The Mathematics of Levi ben Gershon, the Ralbag

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Levi ben Gershon (1288-1344) was a great *talmid hakham*, as well as one of the great secular thinkers of his generation. His rationalist ideas were ahead of their time, and not well received by many in his own Jewish community. We present an overview of his mathematical work with a detailed example of a rare bit of mathematics from his commentary on *Tanakh*. The portion presented is a discussion on the value of π, which hints that Levi understood that π is irrational. The discussion also implies a formula for calculating π from personal body measurements, relating π to the ratio of the length of one's arm to the length of one's hand.

לוי בן גרשון היה תלמיד חכם ואחד מגדולי דורו בחשיבה כללית. הוא היה רציונלי, ולכן הרעיונות שלו היו בלתי מקובלים על ידי הרבה יהודים בדורו. אנחנו מציגים סקירה כללית של עבודתו המתמטית כולל דוגמא מפורטת ונדירה בהקשר לנושא תנ"כי. הדוגמא עוסקת בעניין הערך של π, ומרמזת שלוי הבין כי π מספר אי-רציונלי. הדיון גם מרמז על נוסחא לחישוב π לפי ממדי הגוף האנושי. הערך תלוי ביחס בין האמה לטפח.

Introduction

Levi ben Gershon (1288-1344), rabbi, philosopher, astronomer, scientist, biblical commentator and mathematician, was born in Provence (South France) and lived there all his life. Through his writings, he distinguished himself as one of the great medieval scientists and a major philosopher. He wrote more than a dozen books of commentary on the Old Testament, a major philosophical work *MilHamot Adonai* (Wars of God), a book on logic, four treatises on mathematics, and a variety of other scientific and philosophical commentaries. *MilHamot Adonai* has a long section on astronomy, including: a subsection on trigonometry, original theories on the motion of the moon and planets, a discussion of the *camera obscura,* and the invention of the Jacob's Staff - a device to measure angles between heavenly bodies used for centuries by European sailors for navigation. A complete bibliography on Levi and his work can be found in Kellner[1].

Much of Levi's mathematics and science was ahead of its time. In particular, the material on what we now call discrete mathematics and combinatorics, was unlikely to have been appreciated by his audience. Furthermore, Levi's audience was restricted mainly to Hebrew readers. Although some of Levi's works were translated by Christian scholars into Latin even during his own lifetime, most of his work was available only in Hebrew. Therefore it is not so much of a surprise that Levi's scientific influence was limited.

Levi wrote exclusively in Hebrew, and scholars believe that he did not read Greek, Latin or Arabic. The evidence is from a list of books in his personal library which has survived. All the books are in Hebrew, including Hebrew translations of many major scientific and philosophical works available at the time. The works of Euclid in particular are listed, which he references many times in his own work. The list of books is in Levi's own hand, providing the only known sample of his handwriting. Besides this list, all other extant manuscripts of Levi are copies by later scribes.

Although Levi is well studied today, his influence among Jews in his day was limited. Among Jews in the 14th century, Levi, also known as the *Ralbag*, was a controversial figure because of his rationalist philosophy, which foreshadowed the Renaissance by two centuries. Levi was a deeply committed Jew with a rationalist viewpoint, who did not fit in well in his own era. His philosophical works were unpopular and even shunned by some. For example, his major philosophical work, *MilHamot Adonai* (Wars of God), was dubbed by critics, *MilHamot Im Adonai* (Wars with God). Today, the *Ralbag* is well accepted by the *dati* community. His commentary on *Nakh* is included in most editions of *Mikraot G'dolot,* and editions of his commentary on the *Torah*, as well as his scientific, philosophical and mathematical works are available.

Levi's Life and Times

There is little information about Levi's personal life except that he came from a family of scholars and learned men. In the literature, one can find many references to possible identities of his immediate family, however most of these are not completely established, and some information is speculative or wrong. As an exercise in critical reading, the reader should try to find inconsistencies in the information that follows, complied from a variety of sources. Levi's brother was physician to the Pope in Avignon. Their father was Gershon ben Shlomo of Arles, a Provence scientist famous for his comprehensive book on natural science called *Shaar Hashamayim* (*The Gate of Heaven*). (This book was the standard text on natural science for centuries among Hebrew readers.) Levi's grandfather was Levi ben Abraham, who wrote an eclectic and eccentric book on philosophy, mysticism, ethics, astrology, science and mathematics called *Bottei Hanefesh* (Houses of the Soul), in which the mathematics is presented alternately in verse and in puzzle form[2]. Levi's other grandfather was the Spanish philosopher and Biblical scholar Ramban (Moses ben NaHman, a.k.a. NaHmonides)[3].

The careful reader will note that if Levi's father was really Gershon *ben* Shlomo (Gershon son of Shlomo), then at least one of his grandfathers should have been named Shlomo, and not therefore Abraham or Moses. Hence the information above is not consistent. It is ironic that in a culture where names are identified by a person's father, we know so little for certain about such a great scholar. It is without a doubt, however, that he was surrounded by scholars in the family.

There is little established information regarding Levi's occupation, whom he influenced, who influenced him, or what he was like as an individual. It is known, however, that he corresponded with important Christian scholars, and that he had a school of students or followers[4]. He dedicated his book on trigonometry to Pope Clement VI, and he was commissioned by Phillip of Vitry, Bishop of Meaux, to write his book *On Harmonic Numbers*, which was immediately translated from Hebrew to Latin. He was referred to as Leo Hebraus or Maestro Leon by his Christian colleagues, *RaLBaG* (the capital letters stand for Rabbi Levi Ben Gershon) by Jewish Biblical scholars, and Gersonides by modern scholars. Gersonides (pronounced Ger-Son-i-Dees) for Levi ben Gershon is like Maimonides for the *RaMBaM*, Moses ben Maimon. It is a common *Grecification* of Jewish names.

Life for Jews in most of 14th century France was very difficult. France (and most of Europe) at the time was a collection of semi-independent regions (dukedoms, city states, Papal states) which were often at war with each other. The King of France had loose control over all the provinces but local control varied. The 14th century was a period of economic hardship, plagues and famines. In most of these regions, Jews were blamed for everything from the bad economy to the Black Plague. Jews were expelled, libeled, tortured and killed. It was only in Provence that life was relatively good. In Provence, the Pope was self-exiled in a large city called Avignon, where he protected the Jews and allowed them to live with little or no oppression. There were 15,000 Jews in 14th century Provence out of a total population of approximately two million. Jews worked as money lenders, physicians, craftsmen and merchants. Agriculture and farming was not a political possibility for them due to legal restrictions on who could control land.

Highlights of Levi's Mathematical and Scientific Work

We now outline Levi's mathematical and scientific contributions, concentrating on his major mathematical work *Maaseh Hoshev,* and concluding with a rare and interesting example of his mathematics from his commentary on the *Tanakh*.

This list exhibits the diversity and scope of Levi's work, while providing an overview of his mathematical and scientific accomplishments.

1. The earliest rigorous use of mathematical induction.
2. Pioneer work in the field of combinatorics.
3. A commentary on Euclid including an attempted *proof* of the 5th postulate.
4. A treatise on trigonometry.
5. An elementary proof that the only pairs of harmonic numbers (numbers in the form $2^n 3^m$) that differ by 1 are the pairs (1, 2), (2, 3), (3, 4) and (8, 9).
6. Important astronomical observations and theories about the motions of the moon, earth and sun.
7. Invention of the Jacob's Staff, a device to measure angles between heavenly bodies, which was used for centuries by European sailors for navigation. Figure 1 depicts the use of the device, without showing the detailed and complex measurement markings that were etched upon it.
8. Description of the principles of the *camera obscura* (dark chamber), the forerunner of our modern camera.

Figure 1. The Jacob's Staff, as illustrated in *The Mariner's New Calendar* by Nathaniel Colson (London 1953), page 64.

Levi's Mathematics

Levi's mathematics comprises four major works.

1. *Maaseh Hoshev*, Levi's first and largest mathematics book is extant in two editions, the first completed in 1321, and the second in 1322.

2. A commentary on Euclid was completed in the early 1320's.

3. *De Sinibus, Chordis et Arcubus*, a treatise on trigonometry in *MilHamot Adonai*, was completed in 1342 and dedicated to the Pope.

4. *De Numeris Harmonicis,* completed in 1343, was commissioned by Phillip de Vitry, Bishop of Meaux.

Maaseh Hoshev

Maaseh Hoshev (The Art of Calculation), 1321, is Levi's first book, and a major mathematical work which deserves a detailed discussion. It is best known for its early illustrations of proofs by mathematical induction[5].

The name comes from *Shemot*, the Biblical book of Exodus, where the phrase is used in a number of different places to describe the type of work necessary to construct the *mishkan* or tabernacle. Although it literally means "A Work of Calculation", its true meaning is more subtle.

Exodus 26:1
Moreover thou shalt make the tabernacle with ten curtains of fine twisted linen, and blue and purple and scarlet: with cherubims of **cunning work** *shalt thou make them.*[6]

Exodus 39:2-3
And he made the ephod of gold, blue, purple, and scarlet, and fine twisted linen. And they did beat the gold into thin plates, and cut it into wires, to work it in the blue, and in the purple, and in the scarlet, and in the fine linen, with **cunning work**.[7]

This *cunning work* is in contrast with other types of work in the construction, which are referred to by the skill required, such as *maaseh rokem* (the work of an embroiderer), *maaseh oreg* (the work of a weaver), *maaseh harash* (the work of an engraver), and *maaseh avot* (the work of a braider). The intention is that *maaseh hoshev* denotes a skill that requires more than just technical craftsmanship. It is the skill of an architect: requiring thought, cunning, planning and calculation. Perhaps this is why Levi chose it for the title of his book.

The title is also play on words for theory and practice, *Maaseh* corresponding to practice and *Hoshev* corresponding to theory. Levi writes in his introduction to the book:

"It is only with great difficulty that one can master the art of calculation, without knowledge of the underlying theory. However, with the knowledge of the underlying theory, mastery is easy... and since this book deals with the practice and the theory, we call it Maaseh Hoshev".

Maaseh Hoshev is a major work in two parts. The first part is a collection of 68 theorems and proofs in Euclidean style about arithmetic, algebra and combinatorics. (Levi lists Euclid books 7-9, as prerequisite reading). The second part, about 10% longer than the first, contains algorithms for calculation and is subdivided into six sections:

a. Addition and Subtraction
b. Multiplication
c. Sums
d. Combinatorics
e. Division, Square Roots, Cube Roots.
f. Ratios and Proportions.

Section f is followed by a long appendix of problems which itself comprises about a third of part two, and a fifth of the entire book. A critical edition of *Maaseh Hoshev* with a translation to German was completed by Lange[8] in 1909, but it is missing the section of problems completely. A critical edition of the problem section can be found in my work[9]. Other papers on *Maaseh Hoshev* include discussions of its content and its use of mathematical induction[10].

The Problem Section of *Maaseh Hoshev*

My own work has focussed on the neglected problem section, which until now had remained unpublished, and which due to scribal errors and a few difficult sections, had probably not been read completely by anyone in hundreds of years. The problems Levi presents are interesting not only for their mathematical content, but also for the light they reflect on the culture and lifestyle of Levi's home. He often refers to particular wages, prices, currency and commerce in his problems, confirming economic details of his life and times, not otherwise easily confirmed.

The unifying theme in the problems is proportions. Levi uses this theme to solve a variety of puzzles and problems, most of which can be reduced to simultaneous linear equations. A simple example of one of these problems is the famous recreational barrel problem, where one is given a barrel with a few holes, each of which can empty the barrel alone in a given amount of time. The question is how long does it take to empty the barrel if all the holes are opened at once?

In the course of my research, I discovered that there were actually two editions of *Maaseh Hoshev,* a fact previously unknown. One edition was completed in Nissan 1321 and the other in Elul 1322. Levi was known to have published successive editions of his other scientific work, so this discovery is exciting but not surprising. There are some major differences between the two editions, showing that Levi saw fit to edit and republish some of his work, providing evidence that he had an avid audience.[11]

Today there are twelve remaining manuscripts of *Maaseh Hoshev*, nine of edition one, and three of edition two. They are located all over the world: New York, Moscow (2), Parma (2), Munich (2), Vienna, London, Paris, Jerusalem, Vatican. Most are dated from the 15th century and were copied in either Italy or France, after Levi died. There may still be more copies buried in some old dusty basement or obscure library; but more likely, any other copies, if there were any, have been lost or destroyed over the years. In fact, of the twelve manuscripts that remain, most are incomplete. It is common for the first or last few pages to just fall off. Worm holes can leave a paragraph unreadable. The pages can just turn yellow or brown and deteriorate. Handwriting, script styles, margin comments, watermarks, paper quality and ink quality are all clues in determining the provenance of a manuscript and its history, but the clearest indication of the date of publication is the colophon.

Although a manuscript generally had no table of contents or index, it often had a colophon. This is a short paragraph at the very end of the manuscript, identifying the author and the date, and providing a brief conclusion to the work. Unfortunately, it is exactly these last pages that tend to fall off and get lost. Nevertheless, there are a number of manuscripts both from edition one and edition two that bear colophons. One manuscript of *Maaseh Hoshev* from the Ginsburg collection in Moscow, bears a colophon that reads:

נשלם השער השישי מזה המאמר; ובהשלמו, נשלם זה הספר. והתהלה לאל לבדו. והיתה השלמתו בחודש אלול של שנת שמונים ושתים לפרט האלף השישי. וברוך העוזר

The sixth section of this part is now complete, and with its completion, the book is finished. All praise goes to God. Its completion was in the month of Elul of the 82nd year of the sixth millenium.

This colophon represents the second edition and provides the date of publication as Elul 1322. It is a bit hard to read because it is written in Provencal style script. The first edition colophon (not shown) includes a reference to Levi's age (33) at the time of publication (1321), giving us documentation of his year of birth (1288). Worth noting is that the colophon is the only place in the book where Levi mentions his faith in God. Otherwise he does not mix his mathematics and religion.

Figure 2: The Colophon from the Moscow Manuscript of *Maaseh Hoshev*

Levi's Commentary on Euclid

Levi's commentary on Euclid, like all his work, was written in Hebrew. He had access to Hebrew translations of Euclid which were readily available in that period.[12] In his commentary, Levi defines an infinite straight line (and plane) and distinguishes between natural bodies which are necessarily finite, and mathematical entities (numbers, lines etc.) which can be infinite.

The most interesting part of his commentary is an attempt to prove the fifth postulate of Euclid, the *parallel* postulate. Attempts to prove this postulate were common and widespread for 2000 years, until the 19th century when it was proved that the fifth postulate is independent of the other four postulates, and hence could not be proved or disproved using just the first four postulates. Levi's proof is of course flawed, yet quite ingenious just the same[13].

De Sinibus, Chordis et Arcubus

This work represents a section of the 136 astronomical chapters from Levi's great work *MilHamot Adonai*. It was translated into Latin and circulated separately under the title *De Sinibus, Chordis et Arcubus, item Instrumento Revelatore Secretorum*. The content of the work is an introduction to trigonometry and a description of the Jacob's staff as an astronomical instrument.

Levi introduces the sine and *versine* (cosine) functions but does not mention the tangent function. He describes a method for deriving a sine table on intervals of 15′. His methods are based on the fundamental trigonometric identities and theorems taught today in high school. These include formulae for the sine and cosine of the sum of two angles, the difference of two angles, half an angle, complement of an angle, and the related Pythagorean identities. Of course the details of his book would look strange to a modern reader, because he (as everyone else in that period) refers frequently to *chords* which are straight lines subtending an angle, and not just to the angle itself.

It is notable that the rigor and preciseness of *Maaseh Hoshev* are also present in the theorems, proofs, table compilation and calculations of *De Sinibus, Chordis et Arcubus*. Further details on this work can be found in a number of places.[14]

De Numeris Harmonicis

This book was written in 1343, a year before Levi died, at the request of Philip de Vitry, Bishop of Meaux. Presumably, Levi's reputation among Christian scholars was by this time well established even in northern sections of France.

The work is relatively short and Philip immediately had it translated from Hebrew into Latin. The original Hebrew is now lost.

Philip was a musicologist interested in numbers of the form $2^n 3^m$ called harmonic numbers. In particular, he was interested in pairs of these numbers. He noticed that the musical interval of an octave represents a 1:2 ratio in frequencies, while a *fifth* interval represents a 2:3 ratio, a *fourth* interval represents a 3:4 ratio, and a whole note interval represents a 9:8 ratio. He asked Levi to prove (or disprove) that these four pairs of harmonic numbers are the only pairs where the two numbers differ exactly by one.

Levi's proof is elegant, rigorous but longwinded. It is worth going into details because the main idea is both comprehensible and compact. It provides a glimpse into Levi's genius. Here is a brief sketch[15]:

Levi first proves that if both of the numbers have a factor of two then the difference between them must be at least two. Similarly if both have a factor of three then their difference must be at least three.

Therefore, he only considers pairs of numbers where one is a power of two and the other is a power of three.

He then proves that when a power of three is divided by eight, it leaves a remainder of one if and only if the power is even, and a remainder of three if and only if the power is odd. He notes that when a power of two is divided by eight, it leaves a remainder of one, two and four for those first three powers, and zero for eight and higher. You will see where he is headed with this in the next paragraph.

Levi then analyzes all possible cases by considering whether or not the power of three is one less or greater than the power of two.

a. If the power of three is one less than the power of two, this can be only one of two cases: (1,2) and (3,4). This is because all powers of two greater than four, have a remainder of zero when dividing by eight and there are no powers of three with a remainder of seven.
b. If the power of three is one greater than the power of two, and if the power of three leaves a remainder of three after division by eight, then the only case is (2,3). This is because two is the only power of two that has remainder two after dividing by eight.
c. If the power of three is one greater than the power of two, and leaves a remainder of one after division by eight, then at first glance it seems that there are an infinite number of possible cases, namely all powers of two greater than or equal to eight. Levi reduces the infinite to one. He reminds us that if a power of three leaves a remainder of one after division by eight, then the power must be even. Since we assumed that the power of three is one larger than the power of two, we get $3^{2k} - 1 = (3^k - 1)(3^k + 1)$ must be a power of two. This means that a power of two must equal the product of two powers of two that differ by two. But the only two powers of two that differ by two, are two and four. This gives $3^{2k} - 1 = 8$, and the last possible pair is therefore (8,9).

The sketch above omits all of Levi's rigorous details and proofs, and is not completely faithful to Levi's argument, which of course used no modern algebraic equations or notation. However, it does present the main points in the book and conveys to the reader a sense of Levi's ingenuity.

Levi's Mathematics in his Commentary on the *Tanakh*

It is interesting that Levi's Biblical commentary follows a pattern similar to the structure of *Maaseh Hoshev.* In *Maaseh Hoshev,* he begins with theory and ends with practical applications. In the *Tanakh* commentary, he starts with interpretation and ends with *Toaliyot,* practical lessons for life based on his interpretations of the text. His mathematics rarely contains spiritual discussions, and his Biblical commentary does not often contain mathematics, but there is at least one notable exception.

Levi on the *Yam shel Shlomo*

There is a well known passage in the Bible that seems to imply that the value of π, the ratio of the circumference of a circle to its diameter, is precisely three. The passage describes a *sea* in which the priests would wash themselves. This *sea* is more like a great bathtub inside the Temple in which the *kohanim* could purify themselves before performing ritual functions.

Kings I - 7:23
And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and its height was five cubits: and a line of thirty cubits did compass it round about.[16]

This certainly seems to imply that π is simply 30/10 = 3. Yet the ancient Egyptians and Babylonians knew better than this. Their values for π were $4(8/9)^2 = 3.16049...$ and $3 + 1/8 = 3.125$, respectively[17]. The Rabbis writing in the Talmud (300-600 C.E.), in their discussion of this passage, noted that this Biblical value of 3 is "*by way of approximation*".

Levi ben Gershon the faithful rationalist and biblical scholar, inspired by the Talmudic discussion in Tractate Eruvin (Daf 14), writes:

> And where it is said that a line of thirty cubits surround it about, is by way of approximation. Because the circumference of a circle adds about 1/7 to three measures of the diameter, a close approximation[18].

Levi asserts that the original biblical quote is taken out of context, and that the whole matter can be explained quite logically. He points out that the sea had a thick wall, and the diameter was measured from the edges of the outside of the walls, while the circumference was measured around the inner walls. Indeed if we look just three verses ahead, we find:

Kings I - 7:26
And it was an hand breadth thick, and the brim thereof was wrought like the brim of a cup with flowers of lilies: it contained two thousand baths.[19]

Levi comments:

> And if we say that the measurement around was taken on the inside of the basin, then that would be closer to the truth, but still just a close approximation. Since the width is an hand breadth , the diameter of the inner edge of the basin is ten cubits less one third of a cubit. And its circumference will be approximately one and one third cubits more than thirty cubits.[20]

A Cubit is Six Handbreadths

The Talmud sets the ratio of cubit to hand breadth as 1:6. A cubit is the distance from the elbow to the end of the middle finger, and a hand breadth is the distance across the widest part of the hand. Ostensibly this ratio depends on each particular individual. However, for purposes of legal measurements, the Talmud after a brief debate, establishes the *halakhik* ratio of cubit to hand breadth, as 1:6. A careful look at Levi's argument shows, not surprisingly, that he too uses this ratio. Using this 1:6 ratio, Levi derives that π is equal to $30/(10 - 1/3) = 90/29$ which is approximately 3.1034, see Figure 3. This is indeed in his own words "*closer to the truth but still just a close approximation*". In general, Levi's idea yields that π equals $30C/(10C - 2H)$, where C is the length of a cubit and H is the length of a hand breadth.

Figure 3. Schematic of the Sea Looking Down from the Top.

If we do not use the *halakhik* ratio of 1:6, but instead measure one's own hand breadth and arm length, we can calculate a personal approximation for π based on Levi's interpretation and one's own body measurements. For example, my own cubit is 45 cm and my hand breadth is 9.2 cm. This implies that my personal approximation for π is $30(45)/(10*45 - 2*9.2) = 1350/431.6$ which is about 3.128. When I let my students try this, some like to relate the accuracy of their approximation for π, to the perfection of their body dimensions, but clearly Levi intended to use the 1:6 ratio and would not likely approve of this *irrational* amusement.

Did Levi Know that π is Irrational?

Whether or not Levi suspected that π is an irrational number, a number that cannot be written as a fraction a/b, makes for an interesting discussion. Levi would never make a mathematical claim without a rigorous proof to back it up, and he certainly does not make any explicit claim that π is an irrational number. This is not surprising considering that the first proof that π is an irrational number was by Lambert in 1767, more than 400 years after Levi. On the other hand, Levi was certainly aware of Aristotle's proof that $\sqrt{2}$ is irrational, and Levi's repeated use of the phrase *"a close approximation", qiruv me'at,* makes one suspect that he believed π is also irrational but that he had no proof. Any intelligent argument or speculation on this point needs to be based on what we know historically about mathematical knowledge in Levi's time, and on his particular use of language. There is of course no definitive answer.

The Volume of the *Yam*

Levi is not done yet with this section. He goes on in the next paragraph of his commentary to discuss the shape of the basin given that it held "two thousand baths". He relies on his Talmudic knowledge of the ratios of biblical units, the Talmudic discussion in Eruvin 14, and on the Biblical text.

Kings I   7:25-26
It stood upon 12 oxen, three looking toward the north, and three looking toward the west, and three looking toward the south, and three looking toward the east: and the sea was set above upon them, and all their hinder parts were inward. And it was an hand breadth thick, and the brim thereof was wrought like the brim of a cup with flowers of lilies: it contained two thousand baths.

Levi comments:

And the fountain was circular only on the brim, but below it was square shaped. And this is clear because of the volume it held, two thousand baths. A bath is three se'ah, as it is written: the eifah and the bath hold the same volume. And since the Sages have taught that one by one by three cubits contains forty se'ah, so we have that two thousand baths which is six thousand se'ah, holds four hundred and fifty times a one by one by one cubic cubit... And the number of cubic cubits in the top cubit is approximately seventy four. But if this continued for the whole five cubits, there would be approximately only three hundred and seventy, which is about eighty too small compared to what we mentioned earlier. And so it is necessary to have some of the cubits be square... And the Sages have said that the bottom three cubits of the sea were square, and the top two were circular. And this confirms our own calculations approximately. Furthermore, it is more consistent with the description of the oxen on which the sea is resting, where there are three to each side.

In this last paragraph, I translated only enough to make the idea clear. Had I translated it completely, it would prove to the reader what Goldstein notes in his translation of Levi's Astronomy[21], that although "*his content sparkles with originality, Levi writes in a ponderous Hebrew style*".

To summarize, Levi calculates that the volume of 2000 *baths* makes sense only if the top two cubits are circular, and the bottom three are square shaped. He supports this view mathematically, by converting the units and calculating volumes; and aesthetically, by noting that the description of the oxen on each side makes more sense if you really have *sides*.

He notes that 2000 baths is the same volume as 450 cubic cubits. Since there are six hand breadths in a cubit, the diameter of the circle at the top of the sea is $10 - 1/3$, hence the volume of this top cubit is $\pi(10 - 1/3)/2)^2$, which equals approximately 74. Five cubits of this cross section would give 370 which is 80 short of what we need. Hence Levi agrees with the sages that only the top two cubits had a circular cross section, but the bottom three had a square cross section.

In fact, the volume of the top cubit $\pi(10 - 1/3)/2)^2$ equals more like 73.4. To get 74 Levi had to use a value of π equal to approximately 3.1676. This estimate is a bit high, while the estimate he derived before was a bit low. It seems he was just approximating in order to make the number of missing cubic cubits a nice round 80. Note that his calculations here repeatedly make use of the word *approximately*, implying once more that he knows π does not have a precise fractional value.

How close is the volume of the sea to 2000 baths, assuming Levi's shape is correct? The volume in cubic cubits would be $2\pi(10 - 1/3)/2)^2$ for the two cubits with a circular cross section, plus $3(10 - 1/3)^2$ for the three cubits with a square cross section, totaling approximately 427.1 cubic cubits. Since three cubic cubits are 40 *se'ah,* and three *se'ah* is one *bath,* the volume is approximately 1898 *baths.* If Levi hoped to get closer to the 2000 *baths,* he would need to have the circular cross section be only about .86 cubits high, and the square cross section 4.14 cubits high. Presumably he was well aware of this, but could not find any *Hazal* to back him up, hence he went with the best he could find, namely two circular and three square, and warns us that it only *confirms our calculations approximately.*

There are other opinions of how to calculate the volume of the sea. The variables include: the number of handbreadths in a cubit (5 or 6), the value used for π, and the interpretation of the phrase "And it was an handbreadth thick." For example, if ONLY the walls of the circular cross section were a handbreadth thick, then we must subtract off the 1/3 only in the circular cross section but NOT in the square cross section, and the volume is approximately $300 + 146.8 = 446.8$, a great deal closer to the correct number[22].

Note that the picture in Figure 4 does not bear a good likeness to the *yam* as Levi describes it, because there is no indication of a square cross section anywhere.

Figure 4. Artist's Rendering of the Sea, as shown in *History of π* by Petr Beckmann, (New York, 1971), page 16.

Conclusion

Ralbag was a genius and *talmid hakham* with a broad range of interests. His influence was not as great as it might have been, due to the complexity of his scholarship, his longwinded writing style and his rationalist philosophy. His rationalist views made him controversial in his own time, but anticipated some of the important philosophical dilemmas of the observant Jew in the 20th century. He did not usually mix his religious and scientific writing, but did not hesitate to do so when the opportunity presented itself. His *emunah,* coexistent with his analytic rationality provides a modern model for the observant Jew.

Notes

[1] Kellner, Menachem, *An Annotated List of Writings by and about R. Levi ben Gershom,* in Gad Freudenthal (ed.), **Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist,** Leiden: Brill, 367-414 (1992).

[2] Davidson, Israel, *Levi ben Abraham ben Hayim, A Mathematician of the XIIIth century,* **Scripta Mathematica 6,** 57-65 (1936).

[3] Most of these claims are unsubstantiated. See Encyclopedia Judaica, Volume 11, Macmillan and Company. 91-94 (1971). The Ramban (NaHmonides) claim appears in: Sarton, George, **Introduction to the History of Science,** Vol. III, pt.1, Washington: Carnegie Institute, 594-607 (1947).

[4] Glasner makes a good case for this in: Glasner, Ruth, Levi ben Gershom and the Study of Ibn Rushd in the Fourteenth Century, *The Jewish Quarterly Review***, LXXXVI,** Nos. 1-2, (1995), 51-90.

[5] The early use of induction was first pointed out by Rabinovitch in: Rabinovitch, Nahum L., *Rabbi Levi ben Gershon and the Origins of Mathematical Induction,* **Archive for History of Exact Sciences,** 6, 237-248 (1970).

[6] א וְאֶת־הַמִּשְׁכָּן תַּעֲשֶׂה עֶשֶׂר יְרִיעֹת שֵׁשׁ מָשְׁזָר וּתְכֵלֶת וְאַרְגָּמָן וְתֹלַעַת שָׁנִי כְּרֻבִים מַעֲשֵׂה חֹשֵׁב תַּעֲשֶׂה אֹתָם׃

[7] ב וַיַּעַשׂ אֶת־הָאֵפֹד זָהָב תְּכֵלֶת וְאַרְגָּמָן וְתוֹלַעַת שָׁנִי וְשֵׁשׁ מָשְׁזָר׃ ג וַיְרַקְּעוּ אֶת־פַּחֵי הַזָּהָב וְקִצֵּץ פְּתִילִם לַעֲשׂוֹת בְּתוֹךְ הַתְּכֵלֶת וּבְתוֹךְ הָאַרְגָּמָן וּבְתוֹךְ תּוֹלַעַת הַשָּׁנִי וּבְתוֹךְ הַשֵּׁשׁ מַעֲשֵׂה חֹשֵׁב׃

[8] Lange, Gerson, **Sefer Maassei Chosheb-Die Praxis des Rechners, Ein hebraeisch arithmetisches Werk des Levi ben Gerschom aus dem Jahre 1321**, Franfurt am Main: Louis Golde, (1909).

[9] The critical edition will appear as two papers: *The Missing Problems of Gersonides, A Critical Edition, Part I and The Missing Problems of Gersonides, A Critical Edition, Part II*, in consecutive issues of Historia Mathematica (2000).

[10] A very comprehensive paper in French is: Chemla, Karine and Pahaut, Serge, *Remarques sur les Ouvrages Mathematiques de Gersonide,* in Gad Freudenthal (ed.), **Studies on Gersonides-A 14th Century Jewish Philosopher-Scientist,** Leiden: Brill, 149-191 (1992). Victor Katz discusses Levi in: Katz, Victor J., *Combinatorics and Induction in Medieval Hebrew and Islamic Mathematics,* in Ron Calinger (editor), **Vita Mathematica: Historical Research and Integration with Teaching,** Mathematical Association of America, 99-107 (1996), and in his text on the history of Mathematics: Katz, Victor J., **A History of Mathematics: An Introduction**, Harper Collins, (1993). A joint paper of mine with Tzvi Langermann concentrating on algebra, square roots and cube roots will appear in *Mathematics Across Cultures: A History of Western Mathematics*, editor Helaine Selin, Kluwer Publishing (2000).

[11] It appears that students of Levi may have critiqued the first edition to help produce the second edition. The interested reader can find the complete text with translation and detailed commentary in my work to appear in Historia Mathematica, see note 9.

[12] Indeed a Hebrew translation of Euclid is on the list of his personal library books. The list is the only extant sample of Levi's own handwriting. The remainder of his extant works are copied by students, scribes or translators. The list can be found in the Microfilmed Manuscripts of the Jewish National Library at Hebrew University on Jerusalem.

[13] There is not enough room here to do justice to the proof. For details please see: Levy, Tony, *Gersonide, le Pseudo-Tusi et le Postulat des Paralleles",* **Arabic Science and Philosophy 2,** 39-82 (1992).

[14] A readable summary and partial translation can be found in: Espenshade, P. H., *A text on trigonometry by Levi ben Gerson (1288-1344)*, **Mathematics Teacher 60**, 628-637 (1967). Goldstein has published many articles on the Astronomy of Levi as well as a critical edition, see note 21.

[15] A more faithful and detailed description can be found in the Chemla and Pahaut article, see note 10. Translations of sections of *De Numeris Harmonicis* and *Maaseh Hoshev* can be found in: Carlebach, Jospeh, **Lewi ben Gerson als Mathematiker,** Berlin: J. Lamm (1910).

[16] כג וַיַּעַשׂ אֶת־הַיָּם מוּצָק עֶשֶׂר בָּאַמָּה מִשְּׂפָתוֹ עַד־שְׂפָתוֹ עָגֹל | סָבִיב וְחָמֵשׁ בָּאַמָּה קוֹמָתוֹ וְקָו [וְקָוֹ] שְׁלֹשִׁים בָּאַמָּה יָסֹב אֹתוֹ סָבִיב׃

[17] The history of π is told in: Beckmann, Petr, **History of π,** St. Martin's Press, New York (1971).

[18] והנה אמרו שקו שלשים באמה יסוב אותו סביב הוא על דרך קירוב כי ההיקף הוא מוסיף על שלשה בשעור בקוטר כמו שביעית הקוטר בקירוב מעט.

[19] כו וְעָבְיוֹ טֶפַח וּשְׂפָתוֹ כְּמַעֲשֵׂה שְׂפַת־כּוֹס פֶּרַח שׁוֹשָׁן אַלְפַּיִם בַּת יָכִיל׃

[20] ואם אמרנו שמדת הקו הסובב היא לקוחה בחלל הים, הנה יהיה יותר קרוב אל האמת אך יהיה בו קירוב מעט, כי עביו היה טפח. ולזה יהיה קטר חללו עשר אמות פחות שלישית אמה, ויהיה הקיפו בקירוב יותר על שלשים אמה, אמה ושלישית אמה.

[21] Goldstein, B. R., **The Astronomy of Levi ben Gerson,(1288-1344): A Critical Edition of Chapters 1-20 with Translation and Commentary***, New York: Springer-Verlag, (1985).

[22] A detailed alternative is discussed in *HaTashbetz, V'Hayam Sh'Osah Shlomo*, by D. Garber and B. Tsaban in **Bar-Ilan Parashat HaShavua**, 1999.