frac12})$$

$$= \sum_{s_1,s_2\le H,(s_1,s_2)=1} f(s_1)g(s_2) \sum_{\substack{q_2s_2-q_1s_1=1,\\ q_1s_1\le x,(q_1,s_1)=(q_2,s_2)=1}} 1$$

$$+ O(x^{1+\varepsilon}H^{-\frac{1}{2}}),$$

where we used $f(n) \ll n^{\varepsilon}, g(n) \ll n^{\varepsilon}$ in estimating the error term. Using again Lemma 1 and proceeding as in the previous proof we obtain (2.3) and (2.4).

3. APPLICATIONS

As already remarked, several common arithmetic functions such as $a(n)$, $\Omega(n) - \omega(n)$ and $\mu^2(n)$ (the characteristic function of squarefree numbers) clearly belong to $\mathcal{B}$. It was also remarked that $f,g \in \mathcal{B}$ implies $f\cdot g \in \mathcal{B}$ and $f^{\alpha} \in \mathcal{B}$ for α a natural number; however for Theorem 2 we may take even $\alpha \in \mathrm{Re}$, since the proof obviously works for the sum $\sum_{n\le x} f^{\alpha_1}(n)g^{\alpha_2}(n+1)$ if $f,g \in \mathcal{B}$, $\alpha_1,\alpha_2 \in \mathrm{Re}$ and f,g, are positive for all n. Thus as corollaries of Theorem 2 we have

$$(3.1) \qquad \sum_{n\le x} a(n)(\Omega(n+1) - \omega(n+1)) = Cx + O(x^{\frac{3}{4}+\varepsilon}) \qquad (C > 0),$$

$$(3.2) \qquad \sum_{n\le x} a^r(n)a^s(n+1) = C_{r,s}x + O(x^{\frac{3}{4}+\varepsilon})$$

with $C_{r,s} > 0$ and $r,s \in \mathrm{Re}$ fixed. Hence by the binomial theorem (3.2) implies

$$(3.3) \qquad \sum_{n\le x} (a(n)-a(n+1))^{2m} = C_m x + O(x^{\frac{3}{4}+\varepsilon})$$

with $C_m > 0$ and $m \in \mathbf{N}$ fixed. From Theorem 1 we have

$$(3.4) \qquad \sum_{n\le x,\, a(n)=a(n+1)} 1 = D_1 x + O(x^{\frac{3}{4}}\log^4 x),$$

$$(3.5) \qquad \sum_{n\le x,\, a(n)=\Omega(n+1)-\omega(n+1)} 1 = D_2 x + O(x^{\frac{3}{4}}\log^4 x),$$

where $D_1, D_2 > 0$, the latter formula being true since $a(s_1) = \Omega(s_2) - \omega(s_2)$ has the solution $s_1 = p_1^2$, $s_2 = p_2^3$ (p's primes). The method of proof of Theorem 1 works also for sums like $\sum_{n\le x,\, rf(n)=g(n+1)} 1$, where $f,g \in \mathbf{B}$ and $r > 0$ is given rational number. It would be too long to state explicitly all interesting applications of our theorems. We mention only one more, namely

$$(3.6) \qquad \sum_{n\le x} \mu^2(n)\mu^2(n+1) = Dx + O(x^{\frac{3}{4}+\varepsilon}), \quad D = \prod_p (1-2p^{-2}),$$

and the sum in (3.6) is interesting, since it represents the number of $n \le x$ such that both n and n+1 are squarefree. However, for this particular problem much better error terms are known: $O(x^{2/3+\varepsilon})$ is due to L. Carlitz [1], $O(x^{2/3}\log^{4/3}x)$ to L. Mirsky [21], [22], $O(x^{2/3}\log^{2/3}x)$ to R.R. Hall [8], and finally $O(x^{7/11}\log^7 x)$ was proved recently by D.R. Heath-Brown [10]. Likewise, the error terms in (3.2) and (3.3) may be replaced by $O(x^{2/3+\varepsilon})$. This follows from Th. 1 of L. Mirsky [24], but his method requires the functions in question to be sub-multiplicative, hence it cannot give e.g. (3.1) nor, in general, Th.2, and it cannot be used at all for the sum in Theorem 1.

There are several more general sums than the ones in (2.1) and (2.2) which may be also evaluated by methods similar to those used in proving Theorem 1 and Theorem 2. Consider, for example, $\sum_{n\le x} f(n)h(n+1)$, where $f(n) \in \mathbf{B}$, and $h(n)$ is a positive, integer-valued arithmetic function for which only $h(n) \ll n^{\varepsilon}$ is supposed to hold (so that $h(n)$ is not necessarily a s-function). If $1 \le H \le x^{\frac{1}{2}}$ is a parameter, than we have

$$(3.7)\qquad \sum_{n\le x} f(n)h(n+1) = \sum_{n\le x, s(n)\le H} f(n)h(n+1) + O(x^{1+\varepsilon}H^{-\frac{1}{2}})$$

$$= \sum_{s\le H} f(s) \sum_{q\le x/s, (q,s)=1} h(qs+1) + O(x^{1+\varepsilon}H^{-\frac{1}{2}})$$

$$= \sum_{s\le H} f(s) \sum_{\substack{d\le (x/s)^{\frac{1}{2}},\\ (d,s)=1}} \mu(d) \sum_{\substack{m\le x/(sd^2),\\ (m,s)=1}} h(msd^2+1) + O(x^{1+\varepsilon}H^{-\frac{1}{2}})$$

$$= \sum_{s\le H} f(s) \sum_{d\le H^{\frac{1}{2}}, (d,s)=1} \mu(d) \sum_{\substack{m\le x/(sd^2),\\ (m,s)=1}} h(msd^2+1) +$$

$$+ O(x^{1+\varepsilon}H^{-\frac{1}{2}}) + O(Hx^{\varepsilon})$$

$$= \sum_{s\le H} f(s) \sum_{d\le H^{\frac{1}{2}},(d,s)=1} \mu(d) \sum_{\delta|s} \mu(\delta) \sum_{k\le x, k\equiv 1 (\mathrm{mod}\ sd^2\delta)} h(k) +$$

$$+ O(x^{1+\varepsilon}H^{-\frac{1}{2}}) + O(Hx^{\varepsilon}),$$

where as before q and s denote squarefree and squarefull numbers, respectively. Therefore if we can find an asymptotic formula for

$$(3.8) \qquad H(x,r) = \sum_{n\le x, n\equiv 1 (\mathrm{mod}\ r)} h(n) \qquad (r = ds\delta^2),$$

which is uniform in $r = r(x)$ for a certain range, then inserting (3.8) in (3.7), simplifying and choosing H optimally we obtain an asymptotic formula for the summatory function of $f(n)h(n+1)$. As an example, consider $h(n) = d(n)$, the number of divisors of n. It was proved by D.R. Heath-Brown [9] that

$$(3.9) \qquad \sum_{n\le x, n\equiv 1 (\mathrm{mod}\ r)} d(n) = \frac{x}{r^2}\left(A(r,1)\left(\log \frac{x}{r^2} + 2\gamma - 1\right) + 2B(r,1)\right) +$$

$$+ O(x^{1/3+\varepsilon})$$

uniformly for $1 \le r \le x^{2/3}$, where γ is Euler's constant and

$$A(r,1) = \sum_{d|r} d\mu\left(\frac{r}{d}\right) \ll rx^{\varepsilon}, \quad B(r,1) = \sum_{d|r} d\mu\left(\frac{r}{d}\right)\log d \ll rx^{\varepsilon}.$$

For $r = sd^2\delta$, $s \le H$, $d^2 \le H, \delta \le H$ the condition $r \le x^{2/3}$ is satisfied if $H = x^{2/9}$, whence from (3.7) and (3.9) we obtain

$$(3.10) \qquad \sum_{n\le x} f(n)d(n+1) = C_1 x \log x + C_2 x + O(x^{8/9+\varepsilon}).$$

In this formula, which appears to be new (and gives (1.5) as a special case), $f \in \mathbf{B}$, $C_1 \ge 0$ and C_2 are two constants which may be explicitly evaluated. The analogous problem with $d(n)$ replaced by $d_3(n)$ may be considered if one appeals to the recent result of J.B. Friedlander and H. Iwaniec [7] concerning the asymptotic formula for $\sum_{n\le x, n\equiv a (\mathrm{mod}\ q)} d_3(n)$, $(a,q)=1$. A corresponding formula for $H(x,r)$ in (3.8) when $h(n) = \omega(n)$ or $\Omega(n)$ may be obtained by the methods of H. Delange [2], and in this case the error term would be uniform for $1 \le r \le \log^C x$, $C > 0$ any fixed number. This would give, for $f \in \mathbf{B}$,

$$(3.11) \qquad \sum_{n\le x} f(n)\omega(n+1) = D_1 x \log\log x + D_2 x + O(x/\log x)$$

where $D_1 \ge 0$ and D_2 may be explicitly evaluated. In (3.11) one may consider $\omega^k(n+1)$ for any fixed $k \in \mathbf{N}$ instead of $\omega(n+1)$ only. In this case the right-hand side of (3.11) would contain as the main term x times a polynomial of degree k in $\log\log x$.

One may also consider the sum

$$(3.12) \qquad \sum_{n\le x} f(n)g(n+k)$$

where $f,g \in B$ and $k > 1$. In fact, k may be assumed to be any fixed natural number, and even $k = k(x)$ may be interesting. It is possible to evaluate the general sum appearing in (3.12) by proceeding as in (3.7). In this way one is led to the estimation of the sums

$$G(x;r,l) = \sum_{n\le x, n\equiv l \pmod r} g(n)$$

where $(l,r) = 1$ does not have to hold, and consequently the estimation of $G(x;r,l)$ may be technically difficult. The same applies to the estimation of the sum

$$(3.13) \qquad S_k(x) = \sum_{n\le x} f_1(n+\ell_1)f_2(n+\ell_2) \ldots f_k(n+\ell_k),$$

where $f_1, \ldots, f_k \in B$, $0 \le \ell_1 \le \ldots \le \ell_k$ are fixed integers. This sum would be asymptotic to Cx, $C = C(f_1,\ldots,f_k,\ell_1,\ldots,\ell_k) \ge 0$, and its evaluation would reduce to $S_{k-1}(x)$ for n in a certain residue class. Under certain conditions which include multiplicativity or sub-multiplicativity of the f_i's, this follows from the works of P. Erdős [3] and L. Mirsky [24], respectively, and the latter gives a good error term. But none of these apply to the case when the f_i's are s-functions. Also one should be careful in treating (3.13), since in the general case it may well turn out that $C = C(f_1, \ldots, \ell_k)$ is zero, or even that the whole sum under consideration is zero. As an example, we recall L. Mirsky's formula (1.13), but note that identically

$$\sum_{n\le x} \mu^2(n)\mu^2(n+1)\mu^2(n+2)\mu^2(n+3) = 0,$$

since one of any four consecutive integers must be divisible by four, and $\mu(4m) = 0$ for any $m \ge 1$. For a generalization of (3.13) when $f_j = \mu^2$, see R.R. Hall [8].

4. THE DISTRIBUTION OF VALUES OF a(n)

As in 1. we let $C(x)$ and $D(x)$ denote the number of distinct values of $a(n)$ for $n \le x$ and the number of $n \le x$ such that $n = a(m)$ for some m, respectively. An unconditional proof of the formulas in (1.9) seems quite difficult, but it seems reasonable to expect that $\lim_{x\to\infty} C(x)/D(x) = 0$ may be at least proved. This follows from

THEOREM 3. For $x \ge x_0$

$$D(x) \ge \tfrac{1}{3}C(x)\log\log x. \tag{4.1}$$

PROOF OF THEOREM 3. Let $n_1 = a(k_1), \ldots, n_t = a(k_t)$ $(k_j \le x)$ be the distinct values of $a(n)$ for $n \le x$. Then $t = C(x)$, and from (1.4) it follows that

$$n_j \le \exp(\log x/(2\log\log x)), (x \ge x_0;\ j=1,\dots,t).$$

Put

$$m_{j,r} = k_j(q_1q_2\dots q_{ru})^2,$$

where $q_1, q_2, \dots$ are distinct primes not dividing any of the k_j's (e.g. take $q_l > x$ for $l = 1,2,\dots$), $r \ge 1$ is an integer and $u = [\log x/(\log 2 \log\log x)]$. Thus

$$a(m_{j,r}) = a(k_j)a(q_1^2)\dots a(q_{ru}^2) = n_j(2^u)^r \le x$$

for $j = 1,\dots,t$ and $1 \le r \le [\frac{1}{2}\log\log x]$, so that there are at least

$$[\tfrac{1}{2}\log\log x]t \ge \tfrac{1}{3}C(x)\log\log x$$

numbers $a(m_{j,r})$ for $x \ge x_0$, and (4.1) follows if they are all distinct. Suppose that this is not true. Then if

$$(4.2) \qquad n_j 2^{ru} = n_k 2^{su}$$

for $s > r$, we have for $x \ge x_0$

$$\exp(\log x/(2\log\log x)) \ge n_j/n_k =$$

$$= 2^{u(s-r)} \ge 2^u \ge \exp(3\log x/(4\log\log x)),$$

which is a contradiction. It follows that in (4.2) we have $s = r$, hence $j = k$, and (4.1) is proved.

Note that if (1.9) is true, then $D(x)$ is roughly $C(x)$ to the power $\log^{1/6}x$, which is incomparably stronger than (4.1). On the other hand, practically the only property of $a(n)$ that was used in the above proof was the bound (1.4). Hence an analogous result may be easily formulated for a wide class of multiplicative, prime--independent functions which take natural numbers as values and satisfy a growth condition similar to (1.4).

To obtain lower bounds for $C(x)$ we proceed as follows. Let $t,r \ge 1$ be integers (to be suitably chosen later), and let p_n denote the n-th prime. Consider numbers of the form

$$(4.3)\qquad n = (p_1\cdots p_{j_1})^{k_1}(p_{r+1}\cdots p_{r+j_2})^{k_2}\cdots$$

$$\cdots(p_{r(t-1)+1}\cdots p_{r(t-1)+j_t})^{k_t},$$

where each j_l $(l=1,\dots,t)$ takes all possible values $j_l = 1,2,\dots,r$ and $1 \le k_1 < k_2 < \dots < k_t$ are suitably chosen integers. By the prime number theorem $p_n = (1+o(1))n\log n$ as $n \to \infty$ and $\theta(x) = \sum_{p\le x} \log p = x + o(x)$ as $x \to \infty$, hence for rt large

$$n \le \exp(k_t\theta(p_{rt})) \le \exp(2k_t rt\log(rt)) \le \exp(\log x) = x$$

if r,t and k_t satisfy

$$(4.4)\qquad 2k_t rt\log(rt) \le \log x \qquad (x \ge x_0).$$

Moreover,

$$(4.5)\qquad a(n) = P^{j_1}(k_1)P^{j_2}(k_2)\ldots P^{j_t}(k_t) ,$$

and we wish all the numbers of the form (4.5) to be distinct. Therefore the sequence $k_1 < \ldots < k_t$ should be chosen in such a way that

$$(4.6)\qquad a(n_1) = P^{j_1}(k_1)\ldots P^{j_t}(k_t) = P^{l_1}(k_1)\ldots P^{l_t}(k_t) = a(n_2)$$

implies $n_1 = n_2$ for $1 \le j_m, l_n \le r$, $1 \le m,n \le t$. If (4.4) and the last condition are satisfied, then there are r^t different numbers of the form (4.3), and it follows that

$$(4.7)\qquad C(x) \ge r^t .$$

Hence the larger $t\log r$ is, the better our lower bound for $C(x)$ will be, and therefore k_t (as a function of t) should be as small as possible. One may think of the numbers k_j as the elements of the (minimal) basis of the multiplicative semigroup generated by $P(1)$, $P(2)$, $P(3)$, ..., so that k_j is the j-th element of the basis. In this context $D(x)$ represents the number of distinct elements of this semigroup which do not exceed x. To obtain a specific, unconditional lower bound for $C(x)$ we shall use the following

LEMMA 2. (A. Schinzel) If $\omega(n)$ is the number of distinct prime factors of n, then

$$(4.8) \qquad \lim_{n\to\infty} \omega\left(\prod_{j=1}^{n} P(j)\right) = \infty \quad .$$

PROOF OF LEMMA 2. For the partition function we shall use the well-known asymptotic formula (see e.g. M. Knopp [18], p. 90)

$$(4.9) \qquad P(n) = \frac{1}{4\sqrt{3}} \cdot \frac{e^{a\lambda_n}}{n - 1/24} \left(1 - \frac{1}{a\lambda_n}\right) + 0\left(e^{5a\lambda_n/8}\right) ,$$

where $\lambda_n = (n - 1/24)^{1/2}$ and $a = \pi(2/3)^{1/2}$. We suppose that the primes $q_1,\ldots,q_r$ constitute the prime factors of $P(n)$ for every $n \geq 2$. We shall use a result of R. Tijdeman [30], which says that there exists a constant $C < 0$ such that if the numbers A and B are composed only of $q_1,\ldots,q_r$ and $|A - B| \leq A\log^C A$, then $A = B$. Let us choose two sets of positive integers $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ such that

$$\sum_{j=1}^{k} a_j^m = \sum_{j=1}^{k} b_j^m$$

for $m = 1,2,\ldots, [c/2]$, but

$$\sum_{j=1}^{k} a_j^{[c/2]+1} \neq \sum_{j=1}^{k} b_j^{[c/2]+1} \quad .$$

Using (4.9), Taylor's expansion and simplifying the resulting expressions we obtain

$$(4.10)\qquad \prod_{j=1}^{k} P(n+a_j) = \prod_{j=1}^{k} P(n+b_j)(1+O(n^{-1/2-[c/2]}))$$

and also

$$(4.11)\qquad \prod_{j=1}^{k} P(n+a_j) = \prod_{j=1}^{k} P(n+b_j)(1+\Omega(n^{-1/2-[c/2]})).$$

Putting in Tijdeman's theorem

$$A = \prod_{j=1}^{k} P(n+a_j),\ B = \prod_{j=1}^{k} P(n+b_j)$$

we find from (4.10)

$$A - B \ll A(\log A)^{-2[c/2]-1},$$

which yields $A = B$. However, this contradicts (4.11), so (4.8) must hold.

From Lemma 2 it follows that there exist arbitrarily large t and integers $2 = k_1 < k_2 < k_t$ such that there exist primes $2 = q_1 < q_2 < \ldots < q_t$ with the property that each q_j divides $P(k_j)$ but does not divide any $P(k_l)$ for $l < j$, i.e. $\lim_{t\to\infty} k_t = \infty$. We shall take $t = [A] + 1$ (where $A > 0$ is any fixed constant), $r = [C\log x/\log\log x]$, $C = 1/(10k_t t)$. Then (4.4) clearly holds, and all numbers of the form (4.5) are distinct. Namely, if (4.6) holds, we must have first $j_t = l_t$

(since q_t does not divide $P(k_m)$ for $1 \le m \le t-1$), likewise $j_{t-1} = l_{t-1}, \ldots,$ until finally we obtain $j_1 = l_1$, that is, $n_1 = n_2$. Consequently (4.7) gives immediately

THEOREM 4. For any fixed $A > 0$ and $x \ge x_0(A)$

$$C(x) > (\log x)^A \,. \tag{4.12}$$

It is clear that any non-trivial bound of the form $k_t \le g(t)$, where $g(t)$ is a positive, increasing function tending to ∞, would improve (4.12). For example, if we had $k_t \ll e^{ct}$ $(c > 0)$, then by the foregoing argument it would follow that $C(x) > \exp(B(\log\log x)^2)$ for some $B > 0$. Actually, it seems not unreasonable to expect a drastically better bound for k_t to be true, namely

$$k_t \ll t^{1+\varepsilon} \,, \tag{4.13}$$

which is (up to ε) best possible. Proving (4.13) seems quite difficult, but perhaps it is not hopeless to expect that $k_t \ll t^B \log^C t$ may be proved with some $B \ge 1$, $C \ge 0$. If this were so, then taking $r = 2$ in our construction we would obtain

$$n \le \exp(2k_t rt\log(rt)) \le \exp(D_1 t^{B+1} \log^{C+1} t) \le x \qquad (D_1 > 0,\ x \ge x_0)$$

for

$$t = [D_2(\log x)^{1/(B+1)}(\log\log x)^{-(C+1)/(B+1)}],$$

$$(D_2 > 0)$$

which exists by Lemma 2. (Note the principle of the construction: if we have a poor bound for k_t, take r as large as possible; if we have a good bound for k_t, take $r = 2$, i.e. as small as possible). Then (4.7) gives

$$(4.14)\qquad C(x) \geq r^t = 2^t \gg \exp(D_3(\log x)^{1/(B+1)}(\log\log x)^{-(C+1)/(B+1)})$$

for some $D_3 > 0$. In this case the numbers of the form (4.5) are all distinct, and in fact (4.5) also gives

$$a(n) \leq P^{rt}(k_t) \leq \exp(C\, k_t^{1/2} rt) \leq$$

$$\leq \exp(C_2 t^{(B+2)/2} \log^{C/2} t) \leq \exp(\log x) = x$$

for some C_1, $C_2 > 0$ if $r = 2$ and

$$t = [D_4(\log x)^{2/(B+2)}(\log\log x)^{-C/(B+2)}], (D_4 > 0).$$

Note that we used here an upper bound for the partition function (follows easily from (4.9)), hence an intrinsic property of $a(n)$. With this t the numbers of the form (4.5) are all counted by $D(x)$, and we obtain

$$(4.15)\qquad D(x) \geq 2^t \gg \exp(D_5(\log x)^{2/(B+2)}(\log\log x)^{-C/(B+2)})$$

$$(D_5 > 0).$$

In particular, if we recall that (1.7) - (1.9) hold unconditionally, then from (4.13) - (4.15) we obtain

THEOREM 5. If the bound (4.13) holds, then as $x \to \infty$

$$C(x) = \exp((\log x)^{1/2+o(1)}),$$

$$D(x) = \exp((\log x)^{2/3+o(1)}).$$

This means that if the conjecture (4.13) is true, then (up to the evaluation of the o(1) terms) we have determined asymptotically the order of magnitude of $C(x)$ and $D(x)$. It is also worth remarking that the above procedure for bounding $C(x)$ can be carried over to count distinct values of other multiplicative, prime-independent, positive, integer-valued functions $f(n)$. Instead of trying to obtain a general result, we shall consider the familiar function $f(n) = d_k(n)$. In this case

$$g(\alpha) = d_k(p^\alpha) = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + k - 1)}{(k - 1)!}$$

plays the role that $P(n)$ had for $a(n)$. Now q divides $g(q - 1)$ for $q(> k-1)$ prime, and for almost all q $g(q - m)$ $(m > 1)$ is not divisible by q. Hence in the above notation $k_t \ll t^B \log^C t$ with $B = C = 1$ for $d_k(n)$. If we recall the classical Hardy-Ramanujan formula for the number of integers of the form $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$

($a_1 \geq a_2 \geq \ldots \geq a_r$, p_j the j-th prime) not exceeding x, used in majorizing C(x) as in (1.8), then from (4.14) with B = C = 1 we obtain unconditionally

$$(4.16) \qquad \exp(E_1(k)(\log x)^{1/2}/\log\log x) \ll C(x) \ll$$
$$\ll \exp(E_2(\log x/\log\log x)^{1/2})$$

with $E_1(k) > 0$, $E_2 = 2\pi 3^{-1/2} + \varepsilon$. This seems new for k > 2, but is was already mentioned (see (1.12)) that P. Erdős and L. Mirsky [5] proved a sharper result than (4.16) for k = 2. It is plausible to conjecture that for $d_k(n)$ ($k \geq 2$ fixed) one has, as $x \to \infty$,

$$C(x) = \exp((E(k) + o(1))(\log x)^{1/2}/\log\log x) \qquad (E(k) > 0).$$

In this case the function D(x) is much easier to handle. For k = 2 one has trivially D(x) = x + O(1) (since n = d(m) has a solution for every $n \geq 1$), and for k > 2 it is not difficult to show that D(x) is of the order $x^{1/(k-1)+o(1)}$.

5. THE VALUES OF a(n) ON LONG BLOCKS OF CONSECUTIVE INTEGERS

In this section we shall consider the values $a(n+1)$, $a(n+2)$, ... on "long" blocks of consecutive integers. More precisely, we shall suppose $x \le n \le 2x$ and will try to find two functions $k = k(x)$ and $t = t(x)$, as large as possible, such that $a(n+1) = \dots = a(n+k)$ for many n and such that the values $a(n+1), \dots, a(n+t)$ are all distinct for many n. Analogous problems may be of course considered for other common arithmetic functions, such as $d_k(n)$ for instance. The first problem is very difficult already for $d(n) = d_2(n)$, and it is known that

$$x/\log^7 x \ll \sum_{n\le x, d(n)=d(n+1)} 1 \ll x(\log\log x)^{-1/2}.$$

The lower bound is due to D.R. Heath-Brown [11], and the upper bound to P. Erdős et al. [6] (A. Hildebrand very recently announced in [14] the lower bound $x(\log\log x)^{-3}$). Nothing non-trivial seems to be known about integers n for which $d(n) = d(n+1) = d(n+2)$ etc. On the other hand, P. Erdős and L. Mirsky [5] showed that the number of $x \le n \le 2x$ such that the values $d(n+1), \dots, d(n+t)$ are all distinct is $\gg x^{1/2}$ if $t = [C(\log x)^{1/2}/\log\log x]$ with a suitable $C > 0$. We are going to prove the same result for general $d_k(n)$ by a method which, in the case of $a(n)$, gives the result with $t = [C(\log x/\log\log x)^{1/2}]$. We shall also be able to find very long blocks of consecutive integers where the values of $a(n)$ are all equal. This is quite a contrast with the case of $d_k(n)$, and is essentally due to the fact that $a(n)$ is an s-function, that is, $a(q) = 1$ if q is squarefree. Our results are

THEOREM 6. There exist at least $x^{1/2}$ numbers n from $[x,2x]$ such that

$$a(n + 1) = a(n + 2) = \dots = a(n + k),$$

$$k = \left[\frac{\log x \log\log\log x}{40(\log\log x)^2}\right]$$

THEOREM 7. There exist at least $x^{1/2}$ numbers n from $[x,2x]$ such that for a suitable $C > 0$ the values $a(n + 1), a(n + 2), \dots, a(n + t)$ are all distinct for

$$t = [C(\log x/\log\log x)^{/2}] .$$

PROOF OF THEOREM 6. If k is as in Theorem 6, then we shall find k integers $n+1, n+2, \dots, n+k$ which all have the same pattern for at least $x^{1/2}$ numbers n from

$[x,2x]$. By this we mean that

$$(5.1) \qquad n + i = p_{i,1}^{\alpha_1} p_{i,2}^{\alpha_2} \cdots p_{i,R}^{\alpha_R} l_i \qquad (i=1,2,\ldots,k),$$

where the p_i's are distinct primes not dividing l_i, l_i is squarefree, $2 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_R$, and we say that $(\alpha_1,\alpha_2,\ldots,\alpha_R)$ is the pattern of $n + i$. From (5.1) it follows that

$$a(n + i) = P(\alpha_1)P(\alpha_2)\ldots P(\alpha_R) \qquad (i=1,2,\ldots,k),$$

and since the right-hand side here does not depend on i, we have $a(n+1) = a(n+2) = \ldots = a(n+k)$ and Theorem 6 follows. Note that this construction does not work for $d(n)$, since

$$d(n + i) = (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_R + 1)2^{\omega(l_i)}$$

depends on i. Therefore our problem is reduced to the purely arithmetic problem of finding long blocks of consecutive integers with the same pattern, which seems to be a problem interesting in itself. First we define (fix) the pattern $(\alpha_1,\alpha_2,\ldots,\alpha_R)$ as follows. Note that if $2^a \le k < 2^{a+1}$, i.e. if $a = [\log k/\log 2]$, then $\alpha_R = [\log k/\log 2]$ and this occurs with multiplicity one in the pattern. The number 2 occurs among the α_r's j_1 times if j_1 is defined by

$$2^2 3^2 \ldots p_{j_1}^2 \le k < 2^2 3^2 \ldots p_{j_1}^2 p_{j_1+1}^2 ,$$

where p_m is the m-th prime. In general, for a given r such that $2 \le r \le \log k/\log 2$, the number r occurs in the pattern j_r times, where j_r (≥ 1) is defined by

$$(5.2) \qquad p_1^r p_2^r \ldots p_{j_r}^r \le k < p_1^r p_2^r \ldots p_{j_r}^r p_{j_r+1}^r .$$

Therefore, roughly speaking, the numbers occurring in the pattern will represent the union of all possible exponents of squarefull numbers not exceeding k. Small α_r's will appear in the pattern with a large multiplicity, and large α_r's with a small one. Namely from (5.2) and the prime number theorem we have, for $2 \le r \le g(k)$, $g(k) = o(\log k/\log\log k)$ as $k \to \infty$,

$$(5.3) \qquad j_r = \frac{(1 + o(1))\log k}{r \log\log k} .$$

In general, for $r_0 \le r \le \log k/\log 2$, (5.2) implies by estimates from elementary prime number theory

$$(5.4) \qquad j_r \le \frac{5\log k}{r\log(\frac{\log k}{r} + 1)} .$$

Our construction gives then the precise pattern of numbers of the form (5.1). Further, for $p_j \le k^{1/2}$ we define c_j (≥ 2) as the integer for which $p_j^{c_j-1} \leq k < p_j^{c_j}$, and we denote by $J = \pi(k^{1/2})$ the largest j such that $p_j \leq k^{1/2}$. We require first

$$(5.5) \qquad n \equiv p_1^{c_1} p_2^{c_2} \dots p_J^{c_J} \pmod{p_1^{c_1+1} p_2^{c_2+1} \dots p_J^{c_J+1}} .$$

Next we require that, for $i = 1,\dots,k$,

$$(5.6) \qquad n + i \equiv p_{i,1}^{\alpha_1} p_{i,2}^{\alpha_2} \dots p_{i,R}^{\alpha_R} \pmod{p_{i,1}^{\alpha_1+1} p_{i,2}^{\alpha_2+1} \dots p_{i,R}^{\alpha_R+1}} ,$$

where $(\alpha_1,\alpha_2,\dots,\alpha_R)$ is the pattern defined above, and the congruence conditions ensure that $p_{i,j}^{\alpha_j} \| n + i$. Here the $p_{i,j}$'s are distinct primes exceeding $k^{1/2}$, to be taken as small as possible. The congruences (5.6) are to hold if no prime powers that divide i divide also $F = p_1^{c_1} \dots p_J^{c_J}$. If this condition is not satisfied, and if e.g. some p^{β} ($\beta \geq 2$) divides i and F, then the corresponding factor $p_{i,1}^{\beta}$ is to be omitted in (5.6). This procedure is necessary, since in view of (5.5) the numbers $n + i$ are automatically divisible by i if $i \mid F$, and therefore the omission of suitable $p_{i,1}^{\beta}$'s will ensure that the pattern $(\alpha_1,\alpha_2,\dots,\alpha_R)$ of each $n + i$ is preserved. To elucidate this idea, take $k = 40$. Then the pattern is $(2,2,3,4,5)$, we choose $n \equiv 2^6 3^4 5^3 \pmod{2^7 3^5 5^4}$,

$$n+1 \equiv 7^2 11^2 13^3 17^4 19^5 \pmod{7^3 11^3 13^4 17^5 19^6},$$

and analogously for $n+2$ and $n+3$ we introduce 10 new consecutive primes. But since 2^2 divides $F = 2^6 3^4 5^3$, we shall have

$$n+4 \equiv p_{4,1}^2 p_{4,2}^3 p_{4,3}^4 p_{4,4}^5 \pmod{p_{4,1}^3 p_{4,2}^4 p_{4,3}^5 p_{4,4}^6},$$

similarly

$$n+32 \equiv p_{32,1}^2 p_{32,2}^2 p_{32,3}^3 p_{32,4}^4 \pmod{p_{32,1}^3 p_{32,2}^3 p_{32,3}^4 p_{32,4}^5}$$

$$n+36 \equiv p_{36,1}^3 p_{36,2}^4 p_{36,3}^5 \pmod{p_{36,1}^4 p_{36,2}^5 p_{36,3}^6},$$

and so on. Note that the moduli in (5.5) and (5.6) are all pairwise coprime, hence by the Chinese remainder theorem the system of congruences will have a unique solution modulo $A(k)$, say. From (5.3) and (5.4) it follow that

$$R = \sum_{2 \le r \le \log k/\log 2} j_r \le 6 \log k \log\log k .$$

For each $n+i$ we introduce at most R new primes, so that $A(k)$ is the product of at most $6k \log k \log\log k$ consecutive primes exceeding $k^{1/2}$, some of which may have exponents as large as $\log k/\log 2$. For our purpose, however, it is desirable to have $A(k)$ as small as possible, and therefore in constructing the congruences

(5.6) we shall make the following stipulation. For numbers r appearing in the pattern $(\alpha_1,\alpha_2,\ldots,\alpha_R)$ such that $2 \le r \le \varepsilon\log k/\log\log k$ (this corresponds to (5.3)), we choose the primes $p_{i,r}$ (for each $i = 1,\ldots,k$) as small as possible, and then we "fill in" the larger primes for the r's appearing with smaller multiplicities (this corresponds to (5.4)). Thus for $r > \varepsilon\log k/\log\log k$ we obtain $j_r \ll_\varepsilon \log\log k$ in (5.4). Moreover we have from (5.3), as $k \to \infty$,

$$\sum_{2\le r\le\varepsilon\log k/\log\log k} j_r \le \sum_{2\le r\le\log k} \frac{(1+o(1))\log k}{r\log\log k} = (1+o(1))\log k .$$

Hence it follows, for some $C = C(\varepsilon) > 0$, that

$$A(k) \le \prod_{p\le(1+o(1))k\log k} p^{\log k/\log\log k} \prod_{p\le(6+o(1))k\log k\log\log k} p^{C\log\log k}$$

$$\le \exp\left(\frac{2\Theta(2k\log k)\log k}{\log\log k} + 2C\Theta(7k\log k\log\log k)\log\log k\right)$$

$$\le \exp\left(\frac{8k\log^2 k}{\log\log k}\right) \le x^{1/4}$$

if $k = [\log x\log\log\log x/(40\log\log x)^2]$ as in theorem 6. All solutions of (5.5) and (5.6) are of the form $n = A(k)m + B(k), 0 \le B(k) < A(k)$, hence there are $x/A(k) + O(1)$ such n from $[x,2x]$. It remains yet to show that at least $x^{1/2}$ of these n are of the form

(5.1) (i.e. that l_i in (5.1) is squarefree). This means that we have to omit those n which are $\equiv -i \pmod{p^2}$, $0 \le i \le k, p > A(k)$. For a fixed prime p ($> A(k)$) we omit $x/(A_k p^2) + 0(1)$ numbers, since p^2 may divide only one $n + i$ for a fixed n. This is so, because if $n + i = m_i p^2$ and $n + j = m_j p^2$, then $A(k) \ge |i - j| = p^2 |m_i - m_j| \ge p^2$, which is obviously impossible. Hence the total number of omitted n's does not exceed

$$\sum_{A(k)<p\le x^{1/2}} \left(\frac{x}{A_k p^2} + 0(1)\right) \ll x/A_k^2 + x^{1/2} = o(x/A_k) ,$$

consequently the number of n's such that (5.1) holds is $(1 + o(1))x/A_k > x^{1/2}$. This completes the proof of Theorem 6.

PROOF OF THEOREM 7. We shall develop first a general method, which gives the conclusion of Theorem 7 with a slightly poorer value $t = [C(\log x)^{1/2}/\log\log x]$ for a large class of arithmetical functions, including $d_k(n)$. Then we shall refine the argument in the case of $a(n)$, where we are dealing with an s-function, and obtain the full assertion of Theorem 7.

By the Chinese remainder theorem the system of congruences

$$
(5.7)\qquad
\begin{aligned}
n+1 &\equiv p_1^2 p_2^2 \cdots p_r^2 \pmod{p_1^3 p_2^3 \cdots p_r^3}\\
n+2 &\equiv p_{r+1}^2 p_{r+2}^2 \cdots p_{3r}^3 \pmod{p_{r+1}^3 p_{r+2}^3 \cdots p_{3r}^3}\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
n+t &\equiv p_{\frac12 t(t-1)r}^2 \cdots p_{\frac12 t(t+1)r}^2 \pmod{p_{\frac12 t(t-1)r}^3 \cdots \\
&\qquad\qquad \cdots p_{\frac12 t(t+1)r}^3)
\end{aligned}
$$

has a unique solution modulo $B = (p_1 p_2 \cdots p_{\frac12 t(t+1)r})^3$, where $r = [C\log\log x]$ with a suitable $C > 0$, which will be specified later. Thus

$$
(5.8)\qquad
\begin{aligned}
n+1 &= m_1 p_1^2 \cdots p_r^2, \qquad (m_1, p_1 \cdots p_r) = 1\\
&\vdots \qquad\qquad \vdots\\
n+t &= m_t p_{\frac12 t(t-1)r}^2 \cdots p_{\frac12 t(t+1)r}^2,\\
&\qquad\qquad (m_t, p_{\frac12 t(t-1)r} \cdots p_{\frac12 t(t+1)r}) = 1,
\end{aligned}
$$

and $n = Bm + A$ for some $0 \le A < B$. Moreover,

$$
B = \exp(3\Theta(p_{\frac12 t(t+1)r})) \le \exp(\tfrac74 t^2 r\log(3t^2)) \le
$$

$$
\le \exp\left(2C_1^2 \frac{\log x}{(\log\log x)^2} C(\log\log x)^2\right) = x^{1/25}
$$

if $t = [C_1(\log x)^{1/2}/\log\log x]$ and $2C_1^2 C = 1/25$. Therefore there are $x/B + O(1) > x^{19/20}$ solutions of (5.7) such that $x \le n \le 2x$. We wish to establish that for sufficiently many of these solutions n each m_j $(j = 1,\ldots,t)$ in (5.8) has relatively few prime factors, since by the classical result of Hardy-Ramanujan $\log\log n$ is the normal order of $\Omega(n)$. By a result of J.-L. Nicolas [27] we have, uniformly for $3 \le k \le \log x/\log 2$,

$$(5.9) \qquad \sum_{n\le x,\Omega(n)\ge k} 1 \ll x2^{-k}\log x.$$

This means that the number of n's from $[x,2x]$ which satisfy (5.7) and (5.8) with $\Omega(m_1) > 100\log\log x$ is $\ll x/(p_1^2\ldots p_r^2\log^{30}x)$, consequently the number of these n for which $\Omega(m_1) \le 100\log\log x$ is

$$\frac{x}{p_1^2\ldots p_r^2}\left(1 + O\left(\frac{1}{\log^{30}x}\right)\right).$$

The n's which satisfy $n + 1 \equiv 0 \pmod{p_1^2\ldots p_r^2}$ and $n + 2 \equiv 0 \pmod{p_{r+1}^2\ldots p_{3r}^2}$ lie in a unique arithmetic progression modulo $p_1^2p_2^2\ldots p_{3r}^2$, hence using again (5.9) it is seen that the number of n's for which both $\Omega(m_1) \le 100\log\log x$ and $\Omega(m_2) \le 100\log\log x$ is

$$\frac{x}{p_1^2\ldots p_{3r}^2}\left(1 + O\left(\frac{2}{\log^{30}x}\right)\right),$$

where the 0-constant is absolute. Continuing this analysis, we finally obtain that the number of n's from $[x, 2x]$ for which (5.7) and (5.8) hold with $\Omega(m_1) \le 100 \log\log x, \ldots, \Omega(m_t) \le 100 \log\log x$ is equal to

$$\frac{x}{B}\left(1 + O\left(\frac{t}{\log^{30} x}\right)\right) \ge \frac{x}{2B} > x^{1/2}$$

Let us define now the function $b(n)$ by the relation $2^{b(n)} \| n$ (i.e., $b(n)$ is the exponent of the highest power of 2 which divides n). We shall prove that

$$(5.10) \qquad b(a(n+1)) < b(a(n+2)) < \ldots < b(a(n+t)),$$

which implies that the values $a(n+1), \ldots, a(n+t)$ are all distinct, hence Theorem 7 is proved.

In view of (5.8) and $a(p^2) = 2$ we have

$$a(n+1) = a(m_1) a(p_1^2 \ldots p_r^2) = a(m_1) 2^r,$$

$$a(n+2) = a(m_2) 2^{2r}, \ldots, a(n+t) = a(m_t) 2^{tr}.$$

But from (4.9) it easily follows that $a(p^\alpha) = P(\alpha) <$ $< \exp(c\alpha^{1/2})$ for $\alpha \ge 1$ and some suitable $c > 0$. Thus if the canonical decomposition of n is $n = q_1^{\alpha_1} \ldots q_k^{\alpha_k}$, then for $n \ge 1$

$$(5.11) \qquad a(n) = P(\alpha_1)\ldots P(\alpha_k) < \exp(c(\alpha_1^{1/2} + \ldots + \alpha_k^{1/2})) <$$

$$< \exp(c\Omega(n)) .$$

Hence $a(m_1) < e^{100c\log\log x} < 2^{r/4}$ if $r = [C\log\log x]$ with with $C > 1000c$, and analogously $a(m_2) < 2^{r/4}, \ldots, a(m_t) < < 2^{r/4}$ for $x \geq x_0$. This implies

$$2^r \leq b(a(n + 1)) \leq 2^{r+r/4} < 2^{2r} \leq b(a(n + 2)) \leq 2^{2r+r/4} <$$

$$< 2^{3r} \leq \ldots < 2^{tr} \leq b(a(n + t)),$$

proving (5.10) and completing the proof of the weaker version of Theorem 7.

Analyzing the above proof it becomes clear that the most important property of $a(n)$ that we used is the bound (5.11). However, this type of bound holds for all positive, integer-valued, multiplicative, prime-independent functions $f(n)$ such that $f(p^\alpha) = g(\alpha)$, $g(\alpha_0) > 1$ for some α_0 and $g(\alpha) < \exp(C\alpha)$ for some $C = C_f > 0$. Repeating the proof of Theorem 7 with obvious modifications we find that there exist at least $x^{1/2}$ numbers from $[x,2x]$ such that all the values $f(n+1), f(n+2), \ldots, \ldots, f(n+t)$ are distinct if $t = [C_1(\log x)^{1/2}/\log\log x]$ with a suitable $C_1 = C_1(f) > 0$. In particular, $d_k(p^\alpha) \ll_k 2^{k\alpha}$, hence this result holds for the function

$d_k(n)$. In the special case $k = 2$ this was already established, by a different method, by P. Erdős and L. Mirsky [5], but for general $d_k(n)$ it appears to be new.

Finally we point out how in the case of $a(n)$ one can obtain that the values $a(n+1),\dots,a(n+t)$ are all distinct for $t = [C(\log x/\log\log x)^{1/2}]$. For this it suffices to note that, analogously to the discussion made in proving Theorem 6, we may take all the m_i's in (5.8) to be squarefree. In view of $a(m_i) = 1$ this immediately implies (5.10), but now we can take $r = r_0$ and not as large as $[C\log\log x]$, which we had before. Thus now $B \le x^{1/25}$ will be satisfied for $t = [C(\log x/\log\log x)^{1/2}]$ and a suitable $C > 0$, and Theorem 7 follows. This improves the previous t by a factor of $(\log\log x)^{1/2}$.

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