

ON GENERALIZATIONS OF GOWERS NORMS.

by

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Abstract

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Inspired by the definition of Gowers norms we study integrals of products of multivariate functions. The L_p norms, certain trace norms, and the Gowers norms are all defined by taking the proper root of one of these integrals. These integrals are important from a combinatorial point of view as inequalities between them are useful in understanding the relation between various subgraph densities. Lovász asked the following questions: (1) Which integrals correspond to norm functions? (2) What are the common properties of the corresponding normed spaces? We address these two questions.

We show that such a formula is a norm if and only if it satisfies a Hölder type inequality. This condition turns out to be very useful: First we apply it to prove various necessary conditions on the structure of the integrals which correspond to norm functions. We also apply the condition to an important conjecture of Erdős, Simonovits, and Sidorenko. Roughly speaking, the conjecture says that among all graphs with the same edge density, random graphs contain the least number of copies of every bipartite graph. This had been verified previously for trees, the 3-dimensional cube, and a few other families of bipartite graphs. The special case of the conjecture for paths, one of the simplest families of bipartite graphs, is equivalent to the Blakley-Roy theorem in linear algebra. Our results verify the conjecture for certain graphs including all hypercubes, one of the important classes of bipartite graphs, and thus generalize a result of Erdős and Simonovits. In fact, for hypercubes we can prove statements that are surprisingly

stronger than the assertion of the conjecture.

To address the second question of Lovász we study these normed spaces from a geometric point of view, and determine their moduli of smoothness and convexity. These two parameters are among the most important invariants in Banach space theory. Our result in particular determines the moduli of smoothness and convexity of Gowers norms. In some cases we are able to prove the Hanner inequality, one of the strongest inequalities related to the concept of smoothness and convexity. We also prove a complex interpolation theorem for these normed spaces, and use this and the Hanner inequality to obtain various optimum results in terms of the constants involved in the definition of moduli of smoothness and convexity.

Dedication

Dedicated to Mahya

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Chapter 1

Overview

For a graph G , its vertex set and edge set will be denoted by $V(G)$ and $E(G)$, respectively. Let H and G be graphs. A **homomorphism** from H to G is a mapping $h : V(H) \rightarrow V(G)$ such that for each edge $\{u, v\}$ of H , $\{h(u), h(v)\}$ is an edge of G . Let $t_H(G)$ be the probability that a random mapping from $V(H)$ to $V(G)$ is a homomorphism. In this thesis we will always think of H as a fixed graph, and of $t_H(\cdot)$ as a function from the set of graphs to the interval $[0, 1]$. As we shall see in Section 2.2, for dense graphs G , $t_H(G)$ is closely related to the density of H in G . In fact, in this chapter the reader is advised to think of $t_H(G)$ as roughly being the density of H in G . The connection to subgraph densities makes understanding the relations between the functions $t_H(\cdot)$ for different graphs H one of the main objectives of extremal graph theory.

If A is the adjacency matrix of the graph G , then it follows immediately from the definition of $t_H(G)$ that

$$t_H(G) = \mathbb{E} \left(\prod_{\{u,v\} \in E(H)} A(x_u, x_v) \right), \quad (1.1)$$

where $\{x_u\}_{u \in V(H)}$ are independent random variables taking values in $V(G)$ uniformly at random. The expression in the right hand side of (1.1) is quite common. Such averages appear as **Mayer** integrals in classical statistical mechanics, **Feynman** integrals in quantum field theory [56] and **multicenter** integrals in quantum chemistry [9].

In recent years, several research programs have been emerging in the direction of developing an efficient framework for studying the relations between t_H functions. Reflection positivity characterizations [21, 57], the language of graph limits [41, 2], and Razborov’s flag algebras [44] are some remarkable examples.

It turns out that for C_4 , the cycle of size 4, the function $t_{C_4}(G)$ carries interesting information about G . Denote by K_n , the complete n -vertex graph. If $t_{C_4}(G)^{1/4}$ is close to $t_{K_2}(G)$, then G “looks random” in certain aspects [7]. Such graphs are usually referred to as **pseudo-random** graphs. These observations belong to the same circle of ideas employed by Szemerédi [58, 59] to prove his famous theorem on arithmetic progressions. In fact, the main idea in the proof of Szemerédi’s theorem led to the establishment of Szemerédi’s regularity lemma in [48], which roughly speaking, says that every graph can be decomposed into a small number of subgraphs such that most of them are pseudo-random. Another interesting fact about t_{C_4} is that it can be used to define the 4-trace norm of a matrix. More generally, for an integer $k > 0$, the $2k$ -trace norm of a matrix can be defined through the function $t_{C_{2k}}$. This fact, which has been known for at least 50 years since Wigner’s work on random matrices [63], gives a combinatorial interpretation of the $2k$ -trace norm with many applications in graph theory (see for example [22, 34]). Inspired by the fact that the cycles of even length correspond to norms, and the numerous applications of these norms in graph theory, László Lovász posed the problem of characterizing all graphs that correspond to norms.

Recently Gowers [26, 27] defined hypergraph generalizations of the C_4 norm. The (k th) **Gowers norm** corresponds to k -uniform hypergraphs. These norms are sometimes referred to as **octahedral norms** as they are defined through the densities of the face hypergraphs of higher dimensional generalizations of octahedra. The first Gowers norm is the usual L_2 norm, and since graphs are 2-uniform hypergraphs, the second Gowers norm coincides with the C_4 norm. Subsequently Gowers [28] used these norms to establish a hypergraph regularity lemma, and a so-called counting lemma which roughly speaking

says that a sufficiently pseudo-random graph contains small subhypergraphs with frequencies that are the same as in a random hypergraph. This counting lemma easily implies Szemerédi's theorem [20] in its full generality, and even stronger theorems such as Furstenberg-Katznelson's multi-dimensional arithmetic progression theorem [55] (see also [28]). The only previously known proof of the latter was through ergodic theory [25]. In fact, the arithmetic version of the Gowers norms has interesting interpretations in ergodic theory, and has been studied from that aspect [35]. The discovery of the Gowers norms led to a better understanding of the concept of pseudo-randomness, and provided strong tools. For example, this norm plays an essential role in Green and Tao's proof [32] that the primes contain arbitrarily long arithmetic progressions. Also, the current best bounds for the quantified version of Szemerédi's theorem is through the so called "inverse theorems" for these norms [31, 29, 30, 27]. With all the known applications for the Gowers norms, it seems natural to believe that studying the norms defined through $t_H(\cdot)$ for other graphs and hypergraphs might also lead to some interesting applications.

The purpose of this thesis, is to develop a common framework to study the norms that are defined in a similar fashion. Our setting will be sufficiently general to include all the L_p norms, arguably the most important class of norms. In fact, we establish that the class of norms studied in this thesis, which we refer to as **hypergraph norms**, are a natural generalization of the L_p norms, and we prove that they share many of the nice properties of the L_p norms.

Among the key tools developed in this thesis is a Hölder type inequality. This inequality is extremely useful in the sequel and shall be applied frequently. One can think of it as a common generalization of the classical Hölder inequality and the Gowers-Cauchy-Schwarz inequality.

A rather surprising application of the above mentioned Hölder type inequality is to a conjecture of Erdős, Simonovits and Sidorenko. Razborov used his flag algebras to prove

a previously conjectured explicit formula for the function

$$f(p) = \inf\{t_{K_3}(G) : t_{K_2}(G) \geq p\},$$

which gives a tight asymptotic lower-bound for the density of triangles in G , in terms of the edge density of G . What about graphs other than K_3 ? In other words given a graph H , what is the explicit formula for $\inf\{t_H(G) : t_{K_2}(G) \geq p\}$? Despite all the powerful machinery that has been developed in recent years [46, 45], finding a complete solution to this question in its full generality seems to be far beyond reach at the moment. But when H is a bipartite graph it seems that there is more hope. The Erdős-Simonovits-Sidorenko conjecture says that for every bipartite graph H ,

$$t_H(G) \geq t_{K_2}(G)^{|E(H)|}.$$

For the random graph $G(n, p)$, with high probability

$$t_H(G(n, p)) = p^{|E(H)|} \pm o(1). \tag{1.2}$$

So the Erdős-Simonovits-Sidorenko conjecture is equivalent to the statement that for every bipartite graph H ,

$$\inf\{t_H(G) : t_{K_2}(G) \geq p\} = p^{|E(H)|},$$

where the infimum is taken over all finite graphs with $t_{K_2}(G) \geq p$. Thus, roughly speaking, the conjecture says that random graphs contain the smallest number of copies of every bipartite graph. This had been verified previously for trees, the 3-dimensional cube, and a few other families of bipartite graphs. As we shall see in Chapter 6, the special case of the conjecture for paths, one of the simplest families of bipartite graphs, is equivalent to the Blakley-Roy theorem [4] about matrices. What follows from our results is that for certain graphs H , including *all hypercubes*, and for every subgraph $K \subseteq H$, we have

$$\inf\{t_H(G) : t_K(G) \geq p\} = p^{|E(H)|/|E(K)|}.$$

This gives a sharp lower-bound for $t_H(G)$ in terms of not only $t_{K_2}(G)$, but in terms of every $t_K(G)$ for every subgraph $K \subseteq H$. For a constant $0 < q \leq 1$, setting $p := q^{1/|E(K)|}$ in (1.2) we have $t_K(G(n, p)) = q \pm o(1)$ and $t_{K_2}(G(n, p)) = q^{|E(H)|/|E(K)|} \pm o(1)$. This shows that random graphs are extremal in this case too. We hope that the application to the Erdős-Simonovits-Sidorenko conjecture promises the discovery of more applications of our results to problems in extremal combinatorics.

We also study the hypergraph norms from a geometric point of view. A large portion of Banach space theory is devoted to the study of local properties of normed spaces. These are the properties that depend only on finite dimensional subspaces of a normed space. The two dual concepts of uniform convexity and uniform smoothness are among the most important invariants in local theory of Banach spaces. Roughly speaking, a normed space is **uniformly convex** if its unit ball is uniformly free of “flat spots”, and a normed space is **uniformly smooth** if its unit ball is uniformly free of “corners”. The notions of uniform smoothness and uniform convexity are local properties as they depend only on the structure of the two-dimensional subspaces of a norm space. These notions are first defined by Clarkson in [8], where he studied the smoothness and convexity of L_p spaces. Clarkson [8] defined the modulus of smoothness and modulus of convexity as quantified versions of the notions of uniform smoothness and uniform convexity, and computed the modulus of smoothness of L_p spaces for $1 < p \leq 2$, and the modulus of convexity of L_p spaces for $2 \leq p < \infty$. Later Hanner [33] completed Clarkson’s work by computing the modulus of convexity of L_p spaces for $1 < p \leq 2$, and the modulus of smoothness of L_p spaces for $2 \leq p < \infty$. For most examples of the natural normed spaces, one can deduce the most important local invariants of a normed space from its moduli of smoothness and convexity. Hence from the perspective of local theory it is very desirable to compute these parameters. We shall discuss this further in Chapter 4.

Gowers norms are a very special case of hypergraph norms. But surprisingly some of their key properties, and ideas from pseudo-randomness theory, will be needed in the

study of the geometric properties of the hypergraph norms.

In Chapter 4 we determine the moduli of smoothness and convexity of hypergraph norms. Our results in particular determine the moduli of smoothness and convexity of Gowers norms. They also provide a unified proof for some previously known facts about L_p and Schatten spaces, and generalize them to a wider class of norms. In certain cases we can show a stronger result. Namely that the corresponding normed space satisfies the so called Hanner inequality. This inequality has been proven to hold only for a few spaces, namely the L_p spaces by Hanner [33], and the Schatten spaces S_p for $p \geq 4$ and $1 \leq p \leq 4/3$ by Ball, Carlen and Lieb [1]. We also prove a complex interpolation theorem, and use it together with the Hanner inequality to obtain various optimum results in terms of the constants involved in the definition of moduli of smoothness and convexity.

The rest of this thesis is organized as follows: Chapter 2 is a brief review of the theorems and results of the elementary functional analysis and graph theory which we will use in the subsequent chapters. In Chapter 3 we define the hypergraph norms, and prove various results including the aforementioned Hölder type inequality in the direction of obtaining a characterization theorem. Chapter 4 is devoted to the study of the geometric properties of hypergraph norms. In Chapter 5 we translate some of the results about hypergraph norms to the language of graphs, and also show that n -dimensional cubes can be used to define norms. In Chapter 6 we apply the results from Chapter 5 to the Erdős-Simonovits-Sidorenko conjecture. In Chapter 7 we discuss some open problems.

Throughout this thesis we assume that the reader is familiar with the very basic concepts and notations of graph theory, real analysis, and probability theory such as graphs and subgraphs, path, cycle, Lebesgue integration, event, and random variables. To see the definitions of the basic concepts of graph theory, refer to any textbook on graph theory (such as [62, 5]), for real analysis see for example [19], and for probability theory see [47].

Chapter 2

Background

The aim of this chapter is to introduce the necessary definitions, notations, and basic results from functional analysis and graph theory for this thesis.

For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. For two *disjoint* sets X and Y , we shall sometimes denote their union by $X \dot{\cup} Y$ to emphasize the fact that they are disjoint.

For two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^+$, we write $f = o(g)$, if and only if

$$\lim_{x \rightarrow \infty} f(x)/g(x) = 0.$$

We write $f = O(g)$, if there exists constants $C, N > 0$ such that $f(x) \leq Cg(x)$, for every $x \geq N$.

2.1 Functional analysis

2.1.1 Measure spaces

A σ -algebra over a set Ω is a nonempty collection \mathcal{F} of subsets of Ω which includes \emptyset , and is closed under complementation and countable unions of its members.

A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$ where \mathcal{F} is a σ -algebra over Ω and $\mu : \mathcal{F} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ satisfies the following axioms:

- Null empty set: $\mu(\emptyset) = 0$.
- Countable additivity: if $\{E_i\}_{i \in \mathcal{I}}$ is a countable set of pairwise disjoint sets in \mathcal{F} , then

$$\mu(\cup_{i \in \mathcal{I}} E_i) = \sum_{i \in \mathcal{I}} \mu(E_i).$$

The function μ is called a **measure**, and the elements of \mathcal{F} are called **measurable sets**.

A measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ is called σ -finite, if Ω is the countable union of measurable sets of finite measure.

Every measure space in this thesis is assumed to be σ -finite.

Definition 2.1.1 For a set Ω , a collection R of subsets of Ω is called a *ring* if

- $\emptyset \in R$.
- $A, B \in R$, then $A \cup B \in R$.
- $A, B \in R$, then $A \setminus B \in R$.

The following theorem, due to Carathéodory, is one of the fundamental theorems in measure theory.

Theorem 2.1.2 (Carathéodory's extension theorem) *Let R be a ring of subsets of a given set Ω . One can always extend every σ -finite measure defined on R to the σ -algebra generated by R ; moreover, the extension is unique.*

Consider two measure spaces $\mathcal{M} := (\Omega, \mathcal{F}, \mu)$ and $\mathcal{N} := (\Sigma, \mathcal{G}, \nu)$. The **product measure** $\mu \times \nu$ on $\Omega \times \Sigma$ is defined in the following way: For $F \in \mathcal{F}$ and $G \in \mathcal{G}$, define $\mu \times \nu(F \times G) = \mu(F) \times \nu(G)$. So far we defined the measure $\mu \times \nu$ on $A := \{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}$. Note that A is a ring in that $\emptyset \in A$, and A is closed under complementation and *finite* unions of its members. However, A is not necessarily a σ -algebra, as it is possible that A is not closed under countable unions of its members. Let $\mathcal{F} \times \mathcal{G}$ be the σ -algebra generated

by A , i.e. it is obtained by closing A under complementation and countable unions. It should be noted that $\mathcal{F} \times \mathcal{G}$ is *not* the cartesian product of the two sets \mathcal{F} and \mathcal{G} , and instead it is the σ -algebra generated by the cartesian product of \mathcal{F} and \mathcal{G} . Theorem 2.1.2 shows that $\mu \times \nu$ extends uniquely from A to a measure over all of $\mathcal{F} \times \mathcal{G}$. We denote the corresponding measure space by $\mathcal{M} \times \mathcal{N}$ which is called the *product measure* of \mathcal{M} and \mathcal{N} .

Consider two measure spaces $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and $\mathcal{N} = (\Sigma, \mathcal{G}, \nu)$. A function $f : \Omega \rightarrow \Sigma$ is called **measurable** if the preimage of every set in \mathcal{G} belongs to \mathcal{F} . This in particular implies that a function $f : \Omega \rightarrow \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is **measurable** if for every $x_0 \in \mathbb{K}$ and $\epsilon > 0$, the preimage of the set $\{x : |x - x_0| < \epsilon\}$ belongs to \mathcal{F} .

2.1.2 Metric spaces

A **metric space** is an ordered pair (M, d) where M is a set and d is a **metric** on M , that is, a function $d : M \times M \rightarrow \mathbb{R}^+$ such that

- Non-degeneracy: $d(x, y) = 0$ if and only if $x = y$.
- Symmetry: $d(x, y) = d(y, x)$, for every $x, y \in M$.
- Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$, for every $x, y, z \in M$.

A sequence $\{x_i\}_{i=1}^{\infty}$ of elements of a metric space (M, d) is called a **Cauchy sequence** if for every $\epsilon > 0$, there exist an integer N_ϵ , such that for every $m, n \geq N_\epsilon$, we have $d(x_m, x_n) \leq \epsilon$. A metric space (M, d) is called **complete** if every Cauchy sequence has a limit in M . A metric space is **compact** if and only if every sequence in the space has a convergent subsequence.

2.1.3 Basic inequalities

The most frequently used inequalities in functional analysis are due to Cauchy and Schwarz, and Hölder.

Theorem 2.1.3 (Cauchy-Schwarz inequality) Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$.

For two measurable functions $f, g : \Omega \rightarrow \mathbb{C}$ we have

$$\left| \int f(x)g(x)d\mu(x) \right| \leq \left(\int |f(x)|^2 d\mu(x) \right)^{1/2} \left(\int |g(x)|^2 d\mu(x) \right)^{1/2}.$$

The Hölder inequality is a generalization of the Cauchy-Schwarz inequality.

Theorem 2.1.4 (Hölder's inequality) Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. For

two functions $f, g : \Omega \rightarrow \mathbb{C}$, and two reals $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left| \int f(x)g(x)d\mu(x) \right| \leq \left(\int |f(x)|^p d\mu(x) \right)^{1/p} \left(\int |g(x)|^q d\mu(x) \right)^{1/q}.$$

2.1.4 Normed spaces

By a normed space we mean a pair $(V, \|\cdot\|)$, where V is a vector space over \mathbb{R} or \mathbb{C} , and $\|\cdot\|$ is a function from V to nonnegative reals satisfying

- **(non-degeneracy)**: $\|x\| = 0$ if and only if $x = 0$.
- **(homogeneity)**: For every scalar λ , and every $x \in V$, $\|\lambda x\| = |\lambda|\|x\|$.
- **(triangle inequality)**: For $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

We call $\|x\|$, the norm of x . A **semi-norm** is a function similar to a norm except that it might not satisfy the non-degeneracy condition.

The spaces $(\mathbb{C}, |\cdot|)$ and $(\mathbb{R}, |\cdot|)$ are respectively examples of 1-dimensional complex and real normed spaces.

Every normed space $(V, \|\cdot\|)$ has a metric space structure where the distance of two vectors x and y is $\|x - y\|$. A complete normed space is called a **Banach space**.

Two norms $\|\cdot\|$ and $\|\cdot\|'$ over a vector space V are called **equivalent**, if there exist constants $C_1, C_2 > 0$ such that for every $x \in V$, we have

$$C_1\|x\| \leq \|x\|' \leq C_2\|x\|.$$

When two norms are equivalent, their metric space structures induce the same topology on the underlying vector space. The following theorem shows that in the finite dimensional case all norms are equivalent.

Theorem 2.1.5 *Let V be a finite dimensional vector space. Then all norms on V are equivalent.*

It can be deduced from Theorem 2.1.5 that every finite dimensional normed space is complete and thus is a Banach space.

Consider two normed spaces X and Y . A **bounded operator** from X to Y , is a *linear* function $T : X \rightarrow Y$, such that

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} < \infty. \quad (2.1)$$

The set of all bounded operators from X to Y is denoted by $B(X, Y)$. Note that the operator norm defined in (2.1) makes $B(X, Y)$ a normed space.

A **functional** on a normed space X over \mathbb{C} (or \mathbb{R}) is a bounded linear map f from X to \mathbb{C} (respectively \mathbb{R}), where bounded means that

$$\|f\| := \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} < \infty.$$

The set of all bounded functionals on X endowed with the operator norm, is called the **dual** of X and is denoted by X^* . So for a normed space X over complex numbers, $X^* = B(X, \mathbb{C})$, and similarly for a normed space X over real numbers, $X^* = B(X, \mathbb{R})$.

For a normed space X , the set $\mathbf{B}_X := \{x : \|x\| \leq 1\}$ is called the **unit ball** of X . Note that by the triangle inequality, \mathbf{B}_X is a convex set, and also by homogeneity it is symmetric around the origin, in the sense that $\|\lambda x\| = \|x\|$ for every scalar λ with $|\lambda| = 1$. The non-degeneracy condition implies that \mathbf{B}_X has non-empty interior.

Every compact symmetric convex subset of \mathbb{R}^n with non-empty interior is called a **convex body**. Convex bodies are in one-to-one correspondence with norms on \mathbb{R}^n . A

convex body K corresponds to the norm $\|\cdot\|_K$ on \mathbb{R}^n , where

$$\|x\|_K := \sup\{\lambda \in \mathbb{R}^+ : \lambda x \in K\}.$$

Note that K is the unit ball of $\|\cdot\|_K$. For a set $K \subseteq \mathbb{R}^n$, define its **polar conjugate** as

$$K^\circ = \{x \in \mathbb{R}^n : \sum x_i y_i \leq 1, \forall y \in K\}. \quad (2.2)$$

The polar conjugate of a convex body K is a convex body, and furthermore $(K^\circ)^\circ = K$.

Consider a normed space X on \mathbb{R}^n . For $x \in \mathbb{R}^n$ define $T_x : \mathbb{R}^n \rightarrow \mathbb{R}$ as $T_x(y) := \sum_{i=1}^n x_i y_i$. It is easy to see that T_x is a functional on X , and furthermore every functional on X is of the form T_x for some $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ define $\|x\|^* := \|T_x\|$. This shows that we can identify X^* with $(\mathbb{R}^n, \|\cdot\|^*)$. Let K be the unit ball of $\|\cdot\|$. It is easy to see that K° , the polar conjugate of K , is the unit ball of $\|\cdot\|^*$.

2.1.5 Hilbert Spaces

Consider a vector space V over \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Recall that an **inner product** $\langle \cdot, \cdot \rangle$ on V , is a function from $V \times V$ to \mathbb{K} that satisfies the following axioms.

- Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- Linearity in the first argument: $\langle ax + z, y \rangle = a\langle x, y \rangle + \langle z, y \rangle$ for $a \in \mathbb{K}$ and $x, y \in V$.
- Positive-definiteness: $\langle x, x \rangle > 0$ if and only if $x \neq 0$, and $\langle 0, 0 \rangle = 0$.

A vector space together with an inner product is called an **inner product space**.

Example 1 Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, and let \mathcal{H} be the space of measurable functions $f : \Omega \rightarrow \mathbb{C}$ such that $\int |f(x)|^2 d\mu(x) < \infty$. For two functions $f, g \in \mathcal{H}$ define

$$\langle f, g \rangle := \int f(x) \overline{g(x)} d\mu(x).$$

It is not difficult to verify that the above mentioned function is indeed an inner product.

An inner product can be used to define a norm on V . For a vector $x \in V$, define $\|x\| = \sqrt{\langle x, x \rangle}$.

Lemma 2.1.6 *For an inner product space V , the function $\|\cdot\| : x \mapsto \sqrt{\langle x, x \rangle}$ is a norm.*

Proof. The non-degeneracy and homogeneity conditions are trivially satisfied. It remains to verify the triangle inequality. Consider two vectors $x, y \in V$ and note that by the axioms of an inner product:

$$0 \leq \langle x + \lambda y, x + \lambda y \rangle = \langle x, x \rangle + |\lambda|^2 \langle y, y \rangle + \lambda \overline{\langle x, y \rangle} + \bar{\lambda} \langle x, y \rangle.$$

Now taking $\lambda := \sqrt{\frac{\langle x, x \rangle}{\langle y, y \rangle}} \times \frac{\langle x, y \rangle}{|\langle x, y \rangle|}$ will show that

$$0 \leq 2\langle x, x \rangle \langle y, y \rangle - 2\sqrt{\langle x, x \rangle \langle y, y \rangle} |\langle x, y \rangle|,$$

which leads to the triangle inequality. ■

A Hilbert space is a complete inner-product space.

2.1.6 The L_p spaces

Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. For $1 \leq p < \infty$, the space $L_p(\mathcal{M})$ is the space of all functions $f : \Omega \rightarrow \mathbb{C}$ such that

$$\|f\|_p := \left(\int |f(x)|^p d\mu(x) \right)^{1/p} < \infty.$$

Strictly speaking the elements of $L_p(\mathcal{M})$ are equivalent classes. Two functions f_1 and f_2 are equivalent and are considered identical, if they agree almost everywhere or equivalently $\|f_1 - f_2\|_p = 0$.

Proposition 2.1.7 *For every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, $L_p(\mathcal{M})$ is a normed space.*

Proof. Non-degeneracy and homogeneity are trivial. It remains to verify the triangle inequality. By applying Hölder's inequality:

$$\begin{aligned}
\|f + g\|_p^p &= \int |f(x) + g(x)|^p d\mu(x) = \int |f(x) + g(x)|^{p-1} |f(x) + g(x)| d\mu(x) \\
&\leq \int |f(x) + g(x)|^{p-1} |f(x)| d\mu(x) + \int |f(x) + g(x)|^{p-1} |g(x)| d\mu(x) \\
&\leq \left(\int |f(x) + g(x)|^p d\mu(x) \right)^{\frac{p-1}{p}} \|f\|_p + \left(\int |f(x) + g(x)|^p d\mu(x) \right)^{\frac{p-1}{p}} \|g\|_p \\
&= \|f + g\|_p^{p-1} (\|f\|_p + \|g\|_p),
\end{aligned}$$

which simplifies to the triangle inequality ■

Consider the set of natural numbers \mathbb{N} with the counting measure. We shall use the notation $\ell_p := L_p(\mathbb{N})$.

2.1.7 Schatten norms

Let A be a real or complex matrix. For $1 \leq p < \infty$, the p -th Schatten norm of A is defined as

$$\|A\|_{S_p} = (\text{Tr}(A^*A)^{p/2})^{1/p}.$$

The p -th Schatten norm is sometimes referred to as the p -th Schatten-von Neumann norm, or p -trace norm. Note that when A is an $n \times n$ matrix, $\|A\|_{S_p}$ is just the ℓ_p norm that is applied to the eigenvalues of $|A| = (A^*A)^{1/2}$. This fact generalizes by the spectral theorem to the infinite case.

It is well-known (but not trivial) that $\|\cdot\|_{S_p}$ is a norm. This can be deduced from Theorem 2.1.8 below due to Schatten and von Neumann [50, 51, 52].

Theorem 2.1.8 *Suppose that $1 \leq p, q, r < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. For two $n \times n$ matrices A and B we have*

$$\|AB\|_{S_r} \leq \|A\|_{S_p} \|B\|_{S_q}.$$

For further reading about the Schatten norms we refer the reader to [10].

2.1.8 The Hahn-Banach theorem

Consider a normed space X . Recall that X^* , the dual of X , is the set of bounded functionals on X . In this section we want to answer the following question: Are there enough elements in X^* to distinguish the elements of X ? Equivalently, given $0 \neq x \in X$, is there an element $f \in X^*$ such that $f(x) \neq 0$? Since X is a normed space one wishes more. We want to obtain information about X by looking at X^* . To be more precise we know that

$$\|x\| \geq \sup\{f(x) : \|f\| = 1, f \in X^*\},$$

but is the right-hand side equal to $\|x\|$? In fact, based on what we have presented so far, we do not even know that for every non-zero normed space X , $X^* \neq \{0\}$.

The key to answer the above questions is the Hahn-Banach theorem.

Theorem 2.1.9 (Hahn-Banach theorem) *Let X be a normed space over a field \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and E be a subspace of X . Suppose that $\phi : E \rightarrow \mathbb{K}$ is a functional with $\|\phi\| \leq 1$. It is possible to extend ϕ to a functional $\psi : X \rightarrow \mathbb{K}$ such that $\|\psi\| \leq 1$.*

The following corollary to Theorem 2.1.9 answers the questions discussed above.

Corollary 2.1.10 *Let X be a normed space over a field \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $x_0 \in X$. There exists a functional $\psi : X \rightarrow \mathbb{K}$ such that $\|\psi\| = 1$, and $\psi(x_0) = \|x_0\|$.*

Proof. Let E be the 1-dimensional subspace of X generated by x_0 . There is a unique $\phi : E \rightarrow \mathbb{K}$ with $\phi(x) = x_0$. Note that $\|\phi\| \leq 1$. Now one can apply Theorem 2.1.9 to extend ϕ to a functional on X that satisfies the requirements of the corollary. ■

2.1.9 Finite representations of normed spaces

For a real $\lambda \geq 1$, a normed space X is said to be λ -finitely representable in a normed space Y , if for every finite dimensional subspace $E \subseteq X$, there exists a linear map $T : E \rightarrow Y$

such that $\|x\| \leq \|Tx\| \leq \lambda\|x\|$, for every $x \in E$. If for every $\lambda > 1$, X is λ -finitely representable in Y , then we simply say X is **finitely representable** in Y . Roughly speaking this means that for every finite dimensional subspace E of X , one can find a subspace of Y that is an “almost” identical copy of E . Finite representations of normed spaces are important when one studies the “local” properties of the normed spaces, i.e. the properties that depend only on finite dimensional subspaces of a normed space. For example, as we shall see in Chapter 4 a normed space is called uniformly convex if for every $\epsilon > 0$,

$$0 < \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1, \|x-y\| \geq 2\epsilon \right\}.$$

Note that being uniformly convex is a property that depends only on two-dimensional subspaces of the normed space, so it is a local property. It is easy to see that if X is not uniformly convex, and X is finitely representable in Y , then Y is also not uniformly convex.

It is not difficult to see that for $1 \leq p < \infty$, every L_p space is finitely representable in ℓ_p . The following beautiful theorem due to Dvoretzky [11] plays an important role in local theory of normed spaces.

Theorem 2.1.11 (Dvoretzky’s theorem) *The space ℓ_2 is finitely representable in every infinite dimensional normed space.*

For a proof we refer the reader to [43].

2.2 Graph theory

A **graph** G is a pair (V, E) comprising a finite set V of **vertices**, and a set E of **edges**, where every edge is a 2-element subset of vertices. Sometimes for an edge $\{u, v\}$ we abbreviate it to just uv . The **adjacency matrix** of a graph G is a $|V(G)| \times |V(G)|$ matrix A_G , where rows and columns are indexed by vertices of G , and $A_G(u, v)$ is equal to 1 if $uv \in E$, and it is equal to 0 otherwise.

For an integer $n \geq 3$, C_n denotes the cycle of length n , and for an integer $n \geq 1$, K_n denotes the complete graph on n vertices.

2.2.1 Graph homomorphisms

For two graphs H and G , a homomorphism from H to G is a mapping $h : V(H) \rightarrow V(G)$ such that for each edge $\{u, v\}$ of H , $\{h(u), h(v)\}$ is an edge of G . An isomorphism from G to H , is a bijection $h : V(G) \rightarrow V(H)$, such that h is a homomorphism from G to H , and h^{-1} is a homomorphism from H to G . In other words $uv \in E(G)$ if and only if $h(u)h(v) \in E(H)$.

Two graphs G and H are called isomorphic if there exists an isomorphism between them. An isomorphism from G to itself is called an automorphism. Note that the set of automorphisms of a graph G with the composition operator constitutes a group $\text{Aut}(G)$ which is called the automorphism group of G .

2.2.2 Graph densities vs homomorphism densities

Let $h_H(G)$ and $h_H^{\text{inj}}(G)$ be respectively the number of homomorphisms, and *injective* homomorphisms from H to G . Since an injective homomorphism is a homomorphism that maps vertices of H to distinct vertices of G , the number of copies of H inside G is equal to $\frac{1}{|\text{Aut}(H)|} h_H^{\text{inj}}(G)$. Many classical theorems in extremal graph theory state some fact about the relation between the functions $h_H^{\text{inj}}(\cdot)$ for different graphs H . For example, Turán's theorem says that the maximum number of edges in an n -vertex triangle-free graph is $\lfloor n^2/4 \rfloor$. In other words if $h_{K_2}^{\text{inj}}(G) > 2\lfloor n^2/4 \rfloor$, then $h_{K_3}^{\text{inj}}(G) > 0$.

We also normalize h_H and h_H^{inj} in the following way.

Definition 2.2.1 For two graphs H and G , $t_H(G)$ is the probability that a random mapping from $V(H)$ to $V(G)$ is a homomorphism, and similarly $t_H^{\text{inj}}(G)$ is the probability that a random injective mapping from $V(H)$ to $V(G)$ defines a homomorphism.

Remark 2.2.2 Note that

$$t_H^{\text{inj}}(G) = \frac{h_H^{\text{inj}}(G)}{|V(G)|(|V(G)| - 1) \dots (|V(G)| - |V(H)| + 1)},$$

which shows that the number of copies of H in G is equal to

$$\frac{(|V(G)|(|V(G)| - 1) \dots (|V(G)| - |V(H)| + 1))}{|\text{Aut}(H)|} t_H^{\text{inj}}(G).$$

Similarly

$$t_H(G) = h_H(G)/|V(G)|^{|V(H)|}.$$

■

The following easy lemma [41] shows that t_H and t_H^{inj} are close up to an error term of $o(1)$:

Lemma 2.2.3 *For every two graphs H and G ,*

$$|t_H(G) - t_H^{\text{inj}}(G)| \leq \frac{1}{|V(G)|} \binom{|V(H)|}{2} = o_{|V(G)| \rightarrow \infty}(1).$$

Proof. Set $n := |V(G)|$ and $k := |V(H)|$. Trivially $h_H^{\text{inj}}(G) \leq h_H(G)$, and

$$t_H(G) = \frac{h_H(G)}{n^k} \geq \frac{h_H^{\text{inj}}(G)}{n^k} = t_H^{\text{inj}}(G) \frac{n(n-1) \dots (n-k+1)}{n^k}.$$

But

$$t_H^{\text{inj}}(G) \frac{n(n-1) \dots (n-k+1)}{n^k} = t_H^{\text{inj}}(G) \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) \geq t_H^{\text{inj}}(G) \left(1 - \binom{k}{2} \frac{1}{n}\right),$$

which shows that

$$t_H(G) \geq t_H^{\text{inj}}(G) \left(1 - \binom{k}{2} \frac{1}{n}\right) \geq t_H^{\text{inj}}(G) - \binom{k}{2} \frac{1}{n}. \quad (2.3)$$

On the other hand, by the beginning of the inclusion-exclusion formula, we have

$$h_H^{\text{inj}}(G) \geq h_H(G) - \sum_{H'} h_{H'}(G),$$

where the sum is over all graphs H' that are obtained from H by identifying two of the vertices. The number of such graphs is $\binom{k}{2}$, and hence

$$t_H^{\text{inj}}(G) \geq \frac{h_H^{\text{inj}}(G)}{n^k} \geq t_H(G) - \sum_{H'} \frac{h_{H'}(G)}{n^k} \geq t_H(G) - \binom{k}{2} \frac{1}{n}. \quad (2.4)$$

The assertion of the lemma follows from (2.3) and (2.4). \blacksquare

Although $t_H(G)$ itself is an object of interest, extremal graph theory more often concerns $t_H^{\text{inj}}(G)$. However, Lemma 2.2.3 shows that in many situations one can study $t_H(\cdot)$ instead, and the results will be automatically translated to statements about $t_H^{\text{inj}}(\cdot)$. There are major advantages in working with t_H as it behaves much more nicely than t_H^{inj} . This will become clear in the sequel.

Let us start by looking at an example. Consider C_3 , the cycle of length 3, and an arbitrary graph G . We have

$$h_{C_3}^{\text{inj}}(G) := \sum \{A_G(x_1, x_2)A_G(x_2, x_3)A_G(x_3, x_1) : x_i \in V(G) \text{ and } x_i \text{'s are distinct}\}, \quad (2.5)$$

while

$$h_{C_3}(G) := \sum_{x_1, x_2, x_3 \in V(G)} A_G(x_1, x_2)A_G(x_2, x_3)A_G(x_3, x_1) = \text{Tr}(A_G^3). \quad (2.6)$$

Since the trace of a symmetric matrix is the sum of its eigenvalues, $\text{Tr}(A_G^3) = \sum \lambda_i^3$, where $\{\lambda_i\}_{i=1}^{|V(G)|}$ are eigenvalues of A_G . Note that (2.6) has a nice formula which, for example, relates $h_{C_3}(G)$ to the eigenvalues of A_G . But the condition that the x_i 's are distinct in (2.5) prevents us from getting such a nice formula.

For graphs H and G , let x_u ($u \in V(H)$) be independent random variables that take values in $V(G)$ uniformly at random. We have

$$t_H(G) := \mathbb{E} \prod_{uv \in E(H)} A_G(x_u, x_v), \quad (2.7)$$

If in (2.7) we take the expectation conditioned on x_u being distinct, then we obtain a formula for $t_H^{\text{inj}}(G)$. However, we then lose the independence of the random variables x_u .

2.2.3 Tensors and Blowups

Consider an arbitrary graph G , and a positive integer k . The k -blowup of G is the graph obtained by replacing every vertex of G with k different vertices, and a copy of u is adjacent to a copy of v in the blowup graph if and only if u is adjacent to v in G .

Let H and G be graphs, and G' be the k -blowup of G , for a positive integer k . Let $\pi : V(G') \rightarrow V(G)$ be the map defined by $\pi(u) := v$, if and only if u is one of the copies of v in G' . Consider a uniform random mapping h from $V(H)$ to $V(G')$. Note that $\pi \circ h$ is a uniform random mapping from $V(H)$ to $V(G)$. Furthermore, h is a homomorphism from H to G' , if and only if $\pi \circ h$ is a homomorphism from H to G . We conclude the following lemma.

Lemma 2.2.4 *Let G' be the k -blowup of a graph G , for a positive integer k . Then for every graph H ,*

$$t_H(G) = t_H(G').$$

Another graph operation that behaves nicely with respect to t_H functions is the **tensor product**. Given two graphs G_1 and G_2 , their tensor product $G_1 \otimes G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, where there is an edge between (u, v) and (u', v') if and only if $uu' \in E(G_1)$ and $vv' \in E(G_2)$. Consider a graph H , and for every $u \in H$ let $x_u = (y_u, z_u)$ be a random variable taking values in $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ uniformly at random. Then

$$\begin{aligned} t_H(G_1 \otimes G_2) &= \mathbb{E} \prod_{uv \in E(H)} A_{G_1 \otimes G_2}(x_u, x_v) = \mathbb{E} \prod_{uv \in E(H)} A_{G_1}(y_u, y_v) A_{G_2}(z_u, z_v) \\ &= \left(\mathbb{E} \prod_{uv \in E(H)} A_{G_1}(y_u, y_v) \right) \left(\mathbb{E} \prod_{uv \in E(H)} A_{G_2}(z_u, z_v) \right) = t_H(G_1) t_H(G_2). \end{aligned}$$

We conclude the following lemma.

Lemma 2.2.5 *Let G_1 and G_2 be graphs. For every graph H ,*

$$t_H(G_1 \otimes G_2) = t_H(G_1) t_H(G_2).$$

In the sequel for a positive integer k and a graph G , we denote

$$G^{\otimes k} := \underbrace{G \otimes \dots \otimes G}_{k \text{ copies}}.$$

2.2.4 Graphons

In [41], Lovász and Szegedy studied limits of dense graphs. They called a sequence of finite graphs $\{G_i\}_{i=1}^{\infty}$ **convergent** if, for every finite graph H , the sequence $\{t_H(G_i)\}_{i=1}^{\infty}$ converges. It is not difficult to construct convergent sequences $\{G_i\}_{i=1}^{\infty}$ such that their limits cannot be recognized as graphs, i.e. there is no graph G , with $\lim_{i \rightarrow \infty} t_H(G_i) = t_H(G)$ for every H . It is exactly for this reason that the extremal solution to a problem is often stated as a sequence of graphs instead of a single graph. Let us give an example to explain this. Let $0 < p < 1$ be a constant. Consider the following problem: Assuming $t_{K_2}(G) \geq p$, what is the infimum of $t_{C_4}(G)$? As we shall see in Chapter 6, $t_{C_4}(G) \geq p^4$, for every graph G satisfying $t_{K_2}(G) \geq p$. For an integer $n > 0$, let G_n be an instance of the Erdős-Rényi random graph $G(n, p)$ where each edge is present independently with probability p . It is standard (see [41]) that almost surely $\lim_{n \rightarrow \infty} t_{K_2}(G_n) = p$ and $\lim_{n \rightarrow \infty} t_{C_4}(G_n) = p^4$. This proves the existence of a sequence of graphs $\{G_n\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} t_{K_2}(G_n) = p$ and $\lim_{n \rightarrow \infty} t_{C_4}(G_n) = p^4$. On the other hand, as we shall see below, there is no graph G with $t_{K_2}(G) = p$ and $t_{C_4}(G) = p^4$. To remedy this and similar situations we have to extend the space of graphs, and represent the limits of convergent sequences of graphs as an object in this extended space. Then in this extension we will be able to find an object w such that $t_{K_2}(w) = p$ and $t_{C_4}(w) = p^4$. Let

$$\mathcal{W}_s := \{w : [0, 1]^2 \rightarrow [0, 1] \mid w \text{ is measurable, and } w(x, y) = w(y, x) \text{ for every } x, y \in [0, 1]\}.$$

The elements of \mathcal{W}_s are called **graphons**. Note that the definition of t_H as stated in (2.7) can be adopted for graphons as well: For graphs H , let x_u ($u \in V(H)$) be independent

random variables that take values in $[0, 1]$ uniformly at random. For $w \in \mathcal{W}_s$ define

$$t_H(w) := \mathbb{E} \prod_{uv \in E(H)} w(x_u, x_v). \quad (2.8)$$

Consider a graph G with vertex set $\{1, 2, \dots, n\}$. Define $w_G : [0, 1]^2 \rightarrow \{0, 1\}$ as follows. Let $w_G(x, y) := A_G(\lceil xn \rceil, \lceil yn \rceil)$ if $x, y \in (0, 1]$, and if $x = 0$ or $y = 0$, set w_G to 0. By comparing (2.7) and (2.8) trivially for every H and G ,

$$t_H(G) = t_H(w_G).$$

This shows that w_G contains sufficient information about G so that one can recover the values of $t_H(G)$ for every H . In that sense one can consider the space of graphons \mathcal{W}_s as an extension of the space of finite graphs. However, one should be careful as it is possible that $w_G = w_{G'}$, for two different graphs $G \neq G'$. Indeed it is not difficult to see that the graphon corresponded to G is the same as the graphon corresponded to its k -blowup.

Similar to the graph case, a sequence of graphons $\{w_i\}_{i=1}^\infty$ is called convergent if for every graph H , $\{t_H(w_i)\}_{i=1}^\infty$ converges. The following theorem, due to Lovász and Szegedy, shows that if a sequence $\{t_H(w_i)\}_{i=1}^\infty$ converges then the limit can be represented as a graphon.

Theorem 2.2.6 (Lovász-Szegedy [41]) *For every convergent sequence $\{w_i\}_{i=1}^\infty$ of graphons, there exists a graphon w such that for every graph H ,*

$$\lim_{i \rightarrow \infty} t_H(w_i) = t_H(w).$$

In the space of graphons, the system of equations $t_{K_2}(w) = p$ and $t_{C_4}(w) = p^4$ has the unique solution $w = p$ (see [40]). Since for $0 < p < 1$, there is no graph G with $w_G = p$, it follows that there is no graph G with $t_{K_2}(G) = p$ and $t_{C_4}(G) = p^4$.

The language of the homomorphism densities and graphons gives a neater nature to some problems in extremal graph theory. To reach a comparison, let us consider the statement that we discussed above: $t_{C_4}(w) \geq p^4$ if $t_{K_2}(w) \geq p$, and moreover the

system of equations $t_{K_2}(w) = p$ and $t_{C_4}(w) = p^4$ has the unique solution $w = p$. The translation of the first part of this statement to the language of the classical extremal graph theory will be the following: “For a constant p and a graph G with n vertices, if $|E(G)| \geq p \frac{n^2}{2}(1 \pm o(1))$, then $\#(C_4, G) \geq p^4 \frac{n^4}{8}(1 \pm o(1))$, where $\#(C_4, G)$ is the number of copies of a cycle of length 4 in G .” Since the statement is asymptotic, to show its tightness one needs to introduce a proper sequence of graphs, while in the graphon case, the graphon $w = p$ shows the tightness of the analogous statement. Finally the second part of the statement which talks about the uniqueness does not translate directly and naturally to the language of the classical extremal graph theory.

The following theorem shows that the set $\{w_G : G \text{ is a finite graph}\}$ is dense in \mathcal{W}_s , or in other words \mathcal{W}_s is the closure of $\{w_G : G \text{ is a finite graph}\}$.

Theorem 2.2.7 (Lovász-Szegedy [41]) *For every graphon w , there exists a sequence of graphs $\{G_i\}_{i=1}^\infty$ such that $\{w_{G_i}\}_{i=1}^\infty$ converges to w .*

2.3 Gowers Norms

In this section we define the Gowers norms. The k th Gowers norm is defined on the set of measurable functions $f : [0, 1]^k \rightarrow \mathbb{R}$. For $k = 1$, the Gowers norm of a function $f : [0, 1] \rightarrow \mathbb{R}$ is its usual L_2 norm. Many key properties of Gowers norms will become apparent for $k = 2$. The second Gowers norm coincides with the C_4 norm.

2.3.1 C_4 norm

For a measurable function $w : [0, 1]^2 \rightarrow \mathbb{R}$, its C_4 norm is defined as

$$\|w\|_{C_4} := (\mathbb{E}w(x_0, y_0)w(x_1, y_0)w(x_1, y_1)w(x_0, y_1))^{1/4}. \quad (2.9)$$

Note that for a symmetric w , we have $\|w\|_{C_4} = (t_{C_4}(w))^{1/4}$. For four measurable functions $w_1, w_2, w_3, w_4 : [0, 1]^2 \rightarrow \mathbb{R}$, we denote

$$\langle w_1, w_2, w_3, w_4 \rangle_{C_4} := \mathbb{E} w_1(x_0, y_0) w_2(x_1, y_0) w_3(x_0, y_1) w_4(x_1, y_1).$$

Lemma 2.3.1 *Consider four measurable functions $w_1, w_2, w_3, w_4 : [0, 1]^2 \rightarrow \mathbb{R}$. Then*

$$\langle w_1, w_2, w_3, w_4 \rangle_{C_4} \leq \|w_1\|_{C_4} \|w_2\|_{C_4} \|w_3\|_{C_4} \|w_4\|_{C_4}. \quad (2.10)$$

Proof. The proof is by applying the Cauchy-Schwarz inequality iteratively.

$$\begin{aligned} \text{L.H.S. of (2.10)} &= \mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_0} w_1(x_0, y_0) w_2(x_1, y_0)) (\mathbb{E}_{y_1} w_3(x_0, y_1) w_4(x_1, y_1)) \\ &\leq \left(\mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_0} w_1(x_0, y_0) w_2(x_1, y_0))^2 \right)^{1/2} \times \\ &\quad \left(\mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_1} w_3(x_0, y_1) w_4(x_1, y_1))^2 \right)^{1/2} \end{aligned} \quad (2.11)$$

Now note that by change of variable

$$\begin{aligned} \mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_0} w_1(x_0, y_0) w_2(x_1, y_0))^2 &= \mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_0} w_1(x_0, y_0) w_2(x_1, y_0)) \times \\ &\quad (\mathbb{E}_{y_1} w_1(x_0, y_1) w_2(x_1, y_1)) \\ &= \mathbb{E} w_1(x_0, y_0) w_2(x_1, y_0) w_1(x_0, y_1) w_2(x_1, y_1) \\ &= \langle w_1, w_2, w_1, w_2 \rangle_{C_4}, \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} (2.12) &= \mathbb{E}_{y_0, y_1} (\mathbb{E}_{x_0} w_1(x_0, y_0) w_1(x_0, y_1)) (\mathbb{E}_{x_1} w_2(x_1, y_0) w_2(x_1, y_1)) \\ &\leq \left(\mathbb{E}_{y_0, y_1} (\mathbb{E}_{x_0} w_1(x_0, y_0) w_1(x_0, y_1))^2 \right)^{1/2} \left(\mathbb{E}_{y_0, y_1} (\mathbb{E}_{x_1} w_2(x_1, y_0) w_2(x_1, y_1))^2 \right)^{1/2} \\ &= \|w_1\|_{U_4}^2 \|w_2\|_{U_4}^2 \end{aligned} \quad (2.13)$$

Similarly

$$\mathbb{E}_{x_0, x_1} (\mathbb{E}_{y_1} w_3(x_0, y_1) w_4(x_1, y_1))^2 \leq \|w_3\|_{U_2}^2 \|w_4\|_{U_2}^2.$$

Putting back everything in (2.11) leads to (2.10). ■

Proposition 2.3.2 *The C_4 norm satisfies the axioms of a norm.*

Proof. The homogeneity is trivial. Suppose that $\|w\|_{C_4} = 0$ for $w : [0, 1]^2 \rightarrow \mathbb{R}$. By (2.10), for $f, g : [0, 1] \rightarrow \mathbb{R}$ we have

$$|\mathbb{E}f(x)w(x, y)g(y)| \leq \|f\|_4 \|w\|_{C_4} \|g\|_4 \|1\|_{C_4} = 0.$$

This implies that $w = 0$. Let us now verify the triangle inequality. For $w_1, w_2 : [0, 1]^2 \rightarrow \mathbb{R}$

$$\begin{aligned} \|w_1 + w_2\|_{C_4}^4 &= \langle w_1 + w_2, w_1 + w_2, w_1 + w_2, w_1 + w_2 \rangle_{C_4} \\ &= \langle w_1, w_1 + w_2, w_1 + w_2, w_1 + w_2 \rangle_{C_4} + \langle w_2, w_1 + w_2, w_1 + w_2, w_1 + w_2 \rangle_{C_4} \\ &\leq \|w_1\|_{C_4} \|w_1 + w_2\|_{C_4}^3 + \|w_2\|_{C_4} \|w_1 + w_2\|_{C_4}^3 \end{aligned}$$

■

The following proposition which we state without a proof reveals the importance of the C_4 norm from the point of view of extremal combinatorics (for a proof see [28]).

Proposition 2.3.3 *Consider two graphons $w_1, w_2 \in \mathcal{W}_s$. For every graph H ,*

$$|t_H(w_1) - t_H(w_2)| \leq |E(H)| \times \|w_1 - w_2\|_{C_4}.$$

The C_4 norm is closely related to Schatten norms discussed in Section 2.1.7. In fact, for a real-valued matrix A , we have

$$\|A\|_{S_4} = (\text{Tr}(A^t A)^2)^{1/4} = \left(\sum A(x_0, y_0) A(x_1, y_0) A(x_1, y_1) A(x_0, y_1) \right)^{1/4}. \quad (2.14)$$

Comparing (2.14) to (2.9) shows that apart from some small technicalities the C_4 norm and the 4-Schatten norm are basically the same norms. In fact, one can define the C_4 norm by defining a proper trace function for measurable functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. However, since that is not relevant to the purpose to this thesis we shall not discuss it further in this section.

The fact that $\|\cdot\|_{C_4}$ is a norm was long known due to its connection to the Schatten norms. To the best of the author's knowledge, most of the known proofs for the fact

that Schatten norms satisfy the axioms of norms are through spectral theory. However, the above proof to Lemma 2.3.1 is more recent [26], and its importance is in that it generalizes to k -variable functions, as we shall see in Section 2.3.2.

2.3.2 Gowers norms

Gowers norms are generalizations of the C_4 norm to k -variable functions. For a function $f : [0, 1]^k \rightarrow \mathbb{R}$, its k -th Gowers uniformity norm is defined as

$$\|f\|_{U_k} := \left(\mathbb{E} \prod_{(i_1, \dots, i_k) \in \{0, 1\}^k} f(x_{1, i_1}, \dots, x_{k, i_k}) \right)^{2^{-k}}.$$

Note that setting $k = 2$ in the above formula gives the C_4 norm defined in (2.9).

For $e \in \{0, 1\}^k$, let $f_e : [0, 1]^k \rightarrow \{0, 1\}$ be measurable functions. Define

$$\langle \{f_e\}_{e \in \{0, 1\}^k} \rangle_{U_k} = \mathbb{E} \prod_{e = (i_1, \dots, i_k) \in \{0, 1\}^k} f_e(x_{1, i_1}, \dots, x_{k, i_k})$$

The following lemma which generalizes Lemma 2.3.1 is called the Gowers-Cauchy-Schwarz inequality as it reminisces the classical Cauchy-Schwarz inequality.

Lemma 2.3.4 (Gowers-Cauchy-Schwarz inequality) *For $e \in \{0, 1\}^k$, let $f_e : [0, 1]^k \rightarrow \{0, 1\}$ be measurable functions. Then*

$$\langle \{f_e\}_{e \in \{0, 1\}^k} \rangle_{U_k} \leq \prod_{e \in \{0, 1\}^k} \|f_e\|_{U_k}.$$

Similar to the proof of Proposition 2.3.2, one can use Lemma 2.3.4 to show that $\|\cdot\|_{U_k}$ satisfies the axioms of a norm. We shall present the definition of Gowers norms for complex functions in Chapter 3.

Chapter 3

Product Norms

Gowers in [26, 27] introduced the Gowers norms as a generalization of the C_4 -norm to measure the amount of pseudo-randomness in the context of hypergraphs. Since then these norms have become common tools in combinatorics, combinatorial number theory, and computer science. Gowers norms, apart from their many remarkable applications, are also interesting from a different point of view. Namely Gowers' proof that they satisfy the axioms of norms, although simple, is different from the common proofs that the C_4 norm is a norm. The C_4 norm of a graph G can be defined as $(\sum \lambda_i^4)^{1/4}$, where λ_i are the eigenvalues of the adjacency matrix of the graph, and most common proofs in this case are based on this spectral interpretation of the norm. But in the context of hypergraphs and the general Gowers norms, one does not have access to spectral theory, and Gowers' proof proceeds by iteratively applying the Cauchy-Schwarz inequality. This new approach was an indication that even in the context of graphs, there might be other norms that are defined in a similar fashion to the Gowers norms.

Recall from Section 2.3.1 that for a symmetric measurable map $w : [0, 1]^2 \rightarrow \mathbb{R}$, the C_4 norm is defined as $\|w\|_{C_4} := t_{C_4}(w)^{1/4}$. Lovász asked for which graphs H , $t_H(\cdot)^{1/|E(H)|}$ defines a norm. In this chapter, we take a more general approach, and shall not restrict ourselves to the norms that are defined through graphs. However, in Chapter 5, we shall

revisit Lovász's question, and state the consequences of the results developed in this chapter to that problem.

Let us start by giving some examples of normed spaces. Consider a measurable function $f : [0, 1] \rightarrow \mathbb{C}$. For $1 \leq p < \infty$, the L_p norm of f is defined as

$$\|f\|_p := \left(\int |f(x)|^p dx \right)^{1/p} = \left(\int f(x)^{p/2} \overline{f(x)^{p/2}} dx \right)^{1/p}. \quad (3.1)$$

Next consider a measurable function $f : [0, 1]^2 \rightarrow \mathbb{C}$. Since f is a complex-valued function the Gowers 2-uniformity norm of f is defined as

$$\|f\|_{U_2} := \left(\int f(x_0, y_0) f(x_1, y_1) \overline{f(x_0, y_1)} \overline{f(x_1, y_0)} dx_0 dx_1 dy_0 dy_1 \right)^{1/4}. \quad (3.2)$$

Note that there are similarities between (3.1) and (3.2): Their underlying vector spaces are function spaces, and the norm of a function f is defined by a formula of the form $(\int \Pi)^{1/p}$, where $p > 0$ and Π is a product which involves different copies of the powers of f and \bar{f} . The purpose of this chapter is to develop a common framework to study the norms that are defined in a similar fashion. We establish that such norms are natural generalizations of the L_p norms, and we prove that they share many of the nice properties of the L_p norms.

For now, let us focus on two-variable functions $f : [0, 1]^2 \rightarrow \mathbb{C}$. For finite sets V_1, V_2 and functions $\alpha, \beta : V_1 \times V_2 \rightarrow \mathbb{R}^+$, consider

$$\|f\|_{(\alpha, \beta)} := \left(\int \prod_{(i, j) \in V_1 \times V_2} f(x_i, y_j)^{\alpha(i, j)} \prod_{(i, j) \in V_1 \times V_2} \overline{f(x_i, y_j)^{\beta(i, j)}} \right)^{1/t}, \quad (3.3)$$

where $t := \sum_{(i, j) \in V} \alpha(i, j) + \beta(i, j)$. The question that we study is: for which α, β , does the function $\|\cdot\|_{(\alpha, \beta)}$ define a norm? For example both formulas

$$\|f\|_{2U_2} := \|f^2\|_{U_2}^{1/2} = \left(\int f(x_0, y_0)^2 f(x_1, y_1)^2 \overline{f(x_0, y_1)^2} \overline{f(x_1, y_0)^2} dx_0 dx_1 dy_0 dy_1 \right)^{1/8}, \quad (3.4)$$

and

$$\left(\int |f(x_0, y_0)|^{\sqrt{2}} |f(x_1, y_1)|^{\sqrt{2}} |f(x_0, y_1)| |f(x_1, y_0)| dx_0 dx_1 dy_0 dy_1 \right)^{1/(2\sqrt{2}+2)}, \quad (3.5)$$

can be defined as $\|\cdot\|_{(\alpha,\beta)}$ for proper choices of functions α and β . They are both always nonnegative, and homogenous with respect to scaling. But do they satisfy the triangle inequality? One of our main results, Theorem 3.4.1, says that if $\|\cdot\|_{(\alpha,\beta)}$ satisfies the triangle inequality, then one of the following two conditions hold:

- **Type I:** There exists a constant $s \geq 1$ such that $\alpha(i, j) = \beta(i, j) \in \{0, s/2\}$, for every $(i, j) \in V_1 \times V_2$;
- **Type II:** For every $(i, j) \in V_1 \times V_2$, $\alpha(i, j) = \beta(i, j) = 0$, $\alpha(i, j) = 1 - \beta(i, j) = 0$, or $1 - \alpha(i, j) = \beta(i, j) = 0$.

It follows from the above result that neither of (3.4) and (3.5) satisfies the triangle inequality. The L_p norm $\|f\|_p = (\int |f(x, y)|^p)^{1/p}$ is an example of a norm of Type I, and $\|\cdot\|_{U_2}$ defined in (3.2) is an example of a norm of Type II.

As we shall see in Remark 3.4.4 it is not true that if (α, β) is of Type I or of Type II, then $\|\cdot\|_{(\alpha,\beta)}$ satisfies the triangle inequality.

3.1 Notations and Definitions

In this section we introduce some notation that will help us to generalize Gowers norms. We have already seen the 2-variable case in (3.3), and the reader is encouraged to have that example in mind while reading the following next few paragraphs. Let $k > 0$ be an integer, V_1, \dots, V_k be finite nonempty sets and $V := V_1 \times \dots \times V_k$. For $\alpha, \beta : V \rightarrow \mathbb{R}$, we call the pair $H = (\alpha, \beta)$ a k -hypergraph pair. The size of H is defined as

$$|H| := \sum_{\omega \in V} |\alpha(\omega)| + |\beta(\omega)|.$$

When we say $H = (\alpha, \beta)$ takes only integer values, we mean that $\text{ran}(\alpha), \text{ran}(\beta) \subseteq \mathbb{Z}$.

Consider two k -hypergraph pairs: $H = (\alpha, \beta)$ over $V = V_1 \times \dots \times V_k$, and $H' = (\alpha', \beta')$ over $W = W_1 \times \dots \times W_k$. An isomorphism from H to H' is a k -tuple $h = (h_1, \dots, h_k)$

such that $h_i : V_i \rightarrow W_i$ are bijections satisfying

$$\alpha(\omega) = \alpha'(h(\omega)), \quad \beta(\omega) = \beta'(h(\omega)),$$

for every $\omega = (\omega_1, \dots, \omega_k) \in V$, where $h(\omega) := (h_1(\omega_1), \dots, h_k(\omega_k))$. We say H is isomorphic to H' , and denote it by $H \cong H'$, if there exists an isomorphism from H to H' .

Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space. Every $\omega = (\omega_1, \dots, \omega_k) \in V$ defines a projection from $\Omega^{V_1} \times \dots \times \Omega^{V_k}$ to Ω^k in the following way: For

$$x = (\{x_{1,v}\}_{v \in V_1}, \{x_{2,v}\}_{v \in V_2}, \dots, \{x_{k,v}\}_{v \in V_k}) \in \Omega^{V_1} \times \dots \times \Omega^{V_k},$$

we have

$$\omega(x) := (x_{1,\omega_1}, \dots, x_{k,\omega_k}) \in \Omega^k.$$

For a measurable function $f : \Omega^k \rightarrow \mathbb{C}$, let $f^H : \Omega^{V_1} \times \dots \times \Omega^{V_k} \rightarrow \mathbb{C}$ be defined as

$$f^H(x) := \left(\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \right) \left(\prod_{\omega \in V} \overline{f(\omega(x))^{\beta(\omega)}} \right),$$

where here, and in the sequel we always assume $0^0 = 1$. As we discussed above we want to use hypergraph pairs to construct normed spaces.

Definition 3.1.1 Consider a k -hypergraph pair $H = (\alpha, \beta)$ with $\alpha, \beta \geq 0$, and a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. Let $L_H(\mathcal{M})$ be the set of functions $f : \Omega^k \rightarrow \mathbb{C}$ with $\| |f| \|_H < \infty$, where for a measurable function $f : \Omega^k \rightarrow \mathbb{C}$,

$$\|f\|_H := \left(\int f^H \right)^{1/|H|}. \quad (3.6)$$

A hypergraph pair is called *norming* (semi-norming), if $\|\cdot\|_H$ defines a norm (semi-norm) on $L_H(\mathcal{M})$ for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$.

Remark 3.1.2 Note that Definition 3.1.1 does not require $L_H(\mathcal{M})$ to be a vector space. However, if $\|\cdot\|_H$ satisfies the axioms of a semi-norm, then $L_H(\mathcal{M})$ will automatically become a vector space. ■

Remark 3.1.3 As the reader might have noticed, the variables and the infinitesimals are missing from the integral in (3.6). To keep the notation simple, here and in the sequel when there is no ambiguity we will omit the variables and infinitesimals from the integrals. ■

Remark 3.1.4 If $H \cong H'$, then for every function f we have $\int f^H = \int f^{H'}$. ■

Lemma 3.1.5 Consider a k -hypergraph pair H . If $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ is a measure space, then for every $f \in L_H(\mathcal{M})$, we have $\|f\|_H \leq \|f\|_p$, where $p := |H|$.

Proof. Suppose that H is defined over $V := V_1 \times \dots \times V_k$. Using Hölder's inequality, we have

$$\begin{aligned} \int f^H &\leq \int \prod_{\omega \in V} |f(\omega(x))|^{\alpha(\omega)+\beta(\omega)} \\ &\leq \prod_{\omega \in V} \left(\int |f(\omega(x))|^{|H|} \right)^{\frac{\alpha(\omega)+\beta(\omega)}{|H|}} = \prod_{\omega \in V} \left(\int |f(x)|^{|H|} \right)^{\frac{\alpha(\omega)+\beta(\omega)}{|H|}} \\ &= \|f\|_p^{|H|}. \end{aligned}$$

■

For a measure space $(\Omega, \mathcal{F}, \mu)$, let us call a complex-valued function $f : \Omega^k \rightarrow \mathbb{C}$ simple, if there exists a partition of Ω into finitely many measurable sets $\Omega_1, \dots, \Omega_m$ such that f is a constant on $\Omega_{i_1} \times \dots \times \Omega_{i_k}$ for every choice of $i_1, \dots, i_k \in \{1, \dots, m\}$.

Corollary 3.1.6 Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space. For every semi-norming k -hypergraph pair H , the set of simple functions is dense in $L_H(\mathcal{M})$.

Proof. Suppose that H is a k -hypergraph pair over $V := V_1 \times \dots \times V_k$. Consider $f \in L_H(\mathcal{M})$. Since f^H is integrable, for every $\epsilon > 0$, there exists a measurable set $\Sigma \subseteq \Omega$ such that $\mu(\Sigma) < \infty$ and

$$\left| \int |f|^H - \int_{\Sigma^V} |f|^H \right| \leq \epsilon.$$

Hence setting $p := |H|$, and denoting by 1_{Σ^k} the indicator function of $\Sigma^k \subseteq \Omega^k$,

$$\begin{aligned} \|f - f1_{\Sigma^k}\|_H &= \left| \int |f(1 - 1_{\Sigma^k})|^H \right|^{1/p} = \left| \int |f|^H - \int |f1_{\Sigma^k}|^H \right|^{1/p} \\ &\leq \left| \int |f|^H - \int_{\Sigma^V} |f|^H \right|^{1/p} \leq \epsilon^{1/p}. \end{aligned}$$

Trivially there exists a simple measurable function $g : \Sigma^k \rightarrow \mathbb{C}$ such that $(\int_{\Sigma^k} |f - g|^p)^{1/p} \leq \epsilon$.

Now by Lemma 3.1.5,

$$\|f1_{\Sigma^k} - g\|_H \leq \|f1_{\Sigma^k} - g\|_p \leq \epsilon.$$

We conclude that

$$\|f - g\|_H \leq \|f - f1_{\Sigma^k}\|_H + \|f1_{\Sigma^k} - g\|_H \leq \epsilon^{1/p} + \epsilon.$$

Since ϵ was an arbitrary constant, the assertion of the corollary follows. \blacksquare

Proposition 3.1.7 *A k -hypergraph pair H is norming (semi-norming), if $\|\cdot\|_H$ defines a norm (semi-norm) on $L_H(\{1, \dots, m\})$, for every positive integer m , where $\{1, \dots, m\}$ is endowed with the counting measure.*

Proof. We only prove the case where H is a semi-norming hypergraph pair, and the proof of the case where H is norming follows from the characterization of semi-norming hypergraphs that are not norming in Section 3.4.3.

Let H be a k -hypergraph pair over $V := V_1 \times \dots \times V_k$. Consider an arbitrary measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$.

If $\int f^H \notin \mathbb{R}^+$, for a function $f \in L_H(\mathcal{M})$, then by Corollary 3.1.6 there exists a simple function $g \in L_H(\mathcal{M})$ with $\int g^H \notin \mathbb{R}^+$. Since g is simple, one can partition Ω into finitely many measurable sets $\Omega_1, \dots, \Omega_m$ such that g is a constant on $\Omega_{i_1} \times \dots \times \Omega_{i_k}$ for every choice of $i_1, \dots, i_k \in \{1, \dots, m\}$. Now define $\tilde{g} : [m]^k \rightarrow \mathbb{C}$ by setting $\tilde{g}(a_1, \dots, a_k)$ to be equal to the value of g on $\Omega_{i_1} \times \dots \times \Omega_{i_k}$ multiplied by $\mu(\Omega_{i_1}) \times \dots \times \mu(\Omega_{i_k})$. Trivially $\int_{[m]^V} \tilde{g}^H = \int g^H \notin \mathbb{R}^+$ which shows that $\|\cdot\|_H$ is not a semi-norm on $L_H(\{1, \dots, m\})$.

Similarly if the triangle inequality fails for $L_H(\mathcal{M})$, then by Corollary 3.1.6, there exist simple functions f, g for which the triangle inequality fails. But similar to the previous case this shows that for some positive integer m , the triangle inequality fails in $L_H(\{1, \dots, m\})$. ■

As one would suspect from Definition 3.1.1, the function $\|\cdot\|_H$ is not a priori a norm. We will pursue the question: “Which hypergraph pairs are norming (semi-norming), and what are the properties of the normed spaces defined by them?”

Remark 3.1.8 Let V_1, \dots, V_k be arbitrary finite sets. For $\psi \in V_1 \times \dots \times V_k$, we denote by 1_ψ the k -hypergraph pair $(\delta_\psi, 0)$, where δ_ψ is the Dirac’s delta function: $\delta_\psi(\omega) = 1$ if $\omega = \psi$, and $\delta_\psi(\omega) = 0$ otherwise.

We will apply arithmetic operations to hypergraph pairs: For example for two hypergraph pairs $H_1 = (\alpha_1, \beta_1)$ and $H_2 = (\alpha_2, \beta_2)$, their sum $H_1 + H_2$ and their difference $H_1 - H_2$ are defined respectively as the pairs $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ and $(\alpha_1 - \alpha_2, \beta_1 - \beta_2)$. For a hypergraph pair $H = (\alpha, \beta)$ define $\overline{H} := (\beta, \alpha)$, and $rH := (r\alpha, r\beta)$ for every $r \in \mathbb{R}$. Now let $H_1 = (\alpha_1, \beta_1)$ be a hypergraph pair over $V_1 \times \dots \times V_k$ and $H_2 = (\alpha_2, \beta_2)$ be a hypergraph pair over $W_1 \times \dots \times W_k$. By considering proper isomorphisms we can assume that W_i and V_i are all disjoint. Then the disjoint union $H_1 \dot{\cup} H_2$ is defined as a hypergraph pair over $(V_1 \dot{\cup} W_1) \times \dots \times (V_k \dot{\cup} W_k)$ whose restrictions to $V_1 \times \dots \times V_k$ and $W_1 \times \dots \times W_k$ are respectively H_1 and H_2 , and is defined to be zero everywhere else. With these definitions, for a measurable function $f : \Omega^k \rightarrow \mathbb{C}$, we have

$$\begin{aligned} f^{H_1+H_2} &= f^{H_1} f^{H_2} \\ f^{H_1-H_2} &= f^{H_1} / f^{H_2} \\ f^{\overline{H}} &= \overline{f^H} \\ f^{rH} &= (f^H)^r = (f^r)^H \\ \int f^{H_1 \dot{\cup} H_2} &= \int f^{H_1} \int f^{H_2}. \end{aligned}$$

■

Consider a hypergraph pair H , and note that $\|\cdot\|_H = \|\cdot\|_{H\dot{\cup}H\dot{\cup}\dots\dot{\cup}H}$. Thus in order to characterize all norming (semi-norming) hypergraph pairs it suffices to consider hypergraph pairs that are minimal according to the following definition:

Definition 3.1.9 *A hypergraph pair H over $V_1 \times \dots \times V_k$ is called minimal if*

- *For every $i \in [k]$ and $v_i \in V_i$, there exists at least one $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega_i = v_i$.*
- *There is no k -hypergraph pair H' such that $H \cong H' \dot{\cup} H' \dot{\cup} \dots \dot{\cup} H'$.*

3.2 Examples

The next couple of examples show that some well-known families of normed spaces fall into the framework defined above.

Example 2 *Let $L_p = (\alpha, \beta)$ be the 1-hypergraph pair defined as $\alpha = \beta = p/2$ over V_1 which contains only one element. Then for a measurable function $f : \Omega \rightarrow \mathbb{C}$, we have*

$$\|f\|_{L_p} = \left(\int f^{p/2} \overline{f^{p/2}} \right)^{1/p} = \left(\int |f|^p \right)^{1/p} = \|f\|_p.$$

Hence in this case the $\|\cdot\|_{L_p}$ norm is the usual L_p norm.

Example 3 *Let $k = 2$, $V_1 = V_2 = \{0, 1, \dots, m-1\}$, for some positive integer m . Define the 2-hypergraph pair $S_{2m} = (\alpha, \beta)$ as*

$$\alpha(i, j) := \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(i, j) := \begin{cases} 1 & i = j + 1 \pmod{m} \\ 0 & \text{otherwise} \end{cases}$$

Let μ be the counting measure on a finite set Ω . Then for $A : \Omega^2 \rightarrow \mathbb{C}$ we have

$$\begin{aligned} \|A\|_{S_{2m}} &= \left(\sum A(x_0, y_0) \overline{A(x_1, y_0)} A(x_1, y_1) \overline{A(x_2, y_1)} \dots A(x_{m-1}, y_{m-1}) \overline{A(x_0, y_{m-1})} \right)^{1/2m} \\ &= (\text{Tr}(AA^*)^m)^{1/2m}, \end{aligned}$$

which shows that in this case the $\|\cdot\|_{S_{2m}}$ norm coincides with the usual $2m$ -trace norm of matrices.

Example 4 Let k be a positive integer and $V_1 = \dots = V_k = \{0, 1\}$, and for $\omega \in V_1 \times \dots \times V_k$,

$$\alpha(\omega) := \sum_{i=1}^k \omega_i \pmod{2}$$

and

$$\beta(\omega) := 1 - \alpha(\omega).$$

Then for the k -hypergraph pair $U_k = (\alpha, \beta)$, $\|\cdot\|_{U_k}$ is called the Gowers k -uniformity norm.

3.3 Constructing norming hypergraph pairs

The following definition introduces the tensor product of two hypergraph pairs.

Definition 3.3.1 Let $H_1 = (\alpha_1, \beta_1)$ be a k -hypergraph pair over $V_1 \times \dots \times V_k$ and $H_2 = (\alpha_2, \beta_2)$ be a k -hypergraph pair over $W_1 \times \dots \times W_k$. Then the tensor product of H_1 and H_2 , is a k -hypergraph pair over $U_1 \times \dots \times U_k$ where $U_i := V_i \times W_i$, defined as

$$H_1 \otimes H_2 := (\alpha_1 \otimes \alpha_2 + \beta_1 \otimes \beta_2, \alpha_1 \otimes \beta_2 + \beta_1 \otimes \alpha_2).$$

We have already seen in Examples 2, 3, 4 that norming hypergraph pairs do exist. Theorem 3.3.2 below shows that it is possible to combine two norming hypergraph pairs to construct a new one.

Theorem 3.3.2 *Let H_1 and H_2 be two hypergraph pairs. If H_1 and H_2 are semi-norming, then $H_1 \otimes H_2$ is also semi-norming. If furthermore at least one of H_1 and H_2 is norming, then $H_1 \otimes H_2$ is norming.*

We shall prove Theorem 3.3.2 in Section 3.4.4. We will also need the following lemma in the sequel.

Lemma 3.3.3 *For positive integers a_1, \dots, a_k , and $\frac{1}{2} \leq p < \infty$, the hypergraph pair $K = (p, p)$ over $[a_1] \times \dots \times [a_k]$ is norming.*

Proof. Let K_{a_1, \dots, a_k} denote the hypergraph pair $(1/2, 1/2)$ over $[a_1] \times \dots \times [a_k]$. First we establish the lemma for the case that $p = 1/2$, $a_1 = \dots = a_{k-1} = 1$. Note that for a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, and $f : \Omega^k \rightarrow \mathbb{C}$,

$$\int f^H = \int_{x_1, \dots, x_{k-1}} \int_{y_i: i \in [a_k]} |f(x_1, \dots, x_{k-1}, y_i)| = \int_{x_1, \dots, x_{k-1}} \left(\int_y |f(x_1, \dots, x_{k-1}, y)| \right)^{a_k}.$$

In order to show that $K_{1,1,\dots,1,a_k}$ is norming, by Lemma 3.4.9 below, we have to show that for $f, g : \Omega^k \rightarrow \mathbb{C}$

$$\int |g(x_1, \dots, x_{k-1}, y_1)| \prod_{v \in [a_k] \setminus \{1\}} |f(x_1, \dots, x_{k-1}, y_i)| \leq \|f\|_{K_{1,1,\dots,1,a_k}}^{a_k-1} \|g\|_{K_{1,1,\dots,1,a_k}}. \quad (3.7)$$

Note that by Hölder's inequality

$$\begin{aligned} \text{L.H.S. of (3.7)} &= \int_{x_1, \dots, x_{k-1}} \left(\int_y |g(x_1, \dots, x_{k-1}, y)| \right) \left(\int_y |f(x_1, \dots, x_{k-1}, y)| \right)^{a_k-1} \\ &\leq \left(\int_{x_1, \dots, x_{k-1}} \left(\int_y |g(x_1, \dots, x_{k-1}, y)| \right)^{a_k} \right)^{1/a_k} \times \\ &\quad \left(\int_{x_1, \dots, x_{k-1}} \left(\int_y |f(x_1, \dots, x_{k-1}, y)| \right)^{a_k} \right)^{(a_k-1)/a_k} \\ &= \|f\|_{K_{1,1,\dots,1,a_k}}^{a_k-1} \|g\|_{K_{1,1,\dots,1,a_k}}. \end{aligned}$$

This establishes the case $p = 1$, $a_1 = \dots = a_{k-1} = 1$. Also note that the case $a_1 = \dots = a_k = 1$, and arbitrary p is trivial. The general case then follows from Theorem 3.3.2 as

$$pK_{a_1, \dots, a_k} \cong pK_{1, \dots, 1} \otimes K_{a_1, 1, \dots, 1} \otimes K_{1, a_2, \dots, 1} \otimes \dots \otimes K_{1, \dots, 1, a_k}.$$

■

3.4 Structure of Norming hypergraph pairs

In this section we study the structure of semi-norming hypergraph pairs. The main result that we prove in this direction is the following.

Theorem 3.4.1 *Let $H = (\alpha, \beta)$ be a semi-norming hypergraph pair. Then $H \cong \overline{H}$, and one of the following two cases hold*

- **Type I:** *There exists a real $s \geq 1$, such that for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\alpha(\psi) = \beta(\psi) = s/2$. In this case, s is called the **parameter** of H .*
- **Type II:** *For every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, we have $\{\alpha(\psi), \beta(\psi)\} = \{0, 1\}$.*

Note that the condition $H \cong \overline{H}$ is trivially satisfied for every hypergraph pair that satisfies the requirements of Type I hypergraph pairs. This is not true for Type II hypergraph pairs, and in this case $H \cong \overline{H}$ implies a further restriction on the structure of the hypergraph pair.

Remark 3.4.2 Note that if H is of Type I, then for every measure space \mathcal{M} and every $f \in L_H(\mathcal{M})$, we have $\|f\|_H = \| |f| \|_H$. This fact will be used frequently in the sequel. ■

Suppose that $H = (\alpha, \beta)$ is a k -hypergraph pair over $V_1 \times \dots \times V_k$. For a subset $S \subseteq [k]$, we use the notation π_S to denote the natural projection from $V_1 \times \dots \times V_k$ to $\prod_{i \in S} V_i$. We can construct a hypergraph pair $H_S := (\alpha_S, \beta_S)$ where $\alpha_S, \beta_S : \prod_{i \in S} V_i \rightarrow \mathbb{C}$ are defined as

$$\alpha_S : \omega \mapsto \sum \{ \alpha(\omega') : \pi_S(\omega') = \omega \},$$

and

$$\beta_S : \omega \mapsto \sum \{ \beta(\omega') : \pi_S(\omega') = \omega \}.$$

We have the following trivial observation:

Lemma 3.4.3 *If $H = (\alpha, \beta)$ is a norming (semi-norming) k -hypergraph pair, then for every $S \subseteq [k]$, H_S is a norming (semi-norming) $|S|$ -hypergraph pair.*

Proof. Suppose that $\|\cdot\|_H$ is a norm (semi-norm) on $L_H(\mathcal{M})$ for a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, and a k -hypergraph pair H over $V := V_1 \times \dots \times V_k$. We might assume that H is minimal according to Definition 3.1.9. Let $h : \Omega \rightarrow \{0, 1\}$ be an arbitrary zero-one integrable function with $\int h = 1$.

Consider a measurable function $f : \Omega^{|S|} \rightarrow \mathbb{C}$ in $L_{H_S}(\mathcal{M})$. Let $S = \{i_1, \dots, i_{|S|}\}$. Define $\tilde{f} : \Omega^k \rightarrow \mathbb{C}$ by $\tilde{f}(x_1, \dots, x_k) = f(x_{i_1}, \dots, x_{i_{|S|}}) \prod_{j \neq S} h(x_j)$. Note that for every $\omega \in V_1 \times \dots \times V_k$, and $x \in \Omega^V$,

$$\begin{aligned} \tilde{f}(\omega(x))^{\alpha(\omega)} \overline{\tilde{f}(\omega(x))^{\beta(\omega)}} &= \tilde{f}(x_{1,\omega_1}, \dots, x_{k,\omega_k})^{\alpha(\omega)} \overline{\tilde{f}(x_{1,\omega_1}, \dots, x_{k,\omega_k})^{\beta(\omega)}} \\ &= f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\alpha(\omega)} \overline{f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\beta(\omega)}} \prod_{j \notin S} h(x_{j,\omega_j}). \end{aligned}$$

Hence using the assumptions that h is a zero-one function and it has integral 1, and also the minimality of H according to the first condition of Definition 3.1.9, we have

$$\begin{aligned} \int \tilde{f}^H &= \int_{\Omega^V} \prod_{\omega \in V} f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\alpha(\omega)} \overline{f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\beta(\omega)}} \prod_{j \notin S} h(x_{j,\omega_j}) \\ &= \int_{\Omega^{V_{i_1} \times \dots \times \Omega^{V_{i_{|S|}}}} \prod_{\omega \in V} f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\alpha(\omega)} \overline{f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\beta(\omega)}} \\ &\quad \times \prod_{j \notin S, v \in V_j} \int_{\Omega} h(x_{j,v}) \\ &= \int_{\Omega^{V_{i_1} \times \dots \times \Omega^{V_{i_{|S|}}}} \prod_{\omega \in V} f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\alpha(\omega)} \overline{f(x_{i_1,\omega_{i_1}}, \dots, x_{i_k,\omega_{i_k}})^{\beta(\omega)}} \\ &= \int_{\Omega^{V_{i_1} \times \dots \times \Omega^{V_{i_{|S|}}}} f^{H_S}, \end{aligned}$$

which shows that $\|f\|_{H_S} = \|\tilde{f}\|_H$. We conclude that $\|\cdot\|_{H_S}$ is a norm (semi-norm) on $L_{H_S}(\mathcal{M})$. ■

Remark 3.4.4 The importance of Lemma 3.4.3 is in that one can apply Theorem 3.4.1 to H_S to deduce more conditions on the structure of the original semi-norming hypergraph pair H . For example applying Theorem 3.4.1 to H_S when S has only one element implies that for every $1 \leq i \leq k$, there exists a number d_i such that for every $v_i \in V_i$, we have $\sum \{\alpha(\omega) : \omega_i = v_i\} = \sum \{\beta(\omega) : \omega_i = v_i\} = d_i$. ■

The next theorem gives another necessary condition on the structure of a semi-norming hypergraph pair.

Theorem 3.4.5 *Suppose that $H = (\alpha, \beta)$ is a semi-norming k -hypergraph pair over $V_1 \times \dots \times V_k$ which is not the disjoint union of any two hypergraph pairs. Let $W_i \subseteq V_i$ for $i = 1, \dots, k$, and H' be the restriction of H to $W_1 \times \dots \times W_k$. Then*

$$\frac{|H'|}{|W_1| + \dots + |W_k| - 1} \leq \frac{|H|}{|V_1| + \dots + |V_k| - 1}.$$

Remark 3.4.6 Note that Theorem 3.4.5 requires that H is not the disjoint union of any two hypergraph pairs. Semi-norming hypergraph pairs that are disjoint unions of two or more hypergraph pairs are studied in Section 3.4.2. ■

We present the proofs of Theorems 3.4.1 and 3.4.5 in Section 3.4.6, but first we need to develop some tools.

3.4.1 Two Hölder type inequalities

One of our main tools in the study of hypergraph norms is the trick of amplification by taking tensor powers. This trick has been used successfully in many places (see for example [49]).

Definition 3.4.7 *For $f, g : \Omega^k \rightarrow \mathbb{C}$, the tensor product of f and g is defined as $f \otimes g : (\Omega^2)^k \rightarrow \mathbb{C}$ where $f \otimes g[(x_1, y_1), \dots, (x_k, y_k)] = f(x_1, \dots, x_k)g(y_1, \dots, y_k)$.*

We have the following trivial observation.

Lemma 3.4.8 *Let H_1, H_2 be two k -hypergraph pairs over the same set $V := V_1 \times \dots \times V_k$, and $f_1, f_2, g_1, g_2 : \Omega^k \rightarrow \mathbb{C}$. Then*

$$\int (f_1 \otimes f_2)^{H_1} (g_1 \otimes g_2)^{H_2} = \left(\int f_1^{H_1} g_1^{H_2} \right) \left(\int f_2^{H_1} g_2^{H_2} \right).$$

Proof. Note that every

$$z = (\{[x_{1,v}, y_{1,v}]\}_{v \in V_1}, \dots, \{[x_{k,v}, y_{k,v}]\}_{v \in V_k}) \in (\Omega^2)^V$$

corresponds to a pair $[x, y] \in \Omega^V \times \Omega^V$, where $x = (\{x_{1,v}\}_{v \in V_1}, \dots, \{x_{k,v}\}_{v \in V_k})$, and $y = (\{y_{1,v}\}_{v \in V_1}, \dots, \{y_{k,v}\}_{v \in V_k})$. Now for every $\omega \in V$,

$$f_1 \otimes f_2(\omega(z)) = f_1(\omega(x))f_2(\omega(y)),$$

and

$$g_1 \otimes g_2(\omega(z)) = g_1(\omega(x))g_2(\omega(y)),$$

which in turn implies the assertion of the lemma. \blacksquare

Now with Lemma 3.4.8 in hand, we can prove our first result about semi-norming hypergraph pairs.

Lemma 3.4.9 *Let $H = (\alpha, \beta)$ be a semi-norming hypergraph pair. Then for every measure space \mathcal{M} , and every $f, g \in L_H(\mathcal{M})$ the following holds. For every $\psi \in \text{supp}(\alpha)$,*

$$\left| \int f^{H-1\psi} g^{1\psi} \right| \leq \|f\|_H^{|H|-1} \|g\|_H, \quad (3.8)$$

and for every $\psi \in \text{supp}(\beta)$

$$\left| \int f^{H-1\bar{\psi}} g^{1\bar{\psi}} \right| \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (3.9)$$

Conversely, if for a measure space \mathcal{M} , and every $f, g \in L_H(\mathcal{M})$, $\int f^H \in \mathbb{R}^+$, and at least one of (3.8) or (3.9) holds for some $\psi \in V_1 \times \dots \times V_k$, then $\|\cdot\|_H$ is a semi-norm on $L_H(\mathcal{M})$.

Proof. First we prove the converse direction which is easier. Consider two measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$ and suppose that (3.8) holds for some $\psi \in V_1 \times \dots \times V_k$. Then

$$\begin{aligned} \|f + g\|_H^{|H|} &= \int (f + g)^H = \int (f + g)^{H-1\psi} (f + g)^{1\psi} \\ &= \int (f + g)^{H-1\psi} f^{1\psi} + \int (f + g)^{H-1\psi} g^{1\psi} \\ &\leq \|f + g\|_H^{|H|-1} \|f\|_H + \|f + g\|_H^{|H|-1} \|g\|_H, \end{aligned}$$

which simplifies to the triangle inequality. The proof of the case where (3.9) holds is similar.

Now let us turn to the other direction. Suppose that H is a semi-norming hypergraph pair. Consider $f, g \in L_H(\mathcal{M})$. We might assume that $\|f\|_H \neq 0$, as otherwise one can instead consider a small perturbation of f . Since $\|\cdot\|_H$ is a semi-norm, for every $t \in \mathbb{R}^+$ and every $f, g : \Omega \rightarrow \mathbb{C}$, we have $\|f + tg\|_H \leq \|f\|_H + t\|g\|_H$ which implies that

$$\left. \frac{d\|f + tg\|_H}{dt} \right|_0 \leq \|g\|_H. \quad (3.10)$$

Computing the derivative

$$\frac{d(f + tg)^H}{dt} = \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi)(f + tg)^{H-1_\psi} g^{1_\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi)(f + tg)^{H-1_{\bar{\psi}}} g^{1_{\bar{\psi}}},$$

shows that

$$\begin{aligned} \frac{d\|f + tg\|_H}{dt} &= \frac{1}{|H|} \|f + tg\|_H^{1-|H|} \times \\ &\quad \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi)(f + tg)^{H-1_\psi} g^{1_\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi)(f + tg)^{H-1_{\bar{\psi}}} g^{1_{\bar{\psi}}} \right). \end{aligned}$$

Thus by (3.10),

$$\frac{1}{|H|} \|f\|_H^{1-|H|} \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) f^{H-1_\psi} g^{1_\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) f^{H-1_{\bar{\psi}}} g^{1_{\bar{\psi}}} \right) \leq \|g\|_H,$$

or equivalently

$$\frac{1}{|H|} \left(\int \sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) f^{H-1_\psi} g^{1_\psi} + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) f^{H-1_{\bar{\psi}}} g^{1_{\bar{\psi}}} \right) \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (3.11)$$

Since (3.11) holds for every measure space and every pair of measurable functions, for every integer $m > 0$, we can replace f and g in (3.11), respectively with $f^{\otimes m} \otimes \bar{f}^{\otimes m}$ and $g^{\otimes m} \otimes \bar{g}^{\otimes m}$, and apply Lemma 3.4.8 to obtain

$$\begin{aligned} \frac{1}{|H|} \left(\sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) \left| \int f^{H-1_\psi} g^{1_\psi} \right|^{2m} + \right. \\ \left. \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) \left| \int f^{H-1_{\bar{\psi}}} g^{1_{\bar{\psi}}} \right|^{2m} \right) \leq \left(\|f\|_H^{|H|-1} \|g\|_H \right)^{2m}. \quad (3.12) \end{aligned}$$

But since (3.12) holds for every m , it establishes (3.8) and (3.9) as

$$\frac{1}{|H|} \left(\sum_{\psi \in \text{supp}(\alpha)} \alpha(\psi) + \sum_{\psi \in \text{supp}(\beta)} \beta(\psi) \right) = 1.$$

■

We have the following corollary to Lemma 3.4.9.

Corollary 3.4.10 *If H is a semi-norming hypergraph pair, then $\alpha(\omega) + \beta(\omega) \geq 1$, for every $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$.*

Proof. Let the underlying measure space be the set $\{0, 1\}$ with the counting measure. Consider $\omega \in \text{supp}(\alpha)$, and note that by (3.8), for every pair of functions $f, g : \{0, 1\}^k \rightarrow \mathbb{C}$, we have

$$\left| \int f^{H-1_\psi} g^{1_\psi} \right| \leq \|f\|_H^{|H|-1} \|g\|_H. \quad (3.13)$$

For every $x = (x_1, \dots, x_k) \in \{0, 1\}^k$, define $g(x) := 1$ and

$$f(x) := \begin{cases} \epsilon & x_1 = \dots = x_k = 1 \\ 1 & \text{otherwise,} \end{cases}$$

where $0 \leq \epsilon \leq 1$. Then $|\int f^{H-1_\psi} g^{1_\psi}| = |\int f^{H-1_\psi}| \geq \epsilon^{\alpha(\omega)+\beta(\omega)-1}$, while $\|f\|_H \leq \|g\|_H = \|1\|_H$, which contradicts (3.13) for sufficiently small $\epsilon > 0$, if $\alpha(\omega) + \beta(\omega) < 1$. ■

Under some extra conditions it is possible to extend (3.8) and (3.9) to a much more powerful inequality.

Lemma 3.4.11 *Let H be a semi-norming hypergraph pair, and H_1, \dots, H_n be nonzero¹ and nonnegative hypergraph pairs satisfying $H_1 + H_2 + \dots + H_n = H$. Then for every measure space \mathcal{M} and functions $f_1, f_2, \dots, f_n \in L_H(\mathcal{M})$, we have*

$$\left| \int f_1^{H_1} f_2^{H_2} \dots f_n^{H_n} \right| \leq \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|},$$

provided that at least one of the following two conditions hold:

¹i.e. $H_i \neq (0, 0)$ for every $1 \leq i \leq n$.

(a) $f_1, \dots, f_n \geq 0$.

(b) For every $H_i = (\alpha_i, \beta_i)$, the functions α_i, β_i take only integer values.

Proof. Let us first assume that $f_1, \dots, f_n \geq 0$. Suppose to the contrary that

$$\int f_1^{H_1} f_2^{H_2} \dots f_n^{H_n} > \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|}. \quad (3.14)$$

After normalization we can assume that $\|f_1\|_H, \|f_2\|_H, \dots, \|f_n\|_H \leq 1$ while the right-hand side of (3.14) is strictly greater than 1. Since (3.14) remains valid after small perturbations of f_i 's, without loss of generality we might also assume that for every $1 \leq i \leq n$, f_i does not take the zero value on any point. Consider a positive integer m , and note that by Lemma 3.4.8

$$\begin{aligned} \int \left(\sum_{i=1}^n f_i^{\otimes m} \right)^H &= \int \prod_{i=1}^n \left(\sum_{i=1}^n f_i^{\otimes m} \right)^{H_i} \\ &= \int (f_1^{\otimes m})^{H_1} \dots (f_n^{\otimes m})^{H_n} \left(\prod_{i=1}^n \left(\frac{f_1^{\otimes m} + \dots + f_n^{\otimes m}}{f_i^{\otimes m}} \right)^{H_i} \right) \\ &\geq \int (f_1^{\otimes m})^{H_1} \dots (f_n^{\otimes m})^{H_n} = \left(\int f_1^{H_1} \dots f_n^{H_n} \right)^m. \end{aligned}$$

On the other hand, Lemma 3.4.8 shows that $\|f_i^{\otimes m}\|_H = \|f_i\|_H^m \leq 1$ for every $i \in [n]$.

Then for sufficiently large m we get a contradiction:

$$\left\| \sum f_i^{\otimes m} \right\|_H \geq \left(\int f_1^{H_1} \dots f_n^{H_n} \right)^{m/|H|} > n.$$

Next consider the case where f_i are not necessarily positive, but we know that α_i, β_i all take only integer values. Again to get a contradiction assume that

$$\left| \int f_1^{H_1} \dots f_n^{H_n} \right| > 1 \geq \|f_1\|_H^{|H_1|} \dots \|f_n\|_H^{|H_n|}.$$

where $\|f_1\|_H, \dots, \|f_n\|_H \leq 1$. In this case for every $i \in [n]$, we will consider $f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m}$.

Let \mathcal{H} denote the set of all n -tuples of nonzero hypergraph pairs $(H'_1, H'_2, \dots, H'_n)$ where H'_i 's take only nonnegative integer values and $H'_1 + H'_2 + \dots + H'_n = H$. By Lemma 3.4.8

$$\int \prod_{i=1}^n (f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m})^{H'_i} = \left| \int \prod_{i=1}^n f_i^{H'_i} \right|^{\otimes 2m} \geq 0.$$

Now by expanding the product defined by H , we have

$$\begin{aligned} \int \left(\sum_{i=1}^n f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m} \right)^H &= \sum_{(H'_1, \dots, H'_n) \in \mathcal{H}} \int \prod_{i=1}^n \left(f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m} \right)^{H'_i} \\ &\geq \int \prod_{i=1}^n \left(f_i^{\otimes m} \otimes \overline{f_i}^{\otimes m} \right)^{H_i} = \left| \int \prod_{i=1}^n f_i^{H_i} \right|^{2m}, \end{aligned}$$

which leads to a contradiction similar to the previous case. \blacksquare

Remark 3.4.12 It is possible to show that Lemma 3.4.11 does not necessarily hold in the general case where none of the two conditions are satisfied. To see this consider S_4 from Example 3. If Lemma 3.4.11 holds for the decomposition $S_4 = \frac{1}{3}S_4 + \frac{1}{3}S_4 + \frac{1}{3}S_4$, then by Lemma 3.4.9 $3S_4$ would be a semi-norming hypergraph pair. But Theorem 3.4.1 implies that $3S_4$ is not a semi-norming hypergraph pair. \blacksquare

Consider a *probability space* $\mathcal{P} = (\Omega, \mathcal{F}, \mu)$. It is well-known that for every $1 \leq p \leq q$, and for every $f \in L_q(\mathcal{P})$, we have $\|f\|_p \leq \|f\|_q$. Indeed by Hölder's inequality $\int |f|^p \leq (\int |f|^q)^{p/q} (\int 1^{q/(q-p)})^{1-p/q} = (\int |f|^q)^{p/q}$ which simplifies to $\|f\|_p \leq \|f\|_q$. The next corollary generalizes this to hypergraph pairs.

Corollary 3.4.13 *Let $H = (\alpha, \beta)$ be a semi-norming k -hypergraph pair. Consider a probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mu)$ and $f \in L_H(\mathcal{P})$. Let $K = (\alpha', \beta')$ be a nonzero k -hypergraph pair over the same domain as H such that $0 \leq \alpha' \leq \alpha$ and $0 \leq \beta' \leq \beta$. Then*

$$\|f\|_K \leq \|f\|_H,$$

provided that at least one of the following three conditions holds:

- (a) $f \geq 0$.
- (b) H is of type I.
- (c) The functions $\alpha, \beta, \alpha', \beta'$ take only integer values.

Proof. Parts (a) and (b) follow from applying Lemma 3.4.11 (a) with parameters $n := 2$, $H_1 := K$, $H_2 := H - K$, $f_1 := |f|$ and $f_2 := 1$.

Part (c) follows from applying Lemma 3.4.11 (b) with parameters $n := 2$, $H_1 := K$, $H_2 := H - K$, $f_1 := f$ and $f_2 := 1$. ■

3.4.2 Factorizable hypergraph pairs

In this section we characterize all norming and semi-norming 1-hypergraph pairs. As it is mentioned before, it suffices to consider the hypergraph pairs that are minimal according to Definition 3.1.9. We have already seen one class of examples of norming 1-hypergraph pairs, namely the 1-hypergraph pairs L_p of Example 2. There exists also a semi-norming 1-hypergraph pair that is not norming: Let $G = (1, 0)$ be the 1-hypergraph pair over a set V_1 of size 1. Then for a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and a measurable $f : \Omega \rightarrow \mathbb{C}$ we have $\|f\|_{G\dot{\cup}\bar{G}} = \int |f|$ which defines a semi-norm. The next proposition shows that these are the only examples.

Proposition 3.4.14 *If H is a minimal norming 1-hypergraph pair, then there exists $1 \leq p < \infty$ such that $H \cong L_p$. If H is a minimal semi-norming 1-hypergraph pair that is not norming, then $H \cong G\dot{\cup}\bar{G}$, where $G = (1, 0)$ is a 1-hypergraph pair over a set V_1 of size 1.*

To prove Proposition 3.4.14 we need to study the hypergraph pairs which are decomposable into disjoint union of other hypergraph pairs.

Definition 3.4.15 *A hypergraph pair $H = (\alpha, \beta)$ is called factorizable, if it is a disjoint union of two hypergraph pairs.*

Remark 3.4.16 Let $H = (\alpha, \beta)$ be a hypergraph pair over $V := V_1 \times V_2 \times \dots \times V_k$. By considering a proper isomorphism we can assume that V_i are mutually disjoint. Then H is *non-factorizable*, if and only if the hypergraph on $V_1 \dot{\cup} \dots \dot{\cup} V_k$ whose

edges are tuples in $\text{supp}(\alpha) \cup \text{supp}(\beta)$ is connected. Trivially in this case for every $1 \leq i \leq j \leq k$, and every $u \in V_i$ and $v \in V_j$, there exists a sequence of elements $(\omega_{1,1}, \dots, \omega_{1,k}), \dots, (\omega_{m,1}, \dots, \omega_{m,k}) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, for some $m \in \mathbb{N}$ such that

- $\omega_{1,i} = u$ and $\omega_{m,j} = v$;
- For every $1 < t \leq m$, $\omega_{t-1,s} = \omega_{t,s}$ for some $1 \leq s \leq k$.

■

The next proposition shows that two non-factorizable hypergraph pairs define identical norms, if and only if they are isomorphic. For the proof, we need an easy fact stated in the following Remark.

Remark 3.4.17 Let x_1, \dots, x_n be n complex variables. Define a term as a product $\prod_{i=1}^n x_i^{p_i} \overline{x_i}^{q_i}$, where p_i, q_i are nonnegative reals. Now let P and Q be two formal finite sums of terms. It is easy to see that P and Q are equal as functions on \mathbb{C}^n , if and only if they are equal as formal sums. ■

Proposition 3.4.18 *Let H_1 and H_2 be two minimal k -hypergraph pairs. Suppose that either H_1 and H_2 are both non-factorizable, or we have $|H_1| = |H_2|$. Then*

- *If for every measure space $(\Omega, \mathcal{F}, \mu)$, and every $f : \Omega^k \rightarrow \mathbb{C}$, $\|f\|_{H_1} = \|f\|_{H_2}$, then $H_1 \cong H_2$.*
- *If for every measure space $(\Omega, \mathcal{F}, \mu)$, and every $f : \Omega^k \rightarrow \mathbb{C}$, $\|f\|_{H_1} = \overline{\|f\|_{H_2}}$, then $H_1 \cong \overline{H_2}$.*

Proof. Suppose that H_1 and H_2 are respectively defined over $V_1 \times \dots \times V_k$ and $W_1 \times \dots \times W_k$. First assume that H_1 and H_2 are both non-factorizable. Let μ be the counting measure on $\Omega = [m]$, where $m > \sum_{i=1}^k |V_i| + |W_i|$ is a positive integer. Suppose that for every $f : \Omega^k \rightarrow \mathbb{C}$ with have $\|f\|_{H_1} = \|f\|_{H_2}$. Then define $f(x_1, \dots, x_k)$ to be

equal to 1, if $x_1 = \dots = x_k$, and equal to 0 otherwise. Consider $x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$. Note that if $\omega = (\omega_1, \dots, \omega_k) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, then $f(\omega(x)) = 1$ implies that $x_{1,\omega_1} = \dots = x_{k,\omega_k}$. Hence since H is non-factorizable, by Remark 3.4.16, for $x = (\{x_{i,v}\}_{v \in V_1}, \dots, \{x_{i,v}\}_{v \in V_k}) \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$, $\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))^{\beta(\omega)}} = 1$ if and only if all coordinates of x are equal. This shows that $\int f^{H_1} = |\Omega|$, and similarly since H_2 is non-factorizable $\int f^{H_2} = |\Omega|$. We deduce from $\|f\|_{H_1} = \|f\|_{H_2}$ that $|H_1| = |H_2|$. So it is sufficient to prove the proposition for the case where $|H_1| = |H_2|$.

Next for every $f : \Omega^k \rightarrow \mathbb{C}$, we assume $\int f^{H_1} = \int f^{H_2}$. For $1 \leq i \leq k$, consider $f_i : \Omega^k \rightarrow \{0, 1\}$ defined as $f_i(x_1, \dots, x_k) = 1$ if and only if $x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_k = 1$. Then using a similar argument to the one in the previous paragraph, by minimality of H_1 (see Definition 3.1.9), we conclude that for $x = (\{x_{i,v}\}_{v \in V_1}, \dots, \{x_{i,v}\}_{v \in V_k}) \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$, $\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))^{\beta(\omega)}} = 1$ if and only if all $\{x_{j,v} : j \neq i, v \in V_j\}$ are equal to 1. This shows that $\int f_i^{H_1} = |\Omega|^{|V_i|}$ and $\int f_i^{H_2} = |\Omega|^{|W_i|}$ which implies $|V_i| = |W_i|$. Thus without loss of generality we may assume that $V_i = W_i = \{1, \dots, |V_i|\}$, for every $1 \leq i \leq k$. Now for every $f : \Omega^k \rightarrow \mathbb{C}$ we have

$$\sum_{x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}} \prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))^{\beta(\omega)}} = \sum_{x \in \Omega^{V_1} \times \dots \times \Omega^{V_k}} \prod_{\omega \in V} f(\omega(x))^{\alpha'(\omega)} \overline{f(\omega(x))^{\beta'(\omega)}} \quad (3.15)$$

Consider $x = [(1, \dots, |V_1|), (1, \dots, |V_2|), \dots, (1, \dots, |V_k|)] \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$. Then $\omega(x) = \omega$ for every $\omega \in V$, and hence

$$\prod_{\omega \in V} f(\omega(x))^{\alpha(\omega)} \overline{f(\omega(x))^{\beta(\omega)}} = \prod_{\omega \in V} f(\omega)^{\alpha(\omega)} \overline{f(\omega)^{\beta(\omega)}}. \quad (3.16)$$

Since (3.16) appears in the sum in the left-hand side of (3.15), by Remark 3.4.17 it must also appear as a term in the right-hand side of (3.15). Hence there exists $y = [(y_{1,1}, \dots, y_{1,|V_1|}), \dots, (y_{k,1}, \dots, y_{k,|V_k|})] \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$ such that

$$\prod_{\omega \in V} f(\omega(y))^{\alpha'(\omega)} \overline{f(\omega(y))^{\beta'(\omega)}} = \prod_{\omega \in V} f(\omega)^{\alpha(\omega)} \overline{f(\omega)^{\beta(\omega)}}. \quad (3.17)$$

By minimality, for every $v \in V_i$, there exists $\omega = (\omega_1, \dots, \omega_k) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega_i = v$. This implies $\{y_{i,1}, \dots, y_{i,|V_i|}\} = V_i$, for every $1 \leq i \leq k$. Now $h = (h_1, \dots, h_k)$

defined as $h_i : j \mapsto y_{i,j}$ (for every $1 \leq i \leq k$ and $1 \leq j \leq |V_i|$) is an isomorphism between H_1 and H_2 .

In the second part of the proposition where it is assumed $\|f\|_{H_1} = \overline{\|f\|_{H_2}}$, instead of (3.15) one obtains that the left-hand side of (3.15) is equal to the conjugate of the right-hand side. The proof then proceeds similar to the previous case. \blacksquare

Theorem 3.4.19 *Let $H = H_1 \dot{\cup} H_2 \dot{\cup} \dots \dot{\cup} H_m$ be a semi-norming hypergraph pair such that H_i are all non-factorizable. Then for every measure space \mathcal{M} and every $f \in L_H(\mathcal{M})$ we have*

$$\|f\|_{H_1 \dot{\cup} H_1} = \|f\|_{H_2 \dot{\cup} H_2} = \dots = \|f\|_{H_m \dot{\cup} H_m} = \|f\|_H.$$

Proof. Let $H = G_1 \dot{\cup} G_2$ be semi-norming, where G_1 and G_2 are not necessarily non-factorizable, $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space, and $f \in L_H(\mathcal{M})$. Since $\int f^H = \int f^{G_1} \int f^{G_2}$, we have

$$\|f\|_H = \|f\|_{G_1}^{\frac{|G_1|}{|H|}} \|f\|_{G_2}^{\frac{|G_2|}{|H|}} = \|f\|_{G_1}^{\frac{|G_1|}{|G_1|+|G_2|}} \|f\|_{G_2}^{\frac{|G_2|}{|G_1|+|G_2|}}.$$

It follows from Theorem 3.4.1 that either H is of Type I, or H and G_1 both take only integer values. Hence by Corollary 3.4.13

$$\left| \|f\|_{G_1}^{|G_1|} \right| \leq \|f\|_H^{|G_1|} = \|f\|_{G_1}^{\frac{|G_1|^2}{|G_1|+|G_2|}} \|f\|_{G_2}^{\frac{|G_1||G_2|}{|G_1|+|G_2|}},$$

which simplifies to

$$\| \|f\|_{G_1} \| \leq \| \|f\|_{G_2} \|.$$

Similarly one can show that $\| \|f\|_{G_2} \| \leq \| \|f\|_{G_1} \|$, and thus $\| \|f\|_{G_1} \| = \| \|f\|_{G_2} \|$.

By induction we conclude that $\| \|f\|_{H_1} \| = \dots = \| \|f\|_{H_2} \|$, for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and every $f \in L_H(\mathcal{M})$, and this completes the proof as $\| \|f\|_{H_i} \| = \| \|f\|_{H_i \dot{\cup} H_i} \|$. \blacksquare

Now we can state the proof of Proposition 3.4.14.

Proof.[Proposition 3.4.14] Consider a semi-norming 1-hypergraph pair H over a set $V_1 = \{v_1, \dots, v_m\}$. Consider the factorization $H = H_1 \dot{\cup} H_2 \dot{\cup} \dots \dot{\cup} H_m$, where H_i is a

1-hypergraph pair over $\{v_i\}$. By Theorem 3.4.19, always $\|f\|_{H_1 \dot{\cup} \overline{H_1}} = \|f\|_{H_2 \dot{\cup} \overline{H_2}} = \dots = \|f\|_{H_m \dot{\cup} \overline{H_m}} = \|f\|_H$. By Theorem 3.4.1, for every $1 \leq i \leq m$, either $H_i \dot{\cup} \overline{H_i} \cong L_p \dot{\cup} \overline{L_p}$ for some $1 \leq p < \infty$, or $H_i \dot{\cup} \overline{H_i} \cong G \dot{\cup} \overline{G}$ which completes the proof. ■

3.4.3 Semi-norming hypergraph pairs that are not norming

In this section we study the structure of the semi-norming hypergraph pairs which are *not* norming. Consider a semi-norming k -hypergraph pair $H = (\alpha, \beta)$ over $V := V_1 \times \dots \times V_k$ of Type I with parameter $s = 2m$, where m is a positive integer. Since H is of Type I, it is trivially norming. Consider an arbitrary positive integer k' . We want to use H to construct a semi-norming $(k + k')$ -hypergraph pair that is not norming. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. For every integrable function $f : \Omega^{k+k'} \rightarrow \mathbb{C}$, define $F_f : \Omega^k \rightarrow \mathbb{C}$ as $F_f(x_1, \dots, x_k) = \int f(x_1, \dots, x_{k+k'}) dx_{k+1} \dots dx_{k+k'}$. Since $\|\cdot\|_H$ is norming, the two identities $F_{\lambda f} = \lambda F_f$ and $F_{f+g} = F_f + F_g$ show that the function $\|\cdot\| : f \mapsto \|F_f\|_H$ satisfies the axioms of a semi-norm. Furthermore, any function f with $F_f = 0$ satisfies $\|f\| = 0$. For a generic measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ trivially there exist functions $f \neq 0$ with $F_f = 0$ which shows that in that case $\|\cdot\|$ is a semi-norm which is not a norm. It remains to show that $\|\cdot\|$ can be formulated with the hypergraph pair

notation. Indeed recalling that H is of Type I with parameter $2m$,

$$\begin{aligned}
 \|f\| &= \|F_f\|_H = \int \prod_{\omega \in \text{supp}(\alpha)} |F_f(\omega(x))|^{2m} \\
 &= \int \prod_{\omega \in \text{supp}(\alpha)} \left| \int f(\omega(x), y_1, \dots, y_{k'}) dy_1 \dots dy_{k'} \right|^{2m} \\
 &= \int \prod_{\omega \in \text{supp}(\alpha)} \left(\prod_{j=1}^m \int f(\omega(x), y_1, \dots, y_{k'}) dy_1 \dots dy_{k'} \right) \\
 &\quad \left(\prod_{j'=m+1}^{2m} \int \overline{f(\omega(x), y_1, \dots, y_{k'})} dy_1 \dots dy_{k'} \right) \\
 &= \int \prod_{\omega \in \text{supp}(\alpha)} \prod_{j=1}^m f(\omega(x), y_{1,(\omega,j)}, \dots, y_{k',(\omega,j)}) \\
 &\quad \overline{f(\omega(x), y_{1,(\omega,m+j)}, \dots, y_{k',(\omega,m+j)})}. \tag{3.18}
 \end{aligned}$$

Note that the right hand side of (3.18) falls into the framework of hypergraph pairs. Indeed for $k+1 \leq i \leq k+k'$, let $V_i := \text{supp}(\alpha) \times \{1, \dots, 2m\}$. Now the hypergraph pair $G = (\alpha', \beta')$ over $V_1 \times \dots \times V_{k+k'}$ is defined by

$$\alpha'(v_1, \dots, v_{k+k'}) := \begin{cases} 1 & v_{k+1} = \dots = v_{k+k'} = ([v_1, \dots, v_k], i) \text{ where } 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta'(v_1, \dots, v_{k+k'}) := \begin{cases} 1 & v_{k+1} = \dots = v_{k+k'} = ([v_1, \dots, v_k], i) \text{ where } m+1 \leq i \leq 2m \\ 0 & \text{otherwise} \end{cases}$$

We have $\|F_f\|_H = \|f\|_G$. The next proposition shows that in fact every semi-norming hypergraph pair which is not norming is of this form.

Proposition 3.4.20 *Let $H = (\alpha, \beta)$ be a semi-norming k -hypergraph pair of Type II over $V := V_1 \times \dots \times V_k$. Define S to be the set of all $1 \leq i \leq k$ such that for every $v \in V_i$,*

$$\sum \{\alpha(\omega) + \beta(\omega) : \omega \in V, \omega_i = v\} = 1.$$

Then $H_{[k] \setminus S}$ is a norming hypergraph pair of Type I.

Proof. We can assume that H is minimal according to Definition 3.1.9. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. Note that if $S \neq \emptyset$, then for every $i \in S$, every $f \in L_H(\mathcal{M})$ with $\int f(x_1, \dots, x_k) dx_i = 0$ satisfies $\|f\|_H = 0$. So if H is norming, then $H_{[k] \setminus S} = H$, and the proposition holds. Consider a k -hypergraph pair $H = (\alpha, \beta)$ over $V := V_1 \times \dots \times V_k$ which is not norming. Then there exists a function $f \in L_H(\mathcal{M})$, for some measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, such that $\int f^H = 0$ and $f \neq 0$. Lemma 3.4.9, then shows that for every $g \in L_H(\mathcal{M})$, and every $\psi \in \text{supp}(\alpha)$,

$$\int g^{H-1_\psi} f^{1_\psi} = 0. \quad (3.19)$$

Since $f \neq 0$, there exists measurable sets $\Gamma_1, \dots, \Gamma_k \subseteq \Omega$ of finite measure such that $\int_{\Gamma_1 \times \dots \times \Gamma_k} f \neq 0$. Define $g : \Omega^k \rightarrow \{0, 1\}$, as

$$g(x_1, \dots, x_k) = \begin{cases} 1 & (x_1, \dots, x_k) \in \Gamma_1 \times \dots \times \Gamma_k \\ 0 & \text{otherwise} \end{cases}$$

Note that for $x = (\{x_{i,v}\}_{v \in V_1}, \dots, \{x_{i,v}\}_{v \in V_k}) \in \Omega^{V_1} \times \dots \times \Omega^{V_k}$ and $\omega = (\omega_1, \dots, \omega_k) \in V$, $g(\omega(x)) = 1$ if and only if $x_{i,\omega_i} \in \Gamma_i$ for $1 \leq i \leq k$. Thus

$$g^{H-1_\psi}(x) = \begin{cases} 1 & x_{i,\omega_i} \in \Gamma_i \text{ for all } \omega \in \text{supp}(\alpha) \cup \text{supp}(\beta) \text{ with } \omega \neq \psi \\ 0 & \text{otherwise} \end{cases} \quad (3.20)$$

Suppose that for every $i \in [k]$, there exists $\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta)$ such that $\omega \neq \psi$ but $\omega_i = \psi_i$. Then by (3.20) and minimality of H ,

$$\int g^{H-1_\psi} f^{1_\psi} = \left(\prod_{i=1}^{|V_i|-1} \mu(\Gamma_i) \right) \int_{\Gamma_1 \times \dots \times \Gamma_k} f \neq 0$$

contradicting (3.19).

It follows from (3.19) and its analogue for $\psi \in \text{supp}(\beta)$ that the following holds: For every $\psi = (\psi_1, \dots, \psi_k) \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, there exists $i \in [k]$ such that

$$\{\omega \in \text{supp}(\alpha) \cup \text{supp}(\beta) : \omega_i = \psi_i \text{ and } \omega \neq \psi\} = \emptyset,$$

or in other words: $\sum\{\alpha(\omega) + \beta(\omega) : \omega \in V, \omega_i = \psi_i\} = 1$. Now Remark 3.4.4 shows that for every $v \in V_i$, $\sum\{\alpha(\omega) + \beta(\omega) : \omega \in V, \omega_i = v\} = 1$ which means that $i \in S$. By Lemma 3.4.3, $H_{[k]\setminus S}$ is semi-norming, but then maximality of S shows that it is also norming. \blacksquare

3.4.4 Proof of Theorem 3.3.2

Without loss of generality suppose that $H_1 = (\alpha_1, \beta_1)$ is a semi-norming hypergraph pair over $V^k = V \times \dots \times V$ and $H_2 = (\alpha_2, \beta_2)$ is a semi-norming hypergraph pair over $W^k = W \times \dots \times W$. Set $U := V \times W$. Note that $U^k \cong V^k \times W^k$, and we can think of the elements of U^k as pairs $\omega \otimes \psi$ where $\omega \in V^k$, and $\psi \in W^k$. Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space. Consider $f, g : \Omega^k \rightarrow \mathbb{C}$, and $\omega' \otimes \psi' \in \text{supp}(\alpha_1 \otimes \alpha_2) \subseteq U^k$.

Denote $\mathcal{N} := \mathcal{M}^V$ and $\mathcal{L} := \mathcal{M}^W$. Note that $f^{H_1} : \mathcal{N}^k \rightarrow \mathbb{C}$ and $f^{H_2} : \mathcal{L}^k \rightarrow \mathbb{C}$. With identifications $(\mathcal{M}^U)^k \cong ((\mathcal{M}^V)^W)^k \cong ((\mathcal{M}^W)^V)^k$ we have

$$f^{H_1 \otimes H_2} = (f^{H_1})^{H_2} = (f^{H_2})^{H_1}.$$

By applying Lemma 3.4.9 we get

$$\begin{aligned} \int f^{H_1 \otimes H_2 - 1_{\omega' \otimes \psi'}} g^{1_{\omega' \otimes \psi'}} &= \int (f^{H_1})^{H_2 - 1_{\psi'}} (f^{H_1 - 1_{\omega'}} g^{1_{\omega'}})^{1_{\psi'}} \\ &\leq \left(\int (f^{H_1})^{H_2} \right)^{\frac{|H_2| - 1}{|H_2|}} \left(\int (f^{H_1 - 1_{\omega'}} g^{1_{\omega'}})^{H_2} \right)^{\frac{1}{|H_2|}} \\ &= \|f\|_{H_1 \otimes H_2}^{|H_1|(|H_2| - 1)} \left(\int (f^{H_2})^{H_1 - 1_{\omega'}} (g^{H_2})^{1_{\omega'}} \right)^{\frac{1}{|H_2|}} \\ &\leq \|f\|_{H_1 \otimes H_2}^{|H_1|(|H_2| - 1)} \left(\int (f^{H_2})^{H_1} \right)^{\frac{|H_1| - 1}{|H_1||H_2|}} \left(\int (g^{H_2})^{H_1} \right)^{\frac{1}{|H_1||H_2|}} \\ &= \|f\|_{H_1 \otimes H_2}^{|H_1 \otimes H_2| - 1} \|g\|_{H_1 \otimes H_2}. \end{aligned}$$

Lemma 3.4.9 shows that $H_1 \otimes H_2$ is semi-norming.

Next suppose that H_1 is norming, and $\int f^{H_1 \otimes H_2} = 0$. Then since $f^{H_1 \otimes H_2} = (f^{H_2})^{H_1}$ and H_1 is norming we conclude that $f^{H_2} = 0$ almost everywhere, which shows that $f = 0$ almost everywhere.

3.4.5 Some facts about Gowers norms

In this section we prove some facts about Gowers norms that are needed in the subsequent sections. These facts are only proved as auxiliary results, and thus our aim is not to obtain the best possible bounds, or to prove them in the most general possible setting.

Let $V_1 = \dots = V_k = \{0, 1\}$, and U_k be the Gowers k -hypergraph pair defined in Example 4. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and measurable functions $f_\omega : \Omega^k \rightarrow \mathbb{C}$ for $\omega \in V := V_1 \times \dots \times V_k$. The following inequality due to Gowers [28] (see also [60]) can be proven by iterated applications of the Cauchy-Schwarz inequality:

$$\left| \int \prod_{\omega \in V} f_\omega^{1_\omega} \right| \leq \prod_{\omega \in V} \|f_\omega\|_{U_k}. \quad (3.21)$$

Since always $\|f\|_{U_k} \leq \|f\|_\infty$, we have the following easy corollary.

Corollary 3.4.21 *Let $H = (\alpha, \beta)$ be a k -hypergraph pair over $W := W_1 \times W_2 \times \dots \times W_k$, and $\psi \in W$ be such that $\alpha(\psi) = \beta(\psi) = 0$. Then for the measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and every pair of measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$, we have*

$$\left| \int f^H g^{1_\psi} \right| \leq \|g\|_{U_k} \|f\|_\infty^{|H|}. \quad (3.22)$$

Proof. First note that we can normalize f in (3.22) and assume that $\|f\|_\infty \leq 1$. Hence we need to show that for $\|f\|_\infty \leq 1$,

$$\left| \int f^H g^{1_\psi} \right| \leq \|g\|_{U_k}.$$

It suffices to prove that if we fix $x_{i,v} \in \Omega$, for every $1 \leq i \leq k$, and $v \neq \psi_i$, then

$$\left| \int f^H g^{1_\psi} dx_{1,\psi_1} \dots dx_{k,\psi_k} \right| \leq \|g\|_{U_k} \quad (3.23)$$

We will apply (3.21). To this end for every $\omega \in \{0, 1\}^k$, we define a function $f_\omega : \Omega^k \rightarrow \mathbb{C}$ so that (3.21) reduces to (3.23). Set $f_{(1,\dots,1)} := g$, and for $\omega \neq (1, \dots, 1)$, set

$$f_\omega : (x_{1,\psi_1}, \dots, x_{i,\psi_i}) \mapsto \prod_{\phi \in W, \phi_i \neq \psi_i \Leftrightarrow \omega_i = 0} f(x_{1,\phi_1}, \dots, x_{k,\phi_k})^{\alpha(\phi)} \overline{f(x_{1,\phi_1}, \dots, x_{k,\phi_k})^{\beta(\phi)}}.$$

Note that $f_\omega(y_1, \dots, y_k)$ is only a function of y_i where $\omega_i = 1$. In other words,

$$f_\omega(y_1, \dots, y_k) = f_\omega(y'_1, \dots, y'_k),$$

if $y_i = y'_i$ for every i with $\omega_i = 1$. From this and (3.21) we conclude that

$$\left| \int f^H g^{1_\psi} dx_{1,\psi_1} \dots dx_{k,\psi_k} \right| = \left| \int \prod_{\omega \in \{0,1\}^k} f_\omega \right| \leq \prod_{\omega \in V} \|f_\omega\|_{U_k} \leq \|g\|_{U_k},$$

which verifies (3.23). ■

The next Lemma shows that there exists a function g such that its range is $\{-1, 1\}$ but its Gowers norm is arbitrarily small.

Lemma 3.4.22 *For every $\epsilon > 0$, there exists a probability space $(\Omega, \mathcal{F}, \mu)$ and a function $g : \Omega^k \rightarrow \{-1, 1\}$ such that $\|g\|_{U_k} \leq \epsilon$ and $\int g = 0$.*

Proof. Consider a sufficiently large even integer m , set $\Omega = [m]$, and let μ be the uniform probability measure on Ω . Define g randomly so that $\{g(x)\}_{x \in \Omega^k}$ are independent Bernoulli random variables taking values uniformly in $\{-1, 1\}$. Then

$$\mathbb{E} \left(\int g \right)^2 = \frac{1}{m^{2k}} \mathbb{E} \left(\sum_{x \in [m]^k} g(x) \right)^2 = \frac{1}{m^{2k}} \mathbb{E} \left(\sum_{x, y \in [m]^k} g(x)g(y) \right) = \frac{1}{m^k} = o_{m \rightarrow \infty}(1),$$

and using a similar argument

$$\mathbb{E} \left(\int g^{U_k} \right)^2 = o_{m \rightarrow \infty}(1).$$

Hence for sufficiently large m , there exists $g_0 : \Omega^k \rightarrow \{-1, 1\}$ such that $|\int g_0| \leq (\epsilon/4)^{2^k}$ and $\|g_0\|_{U_k} \leq \epsilon/2$. Since $|\int g_0| \leq (\epsilon/4)^{2^k}$, there exists $g_1 : \Omega^k \rightarrow \{-1, 1\}$ such that $\int g_1 = 0$ and $\int |g_1 - g_0| \leq (\epsilon/4)^{2^k}$. Then by Hölder's inequality

$$\|g_0 - g_1\|_{U_k} = \left(\int (g_0 - g_1)^{U_k} \right)^{-2^k} \leq \|g_0 - g_1\|_{2^k} \leq 2(\epsilon/4) = \epsilon/2,$$

where in the last inequality we used the fact that the range of $g_0 - g_1$ is $\{-2, 0, 2\}$. Now

$$\|g_1\|_{U_k} \leq \|g_0\|_{U_k} + \|g_0 - g_1\|_{U_k} \leq \epsilon,$$

which shows that g_1 is the desired function. ■

Lemma 3.4.23 *For a k -hypergraph pair H over $V := V_1 \times \dots \times V_k$, a probability space \mathcal{P} , and a zero-one function $f \in L_H(\mathcal{P})$ we have*

$$\int f^H \geq \|f\|_1^{-|V_1| \dots |V_k|}.$$

Proof. Consider the k -hypergraph pair $K = (\frac{1}{2}, \frac{1}{2})$ over V . Lemma 3.3.3 shows that K is a norming hypergraph pair. Since f is a zero-one function, we have $f^H \geq f^K$, and thus by Corollary 3.4.13

$$\int f^H \geq \int f^K \geq \|f\|_1^{|K|} \geq \|f\|_1^{-|V_1| \dots |V_k|}.$$

■

Lemma 3.4.24 *Let $f, g : \Omega^k \rightarrow \mathbb{C}$ be two measurable functions with respect to the probability space $(\Omega, \mathcal{F}, \mu)$. Let $H = (\alpha, 0)$ be a hypergraph pair such that $\text{ran}(\alpha) \subseteq \{0, 1\}$. Then*

$$\left| \int f^H - g^H \right| \leq |H| \|f - g\|_{U_k} \max(\|f\|_\infty, \|g\|_\infty)^{|H|-1}.$$

Proof. Let us label the elements of $\text{supp}(\alpha)$ as $\omega_1, \dots, \omega_{|H|}$. Then for $0 \leq i \leq |H|$ define $H_i := \sum_{j=1}^i 1_{\omega_j}$, so that $H_0 = (0, 0)$ and $H_{|H|} = H$. Now by telescoping and applying Corollary 3.4.21, we have

$$\begin{aligned} \left| \int f^H - g^H \right| &\leq \sum_{i=1}^{|H|} \left| \int f^{H-H_{i-1}} g^{H_{i-1}} - f^{H-H_i} g^{H_i} \right| = \sum_{i=1}^{|H|} \left| \int f^{H-H_i} g^{H_{i-1}} (f^{1_{\omega_i}} - g^{1_{\omega_i}}) \right| = \\ &= \sum_{i=1}^{|H|} \left| \int f^{H-H_i} g^{H_{i-1}} (f - g)^{1_{\omega_i}} \right| \leq \sum_{i=1}^{|H|} \|f - g\|_{U_k} \|f\|_\infty^{|H|-i} \|g\|_\infty^{i-1} \leq \\ &\leq |H| \|f - g\|_{U_k} \max(\|f\|_\infty, \|g\|_\infty)^{|H|-1}. \end{aligned}$$

■

3.4.6 Proofs of Theorems 3.4.1 and 3.4.5

Proof.[Theorem 3.4.1] Suppose that H is a semi-norming k -hypergraph pair over $V = V_1 \times \dots \times V_k$. The fact that $H \cong \overline{H}$ follows from Proposition 3.4.18 because trivially $|H| = |\overline{H}|$ and $\|f\|_H = \|f\|_{\overline{H}}$.

Now let $\epsilon > 0$ be sufficiently small, and $h : \Omega^k \rightarrow \{-1, 1\}$ be such that $\|h\|_{U_k} \leq \epsilon$ and $\int h = 0$, where here $(\Omega, \mathcal{F}, \mu)$ is a *probability* space. The existence of h is guaranteed by Lemma 3.4.22.

First we show that it is either the case that for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\alpha(\psi) = \beta(\psi)$ or for every $\psi \in \text{supp}(\alpha) \cup \text{supp}(\beta)$, $\{\alpha(\psi), \beta(\psi)\} = \{0, 1\}$, and we will handle the existence of a universal s later. Suppose that this statement fails for some ψ . Note that at least one of $\alpha(\psi)$ or $\beta(\psi)$ is not equal to 0. We will assume that $\alpha(\psi) > \beta(\psi)$, and the proof of the case $\alpha(\psi) < \beta(\psi)$ will be similar. Since it is not the case that $\beta(\psi) = 1 - \alpha(\psi) = 0$, denoting $H - 1_\psi = (\alpha', \beta')$ we have

$$\psi \in \text{supp}(\alpha') \cup \text{supp}(\beta'). \quad (3.24)$$

For $p := \alpha(\psi) - \beta(\psi) \geq 0$, define $g := h^{1/p}$, and

$$f := \begin{cases} 1 & h = 1 \\ 0 & h = -1 \end{cases}.$$

Since $\int h = 0$, we have $\int f = 1/2$ and

$$\int f^{H-1_\psi} g^{1_\psi} = \int f^H \geq 2^{-|V_1| \dots |V_k|}, \quad (3.25)$$

where the equality follows from (3.24) and the definition of f , and the inequality follows from Lemma 3.4.23. Denote by K the hypergraph pair obtained from H by setting $\alpha(\psi) = \beta(\psi) = 0$, i.e. $K := H - \alpha(\psi)1_\psi - \beta(\psi)\overline{1_\psi}$. Now since $|g| = 1$, applying Corollary 3.4.21, we have

$$\left| \int g^H \right| = \left| \int g^K g^{\alpha(\psi)1_\psi + \beta(\psi)\overline{1_\psi}} \right| = \left| \int g^K |g|^{\beta(\psi)1_\psi} g^{p1_\psi} \right| = \left| \int g^K h^{1_\psi} \right| \leq \|h\|_{U_k} \leq \epsilon,$$

which shows that

$$\|f\|_H^{H-1} \|g\|_H \leq \|f\|_H^{H-1} \epsilon^{1/|H|}. \quad (3.26)$$

For sufficiently small ϵ , (3.25) and (3.26) contradict Lemma 3.4.9.

Next we will prove the existence of a universal s . So suppose that $H = (\alpha, \beta)$ is semi-norming and $\alpha = \beta$. Let $s = \max\{\alpha(\omega) + \beta(\omega) : \omega \in V\}$. We will show that $\frac{1}{s}H$ is semi-norming, and then Corollary 3.4.10 implies that $\alpha(\omega) + \beta(\omega) \in \{0, s\}$. Let ψ be such that $\alpha(\psi) + \beta(\psi) = s$, and let $\tilde{H}_\psi = \frac{1_\psi + 1_{\bar{\psi}}}{2}$. Consider a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ and measurable functions $f, g : \Omega^k \rightarrow \mathbb{C}$, and note that

$$\begin{aligned} \left| \int f^{(\frac{1}{s}H)-1_\psi} g^{1_\psi} \right| &\leq \int |f|^{(\frac{1}{s}H)-1_\psi} |g|^{1_\psi} = \int (|f|^{1/s})^{H-s\tilde{H}_\psi} (|g|^{1/s})^{s\tilde{H}_\psi} \leq \\ &\leq \| |f|^{1/s} \|_H^{|H|-s} \| |g|^{1/s} \|_H^s = \|f\|_{\frac{1}{s}H}^{|\frac{1}{s}H|-1} \|g\|_{\frac{1}{s}H}, \end{aligned}$$

where in the second inequality we used Lemma 3.4.11. Now Lemma 3.4.9 shows that $\frac{1}{s}H$ is a semi-norming hypergraph pair, and this finishes the proof. \blacksquare

Next we give the proof of Theorem 3.4.5.

Proof.[Theorem 3.4.5] Let $\Omega := [k]$ be endowed with the uniform probability measure.

Define $f : \Omega^k \rightarrow \mathbb{R}$ as in the following:

$$f(x_1, \dots, x_k) = \begin{cases} 1 & x_1 = \dots = x_k \\ 0 & \text{otherwise} \end{cases}$$

Since H is a non-factorizable hypergraph pair, $f^H(x) = 1$ if and only if all coordinates of x are equal. Hence

$$\int f^H = k \left(\frac{1}{k} \right)^{|V_1|+\dots+|V_k|}. \quad (3.27)$$

Similarly since for $x \in \Omega^{W_1} \times \dots \times \Omega^{W_k}$, if all coordinates of x are equal, then $f^{H'}(x) = 1$, we have

$$\int f^{H'} \geq k \left(\frac{1}{k} \right)^{|W_1|+\dots+|W_k|}. \quad (3.28)$$

Since Ω is a probability space, by Corollary 3.4.13 we have

$$\|f\|_{H'} \leq \|f\|_H. \quad (3.29)$$

Plugging (3.27) and (3.28) into (3.29), and simplifying it, we obtain the assertion of the theorem. ■

Chapter 4

Geometry of the Hypergraph Norms

The two dual concepts of uniform convexity and uniform smoothness play an important role in Banach space theory. After reviewing the basic definitions and the state of the art for L_p norms and trace norms, we shall study these two notions for hypergraph norms.

4.1 Moduli of Smoothness and Convexity

Let us start by recalling the definition of moduli of smoothness and convexity of a normed space. A normed space X is said to be **uniformly smooth** if for all $\epsilon > 0$, there is a $\tau > 0$ such that if x and y have norm 1, and $\|x - y\| \leq 2\tau$, then $\left\|\frac{x+y}{2}\right\| \geq 1 - \epsilon\tau$. A normed space X is called **uniformly convex**, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if x and y have norm 1, and $\|x - y\| \geq 2\epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.

Roughly speaking, a normed space is uniformly convex if its unit ball is uniformly free of “flat spots”, and a normed space is uniformly smooth if its unit ball is uniformly free of “corners”. Since the unit ball of X^* , the dual of X , is the polar conjugate (see (2.2)) of the unit ball of X , it is not difficult to show that X is uniformly convex if and only if X^* is uniformly smooth. We shall see a stronger result below, and thus we will not

elaborate on this.

The modulus of smoothness and modulus of convexity are quantified versions of the notions of uniform smoothness and uniform convexity. For a normed space X , define its **modulus of smoothness** as the function

$$\rho_X(\tau) = \sup \left\{ \frac{\|x - \tau y\| + \|x + \tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}, \quad (4.1)$$

and its **modulus of convexity** as

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1, \|x - y\| \geq 2\epsilon \right\}, \quad (4.2)$$

where $0 \leq \epsilon \leq 1$. It should be noted that the function δ_X is frequently defined with ϵ in place of 2ϵ . The following observation of Lindenstrauss [38] shows that these two functions behave in a dual form via the Legendre transform:

$$\rho_{X^*}(\tau) = \sup \{ \tau\epsilon - \delta_X(\epsilon) : 0 \leq \epsilon \leq 1 \}, \quad (4.3)$$

where X^* is the dual of X .

Note that X is uniformly smooth, if $\lim_{\tau \rightarrow 0} \rho_X(\tau)/\tau = 0$, and it is called uniformly convex, if for every $\epsilon > 0$, $\delta_X(\epsilon) > 0$. For $t \in (1, 2]$, a normed space X is said to be t -uniformly smooth if there exists a constant $C > 0$ such that $\rho_X(\tau) \leq (C\tau)^t$, and for $r \in [2, \infty)$, a normed space is said to be r -uniformly convex if there exists a constant $C > 0$ such that $\delta_X(\epsilon) \geq (\epsilon/C)^r$.

Note that the moduli of uniform smoothness and uniform convexity of a norm space, depend only on the structure of its two-dimensional subspaces. Thus if X is finitely representable in Y , then $\rho(X) \geq \rho(Y)$ and $\delta(X) \leq \delta(Y)$.

It is known that $\rho_{\ell_2}(\tau) = (1 + \tau^2)^{1/2} - 1 = \tau^2/2 + O(\tau^4)$, $\tau > 0$ and $\delta_{\ell_2}(\epsilon) = 1 - (1 - \epsilon^2)^{1/2} = \epsilon^2/2 + O(\epsilon^4)$ for $0 < \epsilon < 1$. Dvoretzky's theorem (Theorem 2.1.11) implies that for every infinite dimensional normed space X , we have $\rho_X(\tau) \geq \rho_{\ell_2}(\tau)$ and $\delta_X(\epsilon) \leq \delta_{\ell_2}(\epsilon)$, and this was the reason for requiring $t \in (1, 2]$ and $r \in [2, \infty)$ in the definition of t -uniform smoothness and r -uniform convexity. The following theorem due

to Ball, Carlen, and Lieb [1] gives an equivalent definition for the notions of t -uniform smoothness and r -uniform convexity.

Theorem 4.1.1 [1] *A normed space X is t -uniformly smooth for $t \in (1, 2]$ if and only if there exists a constant K such that*

$$\frac{\|x + y\|^t + \|x - y\|^t}{2} \leq \|x\|^t + \|Ky\|^t.$$

Similarly a normed space X is r -uniformly convex for $r \in [2, \infty)$ if and only if there exists a constant K such that

$$\frac{\|x + y\|^r + \|x - y\|^r}{2} \leq \|x\|^r + \|K^{-1}y\|^r.$$

4.1.1 A generalization

In this section we obtain a generalization of Theorem 4.1.1. First we need two lemmas.

Lemma 4.1.2 *Let $1 < p \leq q < \infty$ and $\rho = \sqrt{\frac{p-1}{q-1}}$. Then for every two vectors x and y in an arbitrary normed space X , we have*

$$\left(\frac{\|x + \rho y\|^q + \|x - \rho y\|^q}{2} \right)^{1/q} \leq \left(\frac{\|x + y\|^p + \|x - y\|^p}{2} \right)^{1/p}.$$

For the proof of Lemma 4.1.2 see Corollary 1.e.14 in [39].

Lemma 4.1.3 *Let $t \in (1, 2]$, $r \in [2, \infty)$, and $1 < p, q < \infty$. Then there exists constants $C = C(t, p)$ and $C^* = C^*(r, q)$ such that for every $x, y \in \mathbb{C}$,*

$$\left(\frac{|x + y|^p + |x - y|^p}{2} \right)^{1/p} \leq (|x|^t + |Cy|^t)^{1/t}, \quad (4.4)$$

and

$$\left(\frac{|x + y|^q + |x - y|^q}{2} \right)^{1/q} \geq \left(|x|^r + \left| \frac{1}{C^*} y \right|^r \right)^{1/r}. \quad (4.5)$$

Furthermore, for the best constants one can assume $C(t, p) = C^*(r, q)$, if $\frac{1}{r} + \frac{1}{t} = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We only prove (4.4), and (4.5) as well as the last assertion of the lemma will follow from duality by Proposition 4.1.6 below. It suffices to prove the theorem for $t = 2$ as the right-hand side of (4.4) is a decreasing function in t . By Lemma 4.1.2, we have

$$\left(\frac{|x+y|^p + |x-y|^p}{2}\right)^{1/p} \leq \left(\frac{|x+\rho y|^2 + |x-\rho y|^2}{2}\right)^{1/2} \leq (|x|^2 + |\rho y|^2)^{1/2},$$

where $\rho = \max(1, \sqrt{p-1})$. ■

Now for a normed space X , inspired by Lemma 4.1.3, for $1 < t \leq 2 \leq r < \infty$, and $1 < p, q < \infty$, one can investigate the validity of the following two inequalities:

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \leq (\|x\|^t + \|Ky\|^t)^{1/t}, \quad (4.6)$$

and

$$\left(\frac{\|x+y\|^q + \|x-y\|^q}{2}\right)^{1/q} \geq (\|x\|^r + \|K^{-1}y\|^r)^{1/r} \quad (4.7)$$

where K is a constant. We denote the smallest constant K such that (4.6) is satisfied for all $x, y \in X$ by $K_{t,p}(X)$ and similarly the smallest constant such that (4.7) is satisfied by $K_{r,q}^*(X)$. Trivially $K_{t,p}(X) \geq C(t,p)$ and $K_{r,q}^*(X) \geq C^*(r,q)$ where $C(t,p)$ and $C^*(r,q)$ are the constants defined in Lemma 4.1.3.

Remark 4.1.4 In the sequel, $C(t,p)$ and $C^*(r,q)$ always refer to the constants from Lemma 4.1.3. Note that $C(t,p)$ and $K_{t,p}(X)$ are both increasing in t and p , and $C^*(r,q)$ and $K_{r,q}^*(X)$ are both decreasing in r and q . Since Lemma 4.1.2 is valid for every normed space X , for $1 < p_2 \leq p_1 < \infty$,

$$\begin{aligned} \left(\frac{\|x+y\|^{p_1} + \|x-y\|^{p_1}}{2}\right)^{1/p_1} &\leq \left(\frac{\left\|x + \sqrt{\frac{p_1-1}{p_2-1}}y\right\|^{p_2} + \left\|x - \sqrt{\frac{p_1-1}{p_2-1}}y\right\|^{p_2}}{2}\right)^{1/p_2} \\ &\leq \left(\|x\|^t + \left\|K_{t,p_2}(X)\sqrt{\frac{p_1-1}{p_2-1}}y\right\|^t\right)^{1/t}, \end{aligned}$$

which implies $K_{t,p_1}(X) \leq \sqrt{\frac{p_1-1}{p_2-1}} K_{t,p_2}(X)$. Similarly for $1 < q_2 \leq q_1 < \infty$,

$$\begin{aligned} \left(\frac{\|x+y\|^{q_2} + \|x-y\|^{q_2}}{2} \right)^{1/q_2} &\geq \left(\frac{\|x + \sqrt{\frac{q_2-1}{q_1-1}}y\|^{q_1} + \|x - \sqrt{\frac{q_2-1}{q_1-1}}y\|^{q_1}}{2} \right)^{1/q_1} \\ &\geq \left(\|x\|^r + \left\| \frac{1}{K_{r,q_1}^*(X)} \sqrt{\frac{q_2-1}{q_1-1}}y \right\|^r \right)^{1/r}, \end{aligned}$$

which shows that $K_{r,q_2}^*(X) \leq \sqrt{\frac{q_1-1}{q_2-1}} K_{r,q_1}^*(X)$. ■

The following proposition, which is a generalization Theorem 4.1.1, follows from Theorem 4.1.1 using Remark 4.1.4.

Proposition 4.1.5 *Let X be a t -uniformly smooth normed space. Then for every $1 < p < \infty$, we have $K_{t,p}(X) < \infty$. Conversely if $K_{t,p}(X) < \infty$ for some $1 < p < \infty$, then X is t -uniformly smooth.*

Similarly let Y be an r -uniformly convex normed space. Then for every $1 < q < \infty$, we have $K_{r,q}^(Y) < \infty$. Conversely if $K_{r,q}^*(Y) < \infty$ for some $1 < q < \infty$, then Y is r -uniformly convex.*

The constants $K_{t,p}$ and $K_{r,q}^*$ behave nicely with respect to the duality. The proof of the following proposition is parallel to the proof of Lemma 5 from [1]. But we state it here for the sake of completeness.

Proposition 4.1.6 *Consider a normed space X and its dual X^* . Suppose that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{t} + \frac{1}{r} = 1$. Then $K_{t,p}(X) = K_{r,q}^*(X^*)$.*

Proof. Consider $x, y \in X$. By Hahn-Banach theorem (Corollary 2.1.10), there exists $\lambda, \gamma \in X^*$ such that $\lambda(x+y) = \|x+y\|$, and $\gamma(x-y) = \|x-y\|$, and $\|\lambda\| = \|\gamma\| = 1$. Define $\phi, \psi \in X^*$ by $\phi := Z^{-1/q} \|x+y\|^{p-1} \lambda$ and $\psi := Z^{-1/q} \|x+y\|^{p-1} \gamma$ where

$$Z = (\|x+y\|^p + \|x-y\|^p)/2.$$

Then

$$\|\phi\|^q + \|\psi\|^q = Z^{-1} (\|x + y\|^{(p-1)q} + \|x - y\|^{(p-1)q}) = Z^{-1} (\|x + y\|^p + \|x - y\|^p) = 2.$$

Next we have

$$\begin{aligned} \left(\frac{\|x + y\|^p + \|x - y\|^p}{2} \right)^2 &= \frac{\phi(x + y) + \psi(x - y)}{2} = \frac{\phi + \psi}{2}(x) + \frac{\phi - \psi}{2}(y) \\ &\leq \left(\left\| \frac{\phi + \psi}{2} \right\|^r + \left\| \frac{\phi - \psi}{2K_{r,q}^*(X^*)} \right\|^r \right)^{1/r} (\|x\|^t + \|K_{r,q}^*(X^*)y\|^t)^{1/t} \\ &\leq \left(\frac{\|\phi\|^q + \|\psi\|^q}{2} \right)^{1/q} (\|x\|^t + \|K_{r,q}^*(X^*)y\|^t)^{1/t} \\ &= (\|x\|^t + \|K_{r,q}^*(X^*)y\|^t)^{1/t}, \end{aligned}$$

where in the first inequality we used Hölder's inequality, and in the second one the definition of $K_{r,q}^*(X^*)$. We have established that $K_{t,p}(X) \leq K_{r,q}^*(X^*)$. The proof of $K_{r,q}^*(X^*) \leq K_{t,p}(X)$ is similar. ■

The notion of uniform convexity is first defined by Clarkson in [8], where he studied the smoothness and convexity of L_p spaces. To this end he established four inequalities known as the Clarkson inequalities. Let $1 < p \leq 2 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. In our notation the Clarkson inequalities are the following: $K_{p,p}(\ell_p) = 1$, $K_{q,q}^*(\ell_q) = 1$, $K_{q,p}^*(\ell_p) = 1$, and $K_{p,q}(\ell_q) = 1$. The first two are easier to prove and known as the “easy” Clarkson inequalities, and the latter two are known as the “strong” Clarkson inequalities. The following observation shows that the strong Clarkson inequalities imply the easy Clarkson inequalities.

Lemma 4.1.7 *Let $1 < t \leq 2 \leq r < \infty$ be such that $\frac{1}{t} + \frac{1}{r} = 1$. Then $K_{t,r}(X) = 1$ if and only if $K_{r,t}^*(X) = 1$.*

Proof. Suppose that $K_{t,r}(X) = 1$. Then for every $x, y \in X$, we have

$$\left(\frac{\|x + y\|^r + \|x - y\|^r}{2} \right)^{1/r} \leq (\|x\|^t + \|y\|^t)^{1/t}.$$

Now consider $x', y' \in X$. Replacing x and y in the above inequality, respectively with $\frac{x'+y'}{2}$ and $\frac{x'-y'}{2}$ we get

$$\left(\frac{\|x'\|^r + \|y'\|^r}{2} \right)^{1/r} \leq \left(\left\| \frac{x'+y'}{2} \right\|^t + \left\| \frac{x'-y'}{2} \right\|^t \right)^{1/t}.$$

which simplifies to

$$(\|x'\|^r + \|y'\|^r)^{1/r} \leq \left(\frac{\|x'+y'\|^t + \|x'-y'\|^t}{2} \right)^{1/t},$$

showing that $K_{r,t}^*(X) = 1$. The proof of the converse direction is similar. \blacksquare

Consider $1 < p \leq 2 \leq q < \infty$. As we have already seen in Proposition 4.1.5, the Clarkson inequalities imply that L_p and L_q spaces are both p -uniformly smooth and q -uniformly convex. However, this is not in general the best possible. The actual situation is the following. The L_p spaces are p -uniformly smooth and 2-uniformly convex, and the L_q spaces are 2-uniformly smooth and q -uniformly convex. These facts are proved by Hanner [33] through the so called Hanner inequality. For $1 < p \leq 2$, we say that a normed space satisfies the p -Hanner inequality, if

$$\|x+y\|^p + \|x-y\|^p \geq (\|x\| + \|y\|)^p + \left| \|x\| - \|y\| \right|^p,$$

and for $2 \leq q < \infty$, it satisfies the q -Hanner inequality if

$$\|x+y\|^q + \|x-y\|^q \leq (\|x\| + \|y\|)^q + \left| \|x\| - \|y\| \right|^q.$$

It is shown in [1] that if X satisfies the p -Hanner inequality, then X^* satisfies the q -Hanner inequality where $\frac{1}{p} + \frac{1}{q} = 1$. The following proposition reveals the relation between the Hanner inequality and the notions of uniform smoothness and uniform convexity.

Proposition 4.1.8 *If a normed space X satisfies the t -Hanner inequality for $1 < t \leq 2$, then for every $2 \leq q < \infty$, we have $K_{q,t}^*(X) = C^*(q, t)$, and for every $1 < p \leq t'$, we have $K_{t,p}(X) = 1$ where $\frac{1}{t} + \frac{1}{t'} = 1$.*

Similarly if a normed space X satisfies the r -Hanner inequality for $2 \leq r < \infty$, then for every $1 < p \leq 2$, we have $K_{p,r}(X) = C(p, r)$, and for every $r' \leq q < \infty$, we have $K_{r,q}^(X) = 1$, where $\frac{1}{r} + \frac{1}{r'} = 1$.*

Proof. Suppose that X satisfies the t -Hanner inequality for $1 < t \leq 2$. Consider $2 \leq q < \infty$, and $x, y \in X$. By the t -Hanner inequality

$$\left(\frac{\|x + y\|^t + \|x - y\|^t}{2} \right)^{1/t} \geq \left(\frac{(\|x\| + \|y\|)^t + \|\|x\| - \|y\|\|^t}{2} \right)^{1/t} \geq \left(\|x\|^q + \left\| \frac{1}{C^*(q, t)} y \right\|^q \right)^{1/q},$$

which shows that $K_{q, t}^*(X) \leq C^*(q, t)$. But from this, and Lemma 4.1.7 we also get $K_{t, t'}(X) = 1$ as $K_{t', t}^*(X) \leq C^*(t', t) = 1$. Hence for $1 < p \leq t'$ we have $K_{t, p}(X) = 1$. The second assertion follows from the first one by duality. \blacksquare

Inequalities (4.6) and (4.7) are first appeared in [1], where for $q \geq 2$, the equalities $K_{2, q}(\ell_q) = K_{2, q}(S_q) = K_{2, 2}(\ell_q) = K_{2, 2}(S_q) = \sqrt{q-1}$ are proved, where S_q corresponds to the q -trace norm.

Proposition 4.1.9 *For $1 < t \leq 2 \leq r < \infty$, $1 < t_1 \leq 2 \leq r_1 < \infty$, and $1 < p < \infty$, we have*

$$K_{t_1, p}(\ell_r) = \begin{cases} C(t_1, r) & p \leq r \\ C(t_1, p) \leq \cdot \leq C(t_1, r) \sqrt{\frac{p-1}{r-1}} & p \geq r \end{cases} \quad (4.8)$$

and

$$K_{r_1, p}^*(\ell_r) = \begin{cases} C^*(r_1, t) & p \geq t \\ C^*(r_1, p) \leq \cdot \leq C^*(r_1, t) \sqrt{\frac{t-1}{p-1}} & p \leq t \end{cases} \quad (4.9)$$

In particular $K_{2, p}(\ell_r) = \max(\sqrt{p-1}, \sqrt{r-1})$, and $K_{2, p}^*(\ell_t) = \max\left(\sqrt{\frac{1}{p-1}}, \sqrt{\frac{1}{r-1}}\right)$.

Proof. It suffices to prove (4.8), and then (4.9) will follow from duality. Since ℓ_r satisfies the r -Hanner inequality, by Proposition 4.1.8 we have $K_{t_1, r}(\ell_r) = C(t_1, r)$. Then it follows from Lemma 4.1.2 that for $p \geq r$, $K_{t_1, p}(\ell_r) \leq C(t_1, r) \sqrt{\frac{p-1}{r-1}}$. Furthermore, since $K_{t_1, p}(\ell_r)$ is increasing in p , we have $K_{t_1, p}(\ell_r) \leq C(t_1, r)$, for $p \leq r$. It remains to show that $K_{t_1, p}(\ell_r) \geq C(t_1, r)$ for $p \leq r$. Consider two complex numbers a and b , and let $x, y \in \ell_r$ be as $x = (a, a)$ and $y = (b, -b)$. Then since $\|x + y\|_r = \|x - y\|_r = (|a + b|^r + |a - b|^r)^{1/r}$, plugging these two vectors in

$$\left(\frac{\|x + y\|_r^p + \|x - y\|_r^p}{2} \right)^{1/p} \leq (\|x\|_r^{t_1} + \|K_{t_1, p}(\ell_r)y\|_r^{t_1})^{1/t_1},$$

we get

$$\left(\frac{|a+b|^r + |a-b|^r}{2} \right)^{1/r} \leq (|a|^{t_1} + |K_{t_1,p}(\ell_r)b|^{t_1})^{1/t_1},$$

which shows that $K_{t_1,p}(\ell_r) \geq C(t_1, r)$. ■

Let $1 < t \leq 2 \leq r < \infty$ with $\frac{1}{t} + \frac{1}{r} = 1$. The spaces ℓ_t and ℓ_r are respectively 2-uniformly convex and 2-uniformly smooth. Proposition 4.1.9 determines the optimum value of all corresponding constants. In terms of the constants corresponding to t -uniformly smoothness of ℓ_t and r -uniformly convexity of ℓ_r , by Remark 4.1.4 and Clarkson's inequalities we have

$$K_{t,p}(\ell_t) = \begin{cases} 1 & p \leq r \\ C(t,p) \leq \cdot \leq \sqrt{\frac{p-1}{r-1}} & p \geq r \end{cases}$$

and

$$K_{r,p}^*(\ell_t) = \begin{cases} 1 & p \geq t \\ C^*(r,p) \leq \cdot \leq \sqrt{\frac{t-1}{p-1}} & p \leq t \end{cases}$$

4.1.2 Type and Cotype

The moduli of smoothness and convexity of a Banach space are only isometric invariant, and they may change considerably under an equivalent renorming. This leads to the definition of type and cotype. A normed space is of **type** $1 \leq t \leq 2$ if there exists a constant T_t such that for every integer $n \geq 0$, and every set of vectors x_1, \dots, x_n ,

$$\mathbb{E} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^t \leq T_t \left(\sum_{i=1}^n \|x_i\|^t \right)^{1/t},$$

where ϵ_i are independent Bernoulli random variables taking values uniformly in $\{-1, 1\}$. Similarly a normed space is said to be of **cotype** $2 \leq r \leq \infty$ if there exists a constant C_r such that for every integer $n \geq 0$, and every set of vectors x_1, \dots, x_n ,

$$\left(\sum_{i=1}^n \|x_i\|^r \right)^{1/r} \leq C_r \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|,$$

where in the case $r = \infty$ the left hand-side must be replaced by $\max_{i=1}^n \|x_i\|$.

	$1 < p \leq t$	$t \leq p \leq 2$	$2 \leq p \leq r$	$r \leq p < \infty$
$C(2, p)$	1	1	$\sqrt{p-1}$	$\sqrt{p-1}$
$C^*(2, p)$	$\sqrt{\frac{1}{p-1}}$	$\sqrt{\frac{1}{p-1}}$	1	1
$K_{2,p}(\ell_r)$	$\sqrt{r-1}$	$\sqrt{r-1}$	$\sqrt{r-1}$	$\sqrt{p-1}$
$K_{2,p}^*(\ell_t)$	$\sqrt{\frac{1}{p-1}}$	$\sqrt{\frac{1}{t-1}}$	$\sqrt{\frac{1}{t-1}}$	$\sqrt{\frac{1}{t-1}}$
$C(t, p)$	1	1	1	$\leq \sqrt{\frac{p-1}{r-1}}$
$K_{t,p}(\ell_t)$	1	1	1	$\leq \sqrt{\frac{p-1}{r-1}}$
$C^*(r, p)$	$\leq \sqrt{\frac{t-1}{p-1}}$	1	1	1
$K_{r,p}^*(\ell_r)$	$\leq \sqrt{\frac{t-1}{p-1}}$	1	1	1
$K_{t_1,p}(\ell_r)$	$C(t_1, r)$	$C(t_1, r)$	$C(t_1, r)$	$\leq C(t_1, r) \sqrt{\frac{p-1}{r-1}}$
$K_{r_1,p}^*(\ell_t)$	$\leq C^*(r_1, t) \sqrt{\frac{t-1}{p-1}}$	$C^*(r_1, t)$	$C^*(r_1, t)$	$C^*(r_1, t)$

Figure 4.1: Here $1 < t \leq 2 \leq r < \infty$ are such that $\frac{1}{t} + \frac{1}{r} = 1$, and $1 < t_1 \leq 2 \leq r_1 < \infty$ are arbitrary.

Trivially, every normed space is of type 1 and of cotype ∞ . If a normed space is of type t_0 and cotype r_0 , then it is also of type t and cotype r provided that $t \leq t_0 \leq 2 \leq r_0 \leq r$. Note that type and cotype do not change under an equivalent norm. Figiel and Pisier [17, 18] proved that t -uniform smoothness implies type t , and r -uniform convexity implies cotype r . The reverse is of course not true as for example every finite dimensional space is of type and cotype 2.

It is well-known that infinite dimensional L_p spaces are of type $\min(p, 2)$ and cotype $\max(2, p)$, and nothing better. Thus if ℓ_p is λ -finitely representable (see Section 2.1.9) in an space X of type t and cotype r , then $t \leq \min(2, p)$ and $r \geq \max(2, p)$. A beautiful theorem due to Maurey and Pisier [42] says that the converse is also true, i.e. ℓ_p and ℓ_q are finitely representable in X where $p = \sup\{t : X \text{ is of type } t\}$ and $q = \inf\{r : X \text{ is of cotype } r\}$.

Thus in order to study the type, cotype, modulus of smoothness, and modulus of convexity of a normed space X , it is natural therefore to first try to find the smallest $p \geq 1$ and largest q that ℓ_p and ℓ_q are finitely representable in X .

4.1.3 Geometry of ℓ_H spaces

For a hypergraph pair H , define $\ell_H := L_H(\mathbb{N})$ where \mathbb{N} is endowed with the counting measure.

Theorem 4.1.10 *If $H = (\alpha, \beta)$ is a non-factorizable semi-norming hypergraph pair, then $\ell_{|H|}$ is a subspace of ℓ_H . Furthermore, if H is of Type I with parameter $s \leq 2$, then ℓ_s is finitely representable in ℓ_H .*

The first part of the theorem, which is trivial, shows that any infinite dimensional L_H space is not of any cotype $q < \min(2, |H|)$. The second part, which is more interesting, shows that if H is of Type I with parameter $s < 2$, then every infinite dimensional L_H space is not of any type $p > s$. In particular in the case $s = 1$, an infinite dimensional L_H space has no nontrivial type, and is not uniformly smooth and convex. The next theorem shows that every such space is of cotype $\min(2, |H|)$ which is the best possible by Theorem 4.1.10.

Theorem 4.1.11 *Let H be a non-factorizable semi-norming hypergraph pair of Type I, then ℓ_H is of cotype $\min(2, |H|)$.*

In Theorem 4.1.11, only the case $s = 1$ is interesting to us, as for $s > 1$ we will prove something stronger in Theorem 4.1.12. The key to prove Theorem 4.1.11 is the following observation. Consider a non-factorizable semi-norming k -hypergraph pair $H = (\alpha, \alpha)$ of Type I over $V := V_1 \times \dots \times V_k$, and functions $f_1, f_2, \dots, f_n \in \ell_H$. Then

$$\sum_{i=1}^n f_i^H = \sum_{i=1}^n \prod_{\omega \in V} |f_i \circ \omega|^{2\alpha(\omega)} \leq \prod_{\omega \in V} \left(\sum_{i=1}^n |f_i \circ \omega|^{|H|} \right)^{1/|H|} = \left(\sum_{i=1}^n |f_i|^{|H|} \right)^{\frac{H}{|H|}},$$

where in the inequality above we used the classical Hölder inequality. Hence ¹

$$\sum_{i=1}^n \|f_i\|_H^{|H|} \leq \left\| \left(\sum_{i=1}^n |f_i|^{|H|} \right)^{1/|H|} \right\|_H. \quad (4.10)$$

We will also need the following inequality ² in the sequel:

$$\begin{aligned} \left\| \left(\sum_{i=1}^n |f_i|^s \right)^{\frac{1}{s}} \right\|_H &= \left(\int \left(\sum_{i=1}^n |f_i|^s \right)^{\frac{H}{s}} \right)^{1/|H|} = \left\| \sum_{i=1}^n |f_i|^s \right\|_{H/s}^{1/s} \\ &\leq \left(\sum_{i=1}^n \| |f_i|^s \|_{H/s} \right)^{1/s} = \left(\sum_{i=1}^n \|f_i\|_H^s \right)^{1/s}, \end{aligned} \quad (4.11)$$

where we used the fact that H/s is also norming. Now we can state the proof of Theorem 4.1.11.

Proof.[Theorem 4.1.11] Consider functions $f_1, \dots, f_n \in \ell_H$, and let $m := \max(|H|, 2)$. By applying Minkowski's inequality, Khintchine's inequality, and then (4.10), there exists a constant C such that

$$\begin{aligned} \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_H &= \mathbb{E} \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_H \geq \left\| \mathbb{E} \sum_{i=1}^n \epsilon_i f_i \right\|_H \geq C \left\| \left(\sum_{i=1}^n |f_i|^2 \right)^{1/2} \right\|_H \\ &\geq C \left\| \left(\sum_{i=1}^n |f_i|^m \right)^{1/m} \right\|_H \geq C \left(\sum_{i=1}^n \|f_i\|_H^m \right)^{1/m}. \end{aligned}$$

■

Now let us turn to the other hypergraph pairs, i.e. the ones which are not of Type I with parameter 1. From Theorem 4.1.10, in terms of the four parameters type, cotype, modulus of smoothness, and of convexity, the following theorem is the strongest statement one can hope to prove about them, and in particular implies Theorem 4.1.11 for H of Type I with parameter $s > 1$.

Theorem 4.1.12 *Let H be a non-factorizable semi-norming hypergraph pair such that $|H| \geq 2$.*

¹Inequality (4.10) says that ℓ_H is $|H|$ -concave as a Banach lattice when H is of Type I. For the definition of Banach lattice convexity and concavity we refer the reader to [39].

²Inequality (4.11) says that ℓ_H is s -convex as a Banach lattice (see [39]).

- If H is of Type II or Type I with parameter $s \geq 2$, then ℓ_H is 2-uniformly smooth and $|H|$ -uniformly convex;
- If H is of Type I with parameter $1 < s \leq 2$, then ℓ_H is s -uniformly smooth and $|H|$ -uniformly convex.

Remark 4.1.13 If $1 < |H| < 2$, then it is easy to see by the previous results that $\|\cdot\|_H$ corresponds to the L_p norm where $p = |H|$, and thus the Banach space properties of the norm are well-understood. The case $|H| = 1$ is also trivial. ■

As it is discussed above, the notions of t -uniform smoothness and r -uniform convexity can be further refined by looking at the constants $K_{t,p}$ and $K_{r,q}^*$. In proving Theorem 4.1.12 we will try to obtain the best possible constants. This is treated and discussed in more details in Section 4.1.6. Next we prove Theorems 4.1.10.

4.1.4 Proof of Theorem 4.1.10

Define $T : \ell_{|H|} \rightarrow \ell_H$ as $T : a \mapsto f_a$, where for $a = \{a_i\}_{i \in \mathbb{N}}$, $f_a : \mathbb{N}^k \rightarrow \mathbb{C}$ is defined as

$$f_a(i_1, \dots, i_k) = \begin{cases} a_i & i_1 = i_2 = \dots = i_k = i \\ 0 & \text{otherwise} \end{cases}$$

Since H is non-factorizable, it is easy to see that T is an isometry.

Next we show that ℓ_s is finitely representable in ℓ_H . Since $L_H([0, 1])$ is finitely representable in ℓ_H , it suffices to find a map $T : \ell_s([n]) \rightarrow L_H([0, 1])$ with $\|T\| \|T^{-1}\| \leq 1 + \epsilon$, for every $n \in \mathbb{N}$ and every $\epsilon > 0$. To this end we find $f_1, \dots, f_n : [0, 1]^k \rightarrow \mathbb{C}$, such that for every $x = (x_1, \dots, x_n) \in \ell_s([n])$ with $\|x\|_s = n^{1/s}$,

$$1 - \epsilon/4 \leq \left\| \sum_{i=1}^n x_i f_i \right\|_H \leq 1 + \epsilon/4,$$

and then the map $T : \ell_s([n]) \rightarrow L_H([0, 1])$ defined by $T : e_i \mapsto f_i$, for $i \in [n]$, satisfies $\|T\| \|T^{-1}\| \leq \frac{1+\epsilon/4}{1-\epsilon/4} \leq 1 + \epsilon$, for $\epsilon < 1$. An argument similar to the proof of Lemma 3.4.22,

shows that there exists $f_1, \dots, f_n : [0, 1]^k \rightarrow \{0, 1\}$ such that $\sum f_i = 1$, and for every $i \in [n]$, $\int f_i = \frac{1}{n}$ and $\|f_i - \frac{1}{n}\|_{U_k} \leq \delta$. Note that since f_i are zero-one valued functions, $\sum_{i=1}^n f_i = 1$ implies that the supports of f_i are pairwise disjoint. Then we have

$$\int \left(\sum_{i=1}^n x_i f_i \right)^H = \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}},$$

where $\tilde{H} = (\frac{\alpha+\beta}{s}, 0)$. Furthermore, if $\|x\|_s = n^{1/s}$, then

$$\begin{aligned} \left\| \left(\sum_{i=1}^n |x_i|^s f_i \right) - 1 \right\|_{U_k} &= \left\| \sum_{i=1}^n \left(|x_i|^s f_i - \frac{|x_i|^s}{n} \right) \right\|_{U_k} \\ &\leq \sum_{i=1}^n |x_i|^s \left\| f_i - \frac{1}{n} \right\|_{U_k} \leq \delta \|x\|_s = \delta n^{1/s}. \end{aligned}$$

Now by Lemma 3.4.24

$$\begin{aligned} \left| \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}} - 1 \right| &= \left| \int \left(\sum_{i=1}^n |x_i|^s f_i \right)^{\tilde{H}} - 1^{\tilde{H}} \right| \\ &\leq \delta n^{1/s} |\tilde{H}| \max \left(\left\| \sum_{i=1}^n |x_i|^s f_i \right\|_{\infty}, 1 \right)^{|\tilde{H}|-1} \leq \delta n^{|\tilde{H}|} |\tilde{H}|. \end{aligned}$$

Now taking δ sufficiently small finishes the proof.

4.1.5 Complex Interpolation

Let us recall the definition of the complex interpolation spaces. Two topological vector spaces are called **compatible**, if there exists a Hausdorff topological vector space containing both of these spaces as subspaces. Consider two compatible normed spaces X_0 and X_1 and endow the space $X_0 + X_1$ with the norm $\|f\|_{X_0 + X_1} = \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + \|f_1\|_{X_1})$. For every $0 \leq \theta \leq 1$, one constructs the corresponding complex interpolation space $[X_0, X_1]_{\theta}$, as in the following.

Let $\mathcal{F}(X_0, X_1)$ be the set of all analytic functions $v : \{z : 0 \leq \operatorname{Re}(z) \leq 1\} \rightarrow X_0 + X_1$ which are continuous and bounded on the boundary, and moreover such that the function $t \rightarrow v(j + it)$ ($j = 0, 1$) are continuous functions from the real line into X_j which tend to

zero as $|t| \rightarrow \infty$. We provide the vector space \mathcal{F} with a norm

$$\|v\|_{\mathcal{F}} := \max \left\{ \sup_{x \in \mathbb{R}} \|v(ix)\|_{X_0}, \sup_{x \in \mathbb{R}} \|v(1+ix)\|_{X_1} \right\}.$$

Then for every $0 \leq \theta \leq 1$, the complex interpolation space of X_0 and X_1 is a normed space $X_0 \cap X_1 \subseteq [X_0, X_1]_{\theta} \subseteq X_0 + X_1$ defined as

$$[X_0, X_1]_{\theta} := \{f \in X_0 + X_1 : v(\theta) = f \exists v \in \mathcal{F}(X_0, X_1)\},$$

with the following norm:

$$\|f\|_{\theta} = \inf \{ \|v\|_{\mathcal{F}} : f = v(\theta), v \in \mathcal{F}(X_0, X_1) \}.$$

The space $[X_0, X_1]_{\theta}$ has an interesting property. Consider compatible pairs X_0, X_1 and Y_0, Y_1 . Let $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ be a bounded linear map. Then (see [3]),

$$\|T\|_{[X_0, X_1]_{\theta} \rightarrow [Y_0, Y_1]_{\theta}} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\theta} \|T\|_{X_1 \rightarrow Y_1}^{\theta}. \quad (4.12)$$

Theorem 4.1.14 *Let $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space and H be a norming hypergraph pair of Type I with parameter 1. Then for every $0 \leq \theta \leq 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, where $p_0, p_1 \geq 1$,*

$$[L_{p_0 H}(\mathcal{M}), L_{p_1 H}(\mathcal{M})]_{\theta} = L_{p H}(\mathcal{M}).$$

Proof. Let $f : \Omega^k \rightarrow \mathbb{C}$ be a measurable function with $\|f\|_{pH} = 1$. Define

$$v : \{z : 0 \leq \operatorname{Re}(z) \leq 1\} \rightarrow L_{p_0 H}(\mathcal{M}) + L_{p_1 H}(\mathcal{M})$$

by

$$v(z) = |f|^{p \left(\frac{1-z}{p_0} + \frac{z}{p_1} \right)}.$$

Then $v(\theta) = |f|$ which shows that

$$\|f\|_{\theta} \leq \max \left\{ \sup_{x \in \mathbb{R}} \|v(ix)\|_{p_0 H}, \sup_{x \in \mathbb{R}} \|v(1+ix)\|_{p_1 H} \right\}.$$

But note that

$$\|v(ix)\|_{p_0H} = \left(\int |v(ix)|^{p_0H} \right)^{1/p_0H} = \left(\int (|f|^{p/p_0})^{p_0H} \right)^{1/p_0H} = \left(\int |f|^{pH} \right)^{1/p_0H} = 1,$$

and similarly $\|v(1+ix)\|_{p_1H} \leq 1$ which shows that $\|f\|_\theta \leq \|f\|_{pH}$.

Now for the other direction assume that $\|f\|_\theta = 1$. Then for every $\epsilon > 0$, there exists v_ϵ such that $f = v_\epsilon(\theta)$ and $\|v_\epsilon\|_{\mathcal{F}} \leq 1 + \epsilon$. By Hölder's inequality,

$$\|f\|_{pH}^{|H|} = \sup \left\{ \int f^H g^H : \|g\|_{qH} \leq 1 \right\},$$

where $1 = \frac{1}{p} + \frac{1}{q}$. Fix $g : \Omega^k \rightarrow \mathbb{C}$ with $\|g\|_{qH} \leq 1$, and define

$$u : \{z : 0 \leq \operatorname{Re}(z) \leq 1\} \rightarrow L_{q_0H}(\mathcal{M}) + L_{q_1H}(\mathcal{M})$$

by

$$u(z) = |g|^{q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right)},$$

where $\frac{1}{q_0} + \frac{1}{p_0} = 1$ and $\frac{1}{q_1} + \frac{1}{p_1} = 1$. Let

$$F_\epsilon(z) = \int v_\epsilon(z)^H u(z)^H,$$

and notice that

$$|F_\epsilon(ix)| = \int v_\epsilon(ix)^H u(ix)^H \leq \|v_\epsilon(ix)\|_{p_0H}^{|H|} \|u(ix)\|_{q_0H}^{|H|} \leq \|v_\epsilon\|_{\mathcal{F}}^{|H|} \times \|g^{q/q_0}\|_{q_0H}^{|H|} \leq (1 + \epsilon)^{|H|}.$$

Similarly

$$\begin{aligned} |F_\epsilon(1+ix)| &= \int v_\epsilon(1+ix)^H u(1+ix)^H \leq \|v_\epsilon(1+ix)\|_{p_1H}^{|H|} \|u(1+ix)\|_{q_1H}^{|H|} \\ &\leq \|v_\epsilon\|_{\mathcal{F}}^{|H|} \times \|g^{q/q_1}\|_{q_1H}^{|H|} \leq (1 + \epsilon)^{|H|}. \end{aligned}$$

Then

$$\left| \int f^H g^H \right| = |F_\epsilon(\theta)| \leq 1 + \epsilon,$$

which by tending ϵ to zero leads to $\|f\|_{pH} \leq 1$. We conclude that $\|f\|_{pH} = \|f\|_\theta$. \blacksquare

4.1.6 Proof of Theorem 4.1.12

In this section we give sharp bounds on the moduli of smoothness and convexity of the norms defined by semi-norming hypergraph pairs. This of course will prove Theorem 4.1.12.

Consider a non-factorizable semi-norming hypergraph pair H , and an infinite dimensional space L_H . Theorem 4.1.10 shows that L_H contains $\ell_{|H|}$ as a subspace, and thus $K_{t,p}(\ell_{|H|}) \leq K_{t,p}(L_H)$ and $K_{r,q}^*(\ell_{|H|}) \leq K_{r,q}^*(L_H)$, for $1 < t \leq 2 \leq r < \infty$ and $1 < p, q < \infty$. Comparing Proposition 4.1.8 with Figure 4.1 shows that proving the $|H|$ -Hanner inequality for L_H spaces, gives the optimal values of $K_{2,p}(L_H)$ and $K_{|H|,|H|}^*(L_H)$, for every $p > 1$.

Theorem 4.1.15 (Hanner Inequality) *Let H be a non-factorizable semi-norming hypergraph pair which is either of Type II, or of Type I with an even integer parameter. Then for every $f, g \in \ell_H$, we have*

$$\|f + g\|_H^{|H|} + \|f - g\|_H^{|H|} \leq (\|f\|_H + \|g\|_H)^{|H|} + \left| \|f\|_H - \|g\|_H \right|^{|H|}.$$

Proof. Without loss of generality assume that $\|f\|_H \geq \|g\|_H$. Let \mathcal{H} be the set of all pairs (H_1, H_2) such that H_1 and H_2 are hypergraph pairs taking only nonnegative integer values, and furthermore $H_1 + H_2 = H$ and $|H_2|$ is an even integer. Then

$$\begin{aligned} \|f + g\|_H^{|H|} + \|f - g\|_H^{|H|} &= \int (f + g)^H + (f - g)^H = \sum_{(H_1, H_2) \in \mathcal{H}} \int f^{H_1} g^{H_2} \\ &\leq \sum_{(H_1, H_2) \in \mathcal{H}} \|f\|_H^{|H_1|} \|g\|_H^{|H_2|} \\ &= (\|f\|_H + \|g\|_H)^{|H|} + (\|f\|_H - \|g\|_H)^{|H|}, \end{aligned}$$

where in the inequality we used Lemma 3.4.11. This completes the proof as we assumed $\|f\|_H \geq \|g\|_H$. ■

Consider a norming hypergraph pair H of Type I with parameter $s < 2$ and $|H| \geq 2$. Note that for every $2 \leq q < \infty$, ℓ_s does not satisfy the q -Hanner inequality, as otherwise

it would be 2-uniformly convex. Hence it follows from Theorem 4.1.10 that ℓ_H does not satisfy the q -Hanner inequality for any $2 \leq q < \infty$. However, we conjecture the following.

Conjecture 4.1.16 *Let $H = (\alpha, \beta)$ be a non-factorizable semi-norming hypergraph pair of Type I with parameter $s \geq 2$. Then every L_H space satisfies the $|H|$ -Hanner inequality.*

Since we could not establish the $|H|$ -Hanner inequality for all norming hypergraph pairs of Type I we have to treat some of them separately. The next two lemmas which give the optimum bounds for uniform smoothness and convexity constants of ℓ_H when H is a non-factorizable hypergraph pair of Type I with parameter $s \geq 2$ would have followed from a positive answer to Conjecture 4.1.16.

Lemma 4.1.17 (2-Smoothness) *Let $H = (\alpha, \beta)$ be a non-factorizable semi-norming k -hypergraph pair with $|H| \geq 2$. If H is of Type II, or of Type I with parameter $s \geq 2$, then*

$$K_{2,p}(\ell_H) = K_{2,p}(\ell_{|H|}) = \begin{cases} \sqrt{|H| - 1} & p \leq |H| \\ \sqrt{p - 1} & p \geq |H| \end{cases}$$

Proof. It suffices to prove $K_{2,|H|}(\ell_H) \leq \sqrt{|H| - 1}$, and the rest will follow from Remark 4.1.4. Suppose that H is defined over $V := V_1 \times \dots \times V_k$. For $f, g \in \ell_H$, we have to prove

$$\left(\frac{\|f + g\|_H^{|H|} + \|f - g\|_H^{|H|}}{2} \right)^{2/|H|} \leq \|f\|_H^2 + (|H| - 1)\|g\|_H^2. \quad (4.13)$$

Consider the counting measure on $\{-1, 1\}$, and define the two functions $\epsilon_1, \epsilon_2 : \{-1, 1\}^k \rightarrow \{-1, 0, 1\}$ as

$$\epsilon_1(x_1, \dots, x_k) = \begin{cases} 1 & x_1 = \dots = x_k \\ 0 & \text{otherwise} \end{cases},$$

and

$$\epsilon_2(x_1, \dots, x_k) = \begin{cases} x_1 & x_1 = \dots = x_k \\ 0 & \text{otherwise} \end{cases}.$$

Note that since H is non-factorizable, for $x \in \{-1, 1\}^{V_1} \times \dots \times \{-1, 1\}^{V_1}$, we have

$$\epsilon_1^H(x) = \begin{cases} 1 & x = (1, \dots, 1) \\ 0 & \text{otherwise} \end{cases}, \quad (4.14)$$

and

$$\epsilon_2^H(x) = \begin{cases} \eta & x = (\eta, \dots, \eta) \\ 0 & \text{otherwise} \end{cases}, \quad (4.15)$$

Let $\tilde{f} = f \otimes \epsilon_1$ and $\tilde{g} = g \otimes \epsilon_2$. From (4.14) and (4.15) it is easy to see that

$$\int (\tilde{f} + \tilde{g})^H = \int (\tilde{f} - \tilde{g})^H = \int (f + g)^H + (f - g)^H,$$

and $\int \tilde{f}^H = 2 \int f^H$ and $\int \tilde{g}^H = 2 \int g^H$. Hence it suffices to prove

$$\left(\frac{\int (\tilde{f} + \tilde{g})^H}{2} \right)^{2/|H|} \geq \left(\frac{\int \tilde{f}^H}{2} \right)^{2/|H|} + (|H| - 1) \left(\frac{\int \tilde{g}^H}{2} \right)^{2/|H|}.$$

which simplifies to

$$\left(\int (\tilde{f} + \tilde{g})^H \right)^{2/|H|} \geq \left(\int \tilde{f}^H \right)^{2/|H|} + (|H| - 1) \left(\int \tilde{g}^H \right)^{2/|H|}. \quad (4.16)$$

We will show that for $0 \leq t \leq 1$

$$\left(\int (\tilde{f} + t\tilde{g})^H \right)^{2/|H|} \geq \left(\int \tilde{f}^H \right)^{2/|H|} + t^2(|H| - 1) \left(\int \tilde{g}^H \right)^{2/|H|}. \quad (4.17)$$

Note that (4.17) reduces to (4.16) for $t = 1$. Consider the functions $L, R : [0, 1] \rightarrow \mathbb{R}$, defined as

$$L(t) = \left(\int (\tilde{f} + t\tilde{g})^H \right),$$

and

$$R(t) = \left(\int \tilde{f}^H \right)^{2/|H|} + t^2(|H| - 1) \left(\int \tilde{g}^H \right)^{2/|H|}.$$

We have

$$\frac{d}{dt} L(t) = \int \sum_{\psi \in V} \alpha(\psi) (\tilde{f} + t\tilde{g})^{H-1\psi} \tilde{g}^{1\psi} + \beta(\psi) (\tilde{f} + t\tilde{g})^{H-1\bar{\psi}} \tilde{g}^{1\bar{\psi}}.$$

Then

$$\frac{d}{dt}L(t)^{2/|H|} = \frac{2}{|H|} \left(\int \sum_{\psi \in V} \alpha(\psi)(\tilde{f} + t\tilde{g})^{H-1_\psi} \tilde{g}^{1_\psi} + \beta(\psi)(\tilde{f} + t\tilde{g})^{H-\overline{1_\psi}} \tilde{g}^{\overline{1_\psi}} \right) L(t)^{\frac{2-|H|}{|H|}}.$$

We want to compute the second derivative. Denote $\mathcal{H} = \{1_\psi : \psi \in V\} \cup \{\overline{1_\psi} : \psi \in V\}$, and define $\gamma : \mathcal{H} \rightarrow \mathbb{R}$ by $\gamma : 1_\psi \mapsto \alpha(\psi)$ and $\gamma : \overline{1_\psi} \mapsto \beta(\psi)$. We have

$$\begin{aligned} \frac{d^2}{dt^2}L(t)^{2/|H|} &= \frac{2}{|H|} \left(\int \sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2)(\tilde{f} + t\tilde{g})^{H-H_1-H_2} \tilde{g}^{H_1+H_2} \right. \\ &\quad \left. + \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1)(\tilde{f} + t\tilde{g})^{H-2H_1} \tilde{g}^{2H_1} \right) L(t)^{\frac{2-|H|}{|H|}} + \\ &\quad + \left(\frac{d}{dt}L(t) \right)^2 \frac{2(2-|H|)}{|H|^2} L(t)^{\frac{2-2|H|}{|H|}}. \end{aligned}$$

Recalling the definition of \tilde{f} and \tilde{g} , it is easy to see that

$$L(0)^{2/|H|} = R(0),$$

and since $\int \tilde{f}^{H-1_\psi} \tilde{g}^{1_\psi} = \int f^{H-1_\psi} g^{1_\psi} - \int f^{H-1_\psi} g^{1_\psi} = 0$ and $\int \tilde{f}^{H-\overline{1_\psi}} \tilde{g}^{\overline{1_\psi}} = \int f^{H-\overline{1_\psi}} g^{\overline{1_\psi}} - \int f^{H-\overline{1_\psi}} g^{\overline{1_\psi}} = 0$, we have

$$\left. \frac{d}{dt}L(t)^{2/|H|} \right|_{t=0} = \left. \frac{d}{dt}R(t) \right|_{t=0} = 0.$$

Furthermore, since H is of Type II or of Type I with parameter $s \geq 2$, by Lemma 3.4.11, we have

$$\begin{aligned} \frac{d^2}{dt^2}L(t)^{2/|H|} \Big|_{t=0} &= \frac{2}{|H|} \left(\int \sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2) \tilde{f}^{H-H_1-H_2} \tilde{g}^{H_1+H_2} + \right. \\ &\quad \left. \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1) \tilde{f}^{H-2H_1} \tilde{g}^{2H_1} \right) L(0)^{\frac{2-|H|}{|H|}} \\ &\leq \frac{2}{|H|} \left(\sum_{H_1 \neq H_2 \in \mathcal{H}} \gamma(H_1)\gamma(H_2) + \sum_{H_1 \in \mathcal{H}} \gamma(H_1)(\gamma(H_1) - 1) \right) \times \\ &\quad \times (\|\tilde{f}\|_H^{|H|-2} \|\tilde{g}\|_H^2) \|\tilde{f}\|_H^{2-|H|} \end{aligned} \quad (4.18)$$

$$= 2(|H| - 1) \|\tilde{g}\|_H^2 = \frac{d^2}{dt^2}R(t) \Big|_{t=0}. \quad (4.19)$$

Now for every $0 \leq t_0 \leq 1$, one can replace \tilde{f} with $\tilde{f} + t_0\tilde{g}$ in (4.19) and obtain that for every $0 \leq t_0 \leq 1$

$$\frac{d^2}{dt^2}L(t)^{2/|H|}\big|_{t=t_0} \leq \frac{d^2}{dt^2}R(t)\big|_{t=t_0}. \quad (4.20)$$

We conclude (4.17). ■

Next we prove Clarkson's inequalities for ℓ_H when H is a semi-norming hypergraph pair of Type II or of Type I with parameter $s \geq 2$. As it is mentioned above this would follow from Conjecture 4.1.16.

Lemma 4.1.18 (Clarkson's Inequalities) *Let H be a non-factorizable semi-norming hypergraph pair of Type II or Type I with parameter $s \geq 2$ such that $q := |H| \geq 2$. Then*

$$K_{p,q}(\ell_H) = K_{q,p}^*(\ell_H) = K_{q,q}^*(\ell_H) = 1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Recall that always $K_{q,q}^* \leq K_{q,p}^*$. Hence it suffices to prove $K_{p,q}(\ell_H) = 1$, as by Lemma 4.1.7 this would imply $K_{q,p}^*(\ell_H) = 1$. To this end, we need to show that for $f, g \in \ell_H$, we have

$$\left(\frac{\|f+g\|_H^q + \|f-g\|_H^q}{2} \right)^{1/q} \leq (\|f\|_H^p + \|g\|_H^p)^{1/p}, \quad (4.21)$$

which is equivalent to

$$\left(\left\| \frac{f+g}{2} \right\|_H^q + \left\| \frac{f-g}{2} \right\|_H^q \right)^{1/q} \leq \left(\frac{\|f\|_H^p + \|g\|_H^p}{2} \right)^{1/p}. \quad (4.22)$$

Proposition 4.1.8 shows that (4.22) follows from the $|H|$ -Hanner inequality. Hence Theorem 4.1.15 implies (4.22) when H is of Type II or it is of Type I with parameter s where s is an even integer. Next assume that H is of Type I with parameter $s \geq 2$.

For a real $1 \leq t < \infty$, and a norming hypergraph pair G , define the norm $L_t(\ell_G)$ on the set of pairs (f, g) where $f, g \in \ell_G$ as

$$\|(f, g)\|_{L_t(\ell_G)} := (\|f\|_G^t + \|g\|_G^t)^{1/t}.$$

Consider the linear map $T : (f, g) \mapsto (\frac{f+g}{2}, \frac{f-g}{2})$. Then (4.22) says that

$$\|T\|_{L_p(\ell_H) \rightarrow L_q(\ell_H)} \leq 2^{-\frac{1}{p}}. \quad (4.23)$$

We will prove this by interpolation. Let $\tilde{H} = \frac{1}{s}H$, and s_0 and s_1 be two even integers satisfying $2 \leq s_0 \leq s \leq s_1$, and θ be such that $\frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$. Then $\frac{1}{p} = \frac{1-\theta}{t_0} + \frac{\theta}{t_1}$, where $\frac{1}{t_0} + \frac{1}{s_0|\tilde{H}|} = 1$ and $\frac{1}{t_1} + \frac{1}{s_1|\tilde{H}|} = 1$. Theorem 4.1.14 above, together with Theorem 5.1.2 from [3] imply that

$$\left[L_{s_0|\tilde{H}|}(\ell_{s_0\tilde{H}}), L_{s_1|\tilde{H}|}(\ell_{s_1\tilde{H}}) \right]_{\theta} = L_{s|\tilde{H}|}([\ell_{s_0\tilde{H}}, \ell_{s_1\tilde{H}}]_{\theta}) = L_q(\ell_H),$$

and

$$\left[L_{t_0}(\ell_{s_0\tilde{H}}), L_{t_1}(\ell_{s_1\tilde{H}}) \right]_{\theta} = L_p([\ell_{s_0\tilde{H}}, \ell_{s_1\tilde{H}}]_{\theta}) = L_p(\ell_H).$$

Furthermore

$$\left(2^{-\frac{1}{t_0}} \right)^{1-\theta} \left(2^{-\frac{1}{t_1}} \right)^{\theta} = 2^{-\frac{1}{p}}.$$

Now since we know that (4.23) holds for even values of $s \geq 2$, we have

$$\|T\|_{L_{t_0}(\ell_{s_0\tilde{H}}) \rightarrow L_{s_0|\tilde{H}|}(\ell_{s_0\tilde{H}})} \leq 2^{-\frac{1}{t_0}},$$

and

$$\|T\|_{L_{t_1}(\ell_{s_1\tilde{H}}) \rightarrow L_{s_1|\tilde{H}|}(\ell_{s_1\tilde{H}})} \leq 2^{-\frac{1}{t_1}}.$$

Then interpolation (4.12), implies (4.23). ■

Next Lemma determines the moduli of smoothness and convexity of non-factorizable semi-norming hypergraph pairs of Type I with parameter $1 < s \leq 2$.

Lemma 4.1.19 *Let H be a non-factorizable semi-norming hypergraph pair of Type I with parameter $s > 1$ with $|H| \geq 1$. Then $K_{s,|H|}(\ell_H) = C(s, |H|)$ and $K_{|H|,s}^*(X) = C^*(|H|, s)$.*

Proof. Let $C := C(s, |H|)$ and $C^* := C^*(|H|, s)$. Consider $f, g \in \ell_H$. By (4.10) and

(4.11) we have

$$\begin{aligned}
\left(\frac{\|f + g\|_H^{|H|} + \|f - g\|_H^{|H|}}{2} \right)^{1/|H|} &\leq \left\| \left(\frac{|f + g|^{|H|} + |f - g|^{|H|}}{2} \right)^{1/|H|} \right\|_H \\
&\leq \left\| (|f|^s + |Cg|^s)^{1/s} \right\|_H \\
&\leq (\|f\|_H^s + \|Cg\|_H^s)^{1/s},
\end{aligned}$$

which shows that $K_{s,|H|}(\ell_H) \leq C$. To prove $K_{|H|,s}^* = C^*$, note that by (4.11) and (4.10)

we have

$$\begin{aligned}
\left(\frac{\|f + g\|_H^s + \|f - g\|_H^s}{2} \right)^{1/s} &\geq \left\| \left(\frac{|f + g|^s + |f - g|^s}{2} \right)^{1/s} \right\|_H \\
&\geq \left\| \left(|f|^{|H|} + \left| \frac{1}{C^*} g \right|^{|H|} \right)^{1/|H|} \right\|_H \\
&\geq \left(\|f\|_H^{|H|} + \left\| \frac{1}{C^*} g \right\|_H^{|H|} \right)^{1/|H|}.
\end{aligned}$$

■

Remark 4.1.20 Note that all results in Section 4.1.6 are stated for *non-factorizable* semi-norming hypergraph pairs. Consider a semi-norming hypergraph pair $H = H_1 \dot{\cup} \dots \dot{\cup} H_m$, where H_i 's are non-factorizable. If H is of Type I, then by Theorem 3.4.19, $\|\cdot\|_H = \|\cdot\|_{H_1}$, and thus one can apply the results of Section 4.1.6 to H_1 instead. However, some of our results do not cover the case where H is factorizable and of Type II. ■

Chapter 5

Graph norms

Recall from Section 2.3.1 that for a symmetric measurable map $w : [0, 1]^2 \rightarrow \mathbb{R}$, the C_4 norm is defined as $\|w\|_{C_4} := t_{C_4}(w)^{1/4}$. Similarly, one can see that for every natural number k , $\|w\|_{C_{2k}} := t_{C_{2k}}(w)^{1/2k}$ is a norm function. The C_4 norm, and the Gowers norms (its generalizations to k -variable functions) play an important role in the study of pseudo-randomness. Inspired by the fact that the cycles of even length correspond to norms, and by the numerous applications of these norms in graph theory, László Lovász posed the problem of characterizing all graphs that correspond to norms. In order to study this question, in Chapter 3 we introduced and studied the normed spaces that are defined through hypergraph pairs. The original question of Lovász is about graphs, and the framework of hypergraph pairs is more general than what one actually needs to study the question. As a consequence some proofs have become more complicated, and also some results do not translate immediately to the language of graphs. As we promised before, in this chapter we revisit the original question of Lovász, and state the consequences of the results developed in Chapter 3 to this question. Hence most of the results in this chapter are adaptations of results of Chapter 3 from hypergraph norms to graph norms. The only completely new result in this chapter is Theorem 5.1.9, whose proof constitutes the bulk of this chapter. We shall see the applications of these results

to an important conjecture due to Erdős, Simonovits and Sidorenko in the next chapter.

5.1 Lovász's question

For a graph $G = (V, E)$, a set $S \subseteq V$ is called an **independent set** if there is no edge with both endpoints in S . For reasons that will soon be apparent, we are mainly concerned with bipartite graphs. A graph $G = (V, E)$ is called a bipartite graph if V can be partitioned into two disjoint independent sets V_1 and V_2 . We call the partition of V into (V_1, V_2) a **bipartition** of G . Note that disconnected bipartite graphs have more than one bipartition. Let

$$\mathcal{WS} := \{w : [0, 1]^2 \rightarrow \mathbb{R} \mid w \text{ is measurable, bounded, and symmetric}\},$$

where symmetric means that $w(x, y) = w(y, x)$ for every $x, y \in [0, 1]$. Recall that the C_4 norm of a $w \in \mathcal{WS}$ is defined by $t_{C_4}(w)^{1/4}$. In an attempt to generalize the C_4 -norm, Lovász asked the following question.

Question 5.1.1 (Lovász) *For which graphs H , does the function $t_H(\cdot)^{1/|E(H)|}$ define a norm on \mathcal{WS} ?*

Consider a non-bipartite graph H . Let $w_1, w_2 \in \mathcal{WS}$ be defined as

$$w_1(x) = \begin{cases} 1 & x_1, x_2 \in [0, 1/2] \\ 1 & x_1, x_2 \in [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

and $w_2 = 1 - w_1$. Note that w_2 is the graphon corresponded to K_2 (See section 2.2.4), and hence $t_H(w_2) = t_H(K_2)$. Note that a graph is homomorphic to K_2 if and only if it is bipartite. Since H is not bipartite, $t_H(w_2) = t_H(K_2) = 0$, and we get

$$t_H(w_1)^{1/|E(H)|} + t_H(w_2)^{1/|E(H)|} < 1 = t_H(1)^{1/|E(H)|} = t_H(w_1 + w_2)^{1/|E(H)|},$$

and so $t_H(\cdot)^{1/|E(H)|}$ is not a norm. This shows that to study Question 5.1.1 it is sufficient to restrict to the case where H is bipartite.

Consider a bipartite graph H , and let $V(H) = X \dot{\cup} Y$ be a bipartition of H . Note that if H_1, \dots, H_k are connected components of H , then for every $w \in \mathcal{WS}$,

$$t_H(w) = t_{H_1}(w) \dots t_{H_k}(w). \quad (5.1)$$

Suppose that $t_H(\cdot)^{1/|E(H)|}$ defines a semi-norm on \mathcal{WS} . Now for every measurable and bounded, but not necessarily symmetric function $w : [0, 1]^2 \rightarrow \mathbb{R}$, we will symmetrize w to obtain a symmetric measurable function $Tw : [0, 1]^2 \rightarrow \mathbb{R}^+$. Figure 5.1 shows that intuitively how Tw is defined according to w .

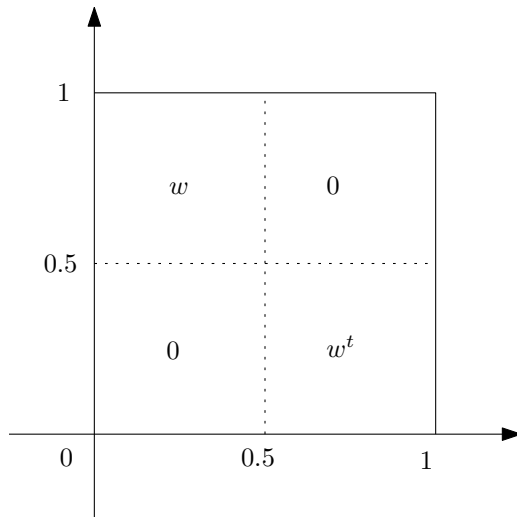


Figure 5.1: This figure shows how Tw is defined according to w . Here w^t is the transpose of w , defined by $w^t(x, y) := w(y, x)$.

More formally define $Tw : [0, 1]^2 \rightarrow \mathbb{R}$ as

$$Tw(x) := \begin{cases} w(2x_1, 2x_2 - 1) & x_1 \in [0, 1/2], x_2 \in (1/2, 1] \\ w(2x_2, 2x_1 - 1) & x_2 \in [0, 1/2], x_1 \in (1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

Note that Tw is measurable, bounded and symmetric. Furthermore, it is not difficult to see that for $j = 1, \dots, k$,

$$t_{H_j}(Tw) = 2^{1-|V(H_j)|} \mathbb{E} \prod_{uv \in E(H_j): u \in X \cap V(H_j), v \in Y \cap V(H_j)} w(x_u, y_v),$$

which together with (5.1) shows that

$$t_H(Tw) = 2^{k-|V(H)|} \mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v).$$

Hence $\left(\mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v) \right)^{1/|E(H)|} = c_H t_H(Tw)^{1/|E(H)|}$, where $c_H = 2^{\frac{k-|V(H)|}{|E(H)|}}$.

Since T is linear and $t_H(\cdot)^{1/|E(H)|}$ is a norm, we conclude that

$$\left(\mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v) \right)^{1/|E(H)|},$$

defines a norm on the space of measurable, bounded functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. On the other hand, if such a w is symmetric, then

$$\left(\mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v) \right)^{1/|E(H)|} = t_H(w)^{1/|E(H)|}.$$

These observations show that to study Question 5.1.1, we can use a more general setting than \mathcal{WS} and remove the condition that w is symmetric. Namely for a given bipartite graph H and its bipartition $V(H) := X \dot{\cup} Y$, the question is whether

$$\left(\mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v) \right)^{1/|E(H)|}$$

defines a norm on the space of measurable, bounded functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. As we shall see below this falls into the framework of hypergraph pairs developed in Chapter 3.

In order to show this we need to identify H and its bipartition with a hypergraph pair so that $\|\cdot\|_H = \left(\mathbb{E} \prod_{uv \in E(H): u \in X, v \in Y} w(x_u, y_v) \right)^{1/|E(H)|}$.

In graph theory, $G = (V, E)$ is called a bipartite graph if V can be partitioned into two disjoint independent sets V_1 and V_2 . In this chapter we use a different definition that

fixes one specific bipartition for G . So in the sequel by a **partitioned bipartite graph** we mean a triple $G = (V_1, V_2; E)$, where V_1 and V_2 are two disjoint sets and E is a subset of $V_1 \times V_2$. Note that here we fix the bipartition (V_1, V_2) as a part of the definition. Also here every edge is an ordered pair, and can be thought of as a directed edge from V_1 to V_2 . Consider a partitioned bipartite graph H . We assign two hypergraph pairs to H :

- We identify H with the hypergraph pair $(\alpha, 0)$ over $V_1 \times V_2$, where α is the indicator function of the edges of H .
- We also define $r(H) := \frac{H + \bar{H}}{2}$.

Furthermore, in this chapter we are only concerned with *real*-valued functions. Note that for a measurable $w : [0, 1]^2 \rightarrow \mathbb{R}$, using the notation of Chapter 3, we have

$$\|w\|_H = \left(\mathbb{E} \prod_{(u,v) \in E} w(x_u, y_v) \right)^{1/|E(H)|},$$

and

$$\|w\|_{r(H)} = \left(\mathbb{E} \prod_{(u,v) \in E} |w(x_u, y_v)| \right)^{1/|E(H)|}.$$

The reason that we defined the hypergraph pair $r(H)$ is that, for some applications, it suffices that $\|\cdot\|_{r(H)}$ defines a norm on the space of bounded measure functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. (See Chapter 6 for one such application.)

Definition 5.1.2 *Consider a partitioned bipartite graph H , and a measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. Let $L_H(\mathcal{M}, \mathbb{R})$ and $L_{r(H)}(\mathcal{M}, \mathbb{R})$ be respectively the sets of measurable functions $f : \Omega^k \rightarrow \mathbb{R}$ with $\| |f| \|_H < \infty$ and $\| |f| \|_{r(H)} < \infty$.*

*A partitioned bipartite graph H is called **norming (semi-norming)**, if $\|\cdot\|_H$ defines a norm (semi-norm) on $L_H(\mathcal{M}, \mathbb{R})$ for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. A partitioned bipartite graph H is called **weakly norming**, if $\|\cdot\|_{r(H)}$ defines a norm on $L_{r(H)}(\mathcal{M}, \mathbb{R})$ for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$.*

We have the following observations:

Lemma 5.1.3 *Let H be a partitioned bipartite graph. Then*

- (i) *If H is semi-norming or norming, then H is weakly norming.*
- (ii) *If H has a vertex of odd degree, then H is not norming.*

Proof. Part (i) is trivial. To prove (ii), define $w : \{0, 1\}^2 \rightarrow \mathbb{R}$ as

$$w(x, y) = \begin{cases} 1 & x = y \\ -1 & x \neq y \end{cases}$$

Let $H = (X, Y; E)$ be a partitioned bipartite graph and $w \in X$ be a vertex of odd degree. For $v \in Y$, fix $y_v \in [0, 1]$. Note that

$$\prod_{(w,v) \in E} w(0, y_v) = (-1)^{\deg(w)} \prod_{(w,v) \in E} w(1, y_v) = - \prod_{(w,v) \in E} w(1, y_v),$$

which shows that

$$\mathbb{E}_x \prod_{(w,v) \in E} w(x, y_v) = 0,$$

and in turn

$$\mathbb{E} \prod_{(u,v) \in E} w(x_u, y_v) = 0.$$

Thus $\|w\|_H = 0$ and so H is not norming. ■

According to Lemma 5.1.3 (i), we have the following implications:

$$\boxed{\text{norming} \Rightarrow \text{semi-norming} \Rightarrow \text{weakly norming}}$$

Theorem 5.1.4 below “almost” follows from Lemmas 3.4.9 and 3.4.11. The only problem is that here the theorem is about real-valued functions as opposed to the complex-valued functions in Lemmas 3.4.9 and 3.4.11. However, since to prove Theorem 5.1.4 one does not need to follow all the steps of the proofs of Lemmas 3.4.9 and 3.4.11, we give a complete proof of Theorem 5.1.4 below. We built most of the results in Chapters 3 and 4

upon Lemmas 3.4.9 and 3.4.11. Theorem 5.1.4 below plays the same role in the context of graph norms, and it is extremely useful in the sequel and shall be applied frequently. One can think of the inequalities (5.2) and (5.3) as common generalizations of the classical Hölder inequality and the Gowers-Cauchy-Schwarz inequality. Gowers [26, 28] proved a similar inequality in the context of Gowers norms, to show that Gowers norms satisfy the axioms of a normed space. Hence it was quite expected that for a graph H such an inequality would imply that $\|\cdot\|_H$ satisfies the axioms of a semi-norm. However, the surprising part of the following theorem is that it shows that every semi-norm $\|\cdot\|_H$ satisfies such an inequality.

Theorem 5.1.4 *Let H be a partitioned bipartite graph.*

- (i) *H is semi-norming iff $\int f^H$ is always positive, and for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, the following Gowers-Cauchy-Schwarz type inequality holds: For functions $\{f_e\}_{e \in E(H)}$, where $f_e : \Omega^2 \rightarrow \mathbb{R}$ are measurable, we have*

$$\int \prod_{e=uv \in E(H)} f_e(x_u, y_v) \leq \prod_{e \in E(H)} \|f_e\|_H. \quad (5.2)$$

- (ii) *H is weakly norming iff for every measure space $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$, the following Gowers-Cauchy-Schwarz type inequality holds: For functions $\{f_e\}_{e \in E(H)}$, where $f_e : \Omega^2 \rightarrow \mathbb{R}$ are measurable, we have*

$$\int \prod_{e=uv \in E(H)} |f_e(x_u, y_v)| \leq \prod_{e \in E(H)} \|f_e\|_{r(H)}. \quad (5.3)$$

Remark 5.1.5 The inequality (5.2) does not imply that H is norming. In order to verify that H is norming, one must also show that $\|f\|_H \neq 0$, for every $f \neq 0$.

Recall that in Proposition 3.4.14 we characterized all 1-hypergraph norms. In Proposition 3.4.20 we characterized all semi-norming hypergraph pairs that are not norming. It follows from these two results that if H is semi-norming but not norming, then there exists an integer $m \geq 1$ such that all components of H are either isolated vertices or isomorphic to $K_{1,m}$. ■

Proof.[Theorem 5.1.4] Let H be a partitioned bipartite graph, and $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose that $\int f^H$ is always nonnegative, and (5.2) is satisfied. Then for $f, g : \Omega^2 \rightarrow \mathbb{R}$, and an edge $e_0 = (u_0, v_0) \in H$, we have

$$\begin{aligned} \|f + g\|_H^{|E(H)|} &= \int \prod_{e=uv \in E(H)} (f + g)(x_u, y_v) \\ &= \int f(x_{u_0}, y_{v_0}) \prod_{e=uv \in E(H) \setminus \{e_0\}} (f + g)(x_u, y_v) \\ &\quad + \int g(x_{u_0}, y_{v_0}) \prod_{e=uv \in E(H) \setminus \{e_0\}} (f + g)(x_u, y_v) \\ &= \|f\|_H \|f + g\|_H^{|E(H)|-1} + \|g\|_H \|f + g\|_H^{|E(H)|-1}, \end{aligned}$$

which simplifies to the triangle inequality. This proves that $\|\cdot\|_H$ is a semi-norm.

Next suppose that (5.2) does not hold. Then there exists $f_e : \Omega^2 \rightarrow \mathbb{R}$, such that

$$\int \prod_{e \in E(H)} f_e > \prod_{e \in E(H)} \|f_e\|_H.$$

After proper normalization we may assume that $\|f_e\|_H \leq 1$, for every $e \in E(H)$, and $\int \prod_{e \in E(H)} f_e = c$, for some $c > 1$. Now by amplification by tensors (see Lemma 3.4.8), for every positive integer n , we have

$$\begin{aligned} \left\| \sum_{e \in E(H)} f_e^{\otimes 2n} \right\|_H^{|E(H)|} &= \int \prod_{e' \in E(H)} \left(\sum_{e \in E(H)} f_e^{\otimes 2n} \right) \\ &= \sum_{\pi: E(H) \rightarrow E(H)} \left(\int \prod_{e \in E(H)} w_{\pi(e)}^{\otimes 2n} \right) \\ &= \sum_{f: E(H) \rightarrow E(H)} \left(\int \prod_{e \in E(H)} w_{\pi(e)} \right)^{2n} \\ &\geq \left(\int \prod_{e \in E(H)} f_e \right)^{2n} = c^{2n}, \end{aligned}$$

while for every $e \in E(H)$, by Lemma 3.4.8, we have that

$$\|f_e^{\otimes 2n}\|_H = \|f_e\|_H^{2n} \leq 1.$$

Thus for large enough n , the triangle inequality fails:

$$\sum_{e \in E(H)} \|f_e^{\otimes 2n}\|_H \leq m < c^{2n/m} \leq \left\| \sum_{e \in E(H)} f_e^{\otimes 2n} \right\|_H. \quad (5.4)$$

The proof of the weakly norming case is similar. ■

Note that for a partitioned bipartite graph H , and a complex valued function f , $\|f\|_H = \| |f| \|_{r(H)}$, and hence for a measure space \mathcal{M} , $\|\cdot\|_{r(H)}$ is a norm on $L_{r(H)}(\mathcal{M}, \mathbb{C})$ iff it is a norm on $L_{r(H)}(\mathcal{M}, \mathbb{R})$. Thus the following necessary conditions for a graph to be weakly norming follow immediately from Remark 3.4.4 and Theorem 3.4.5.

Theorem 5.1.6 *Suppose that G is a weakly norming graph.*

- (i) *If G is connected, then for every subgraph $H \subseteq G$, we have $\frac{|E(H)|}{|V(H)|-1} \leq \frac{|E(G)|}{|V(G)|-1}$.*
- (ii) *If u and v belong to the same part in the bipartition of G , then $\deg(u) = \deg(v)$.*

For two partitioned bipartite graphs $G = (V_1, V_2; E)$ and $H = (W_1, W_2; E')$ define their tensor product $G \otimes H$ to be the partitioned bipartite graph with bipartition $(V_1 \times W_1, V_2 \times W_2)$, and the edges $([v_1, w_1], [v_2, w_2])$ where $(v_1, v_2) \in G$ and $(w_1, w_2) \in H$. The following theorem follows immediately from Theorem 3.3.2 and Lemma 3.3.3.

Theorem 5.1.7 *We have the following:*

- (i) *If G and H are both semi-norming (weakly norming), then so is $G \otimes H$.*
- (ii) *If G is norming and H is semi-norming, then $G \otimes H$ is norming.*
- (iii) *For every $m, n \geq 1$, the graph $K_{m,n}$ is weakly norming. If both m and n are even then $K_{m,n}$ is norming.*

Remark 5.1.8 If G is norming and H is weakly norming, then by Theorem 5.1.7 (iii) $G \otimes H$ is weakly norming. We do not know if the stronger statement follows that $G \otimes H$ is semi-norming.

If G is semi-norming and H is weakly norming, then $G \otimes H$ is weakly norming, but not necessarily semi-norming. Note that $G := K_2 \dot{\cup} K_2$ is semi-norming, and $H := K_{3,3}$ is weakly norming. Then $G \otimes H = K_{3,3} \dot{\cup} K_{3,3}$ is weakly norming, but not semi-norming. Indeed by Lemma 5.1.3 (ii), $K_{3,3} \dot{\cup} K_{3,3}$ is not norming, and then by Remark 5.1.5 it is semi-norming. ■

The n -dimensional hypercube Q_n is the partitioned bipartite graph $(X, Y; E)$ where X is the set of elements of $\{0, 1\}^n$ with an even number of 1's in their coordinates, and $Y = \{0, 1\}^n \setminus X$. Moreover $(x, y) \in E$ if and only if y differs only in one coordinate from x . The main theorem that we prove in this chapter is the following.

Theorem 5.1.9 *The hypercubes Q_n are weakly norming.*

5.1.1 Proof of Theorem 5.1.9

Let Q_n denote the n -dimensional hypercube. We identify the vertices of a hypercube Q_n with the 0-1 strings of length n , where two vertices are adjacent if their strings differ in one bit. With this notation we can concatenate two nodes $s \in V(Q_n)$ and $v \in V(Q_m)$ to obtain the node $sv \in V(Q_{n+m})$. Note that Q_n is bipartite. We use the convention that $X(Q_n)$ is the set of vertices with an even number of 1's in their strings, and $Y(Q_n)$ is the rest of the vertices.

By Theorem 5.1.4, to prove Theorem 5.1.9, it suffices to establish (5.3). We prove something stronger.

Consider a measure spaces $\mathcal{M} = (\Omega, \mathcal{F}, \mu)$. For all $u \in X$ and $v \in Y$, let $f_u : \Omega \rightarrow \mathbb{R}$ and $g_v : \Omega \rightarrow \mathbb{R}$ be measurable functions, and for every edge $e \in Q_n$ let $w_e : \Omega^2 \rightarrow \mathbb{R}$ be measurable functions. We claim the following strengthening of Theorem 5.1.9:

Claim 5.1.10

$$\int \left| \prod_{u \in X(Q_n), v \in Y(Q_n)} f_u(x_u) g_v(y_v) \prod_{e=(a,b) \in E} w_e(x_a, y_b) \right| \leq \prod_{e=(a,b) \in E} \left(\int |R_e| \right)^{1/|E(Q_n)|}, \quad (5.5)$$

where for $e = (a, b)$,

$$R_e := \prod_{u \in X(Q_n), v \in Y(Q_n)} f_a(x_u) g_b(y_v) \prod_{(s,t) \in E} w_e(x_s, y_t).$$

If one substitutes $f_u = 1$, $g_v = 1$ for every $u \in X(Q_n)$, and $v \in Y(Q_n)$, then by Theorem 5.1.4, Claim 5.1.10 reduces to Theorem 5.1.9. So it is sufficient to prove the claim. Before proving Claim 5.1.10 in its general form we prove it for $n = 2$ as a separate lemma. First notice that without loss of generality we can assume that $f_u, g_v \geq 0$ and $w_e \geq 0$ for every $u \in X(Q_n)$, $v \in Y(Q_n)$, and $e \in E(Q_n)$, and drop the absolute value signs from the proof.

Lemma 5.1.11 *Claim 5.1.10 holds for $n = 2$.*

Proof. For an edge $e = (u, v) \in Q_2$, define

$$w'_e : \Omega^2 \rightarrow \mathbb{R},$$

$$w'_e : (x, y) \mapsto \sqrt{f_u(x)} w_e(x, y) \sqrt{g_v(y)}.$$

Now since Q_2 is isomorphic to C_4 , for $n = 2$, we have

$$\text{L.H.S. of (5.5)} = \int \prod_{e=(u,v) \in E(Q_2)} w'_e(x_u, y_v) \leq \prod_{e \in E(C_4)} \|w'_e\|_{C_4} = \prod_{e \in E(Q_2)} \left(\sum R_e \right)^{1/4}.$$

■

We now turn to the proof of Claim 5.1.10 in its general form, and as we discussed above this will imply Theorem 5.1.9. The proof is divided into several steps, so that hopefully the main ideas can be distinguished from technicalities. Let us first introduce some notation that helps us to keep the proof short.

Remark 5.1.12 Let $\phi : V(Q_n) \rightarrow V(Q_n)$ be such that $\phi(u) \in X$ and $\phi(v) \in Y$ for every $u \in X$ and $v \in Y$, and furthermore, $(\phi(u), \phi(v)) \in E$ if $(u, v) \in E$. Define

$$R_\phi = \prod_{u \in X(Q_n), y \in Y(Q_n)} f_{\phi(u)}(x_u) g_{\phi(v)}(y_v) \prod_{(s,t) \in E(Q_n)} w_{(\phi(s), \phi(t))}(x_s, y_t).$$

For example, for $e = (a, b)$, let ϕ_e be defined as $\phi_e(u) = a$, if $u \in X(Q_n)$ and $\phi_e(u) = b$, otherwise. Then R_e as it is defined in Claim 5.1.10 is in fact the same as R_{ϕ_e} , and if we denote by $id.$ the identity map from $V(Q_n)$ to itself, then Claim 5.1.10 says that

$$\int |R_{id.}| \leq \prod_{e \in E(Q_n)} \left(\int |R_{\phi_e}| \right)^{1/|E(Q_n)|}. \quad (5.6)$$

■

Proof.[Claim 5.1.10] We prove the claim by induction. Before engaging in the calculations, let us explain the intuition behind the proof. The variables x_u, y_v assign some values to the vertices. The product in the left-hand side of (5.5) is the product of the functions f_u, g_v and w_e where f_u and g_v depend only on the values that are assigned to the vertices u and v respectively, and w_e depends only on the values that are assigned to the endpoints of e . The first step in the proof is to group these functions together so that they can be interpreted as the same product but for Q_{n-1} instead of Q_n . Then we can apply the induction hypothesis.

Step 1: We regroup the product in the left-hand side of (5.5) in the following way.

$$\prod_{u \in X(Q_n), v \in Y(Q_n)} f_u g_v \prod_{e \in E(Q_n)} w_e = \prod_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} (f_{0u} w_{(0u, 1u)} g_{1u}) (f_{1v} w_{(1v, 0v)} g_{0v}) \prod_{(s, t) \in E(Q_{n-1})} (w_{(0s, 0t)} w_{(1t, 1s)}). \quad (5.7)$$

The left-hand side of (5.7) is the product in the left-hand side of (5.5), and the right-hand side of (5.7) can be interpreted as the same product for Q_{n-1} but on different index sets in the following way: Let the value assigned to the vertices $u \in X(Q_{n-1})$ and $v \in Y(Q_{n-1})$ be the pair $[x_{0u}, y_{1u}]$ and $[x_{1v}, y_{0v}]$ respectively. Note that in the right-hand side of (5.7), $f_{0u} w_{(0u, 1u)} g_{1u}$ depends only on $[x_{0u}, y_{1u}]$, and $f_{1v} w_{(1v, 0v)} g_{0v}$ depends only on $[x_{1v}, y_{0v}]$, and finally $w_{(0s, 0t)} w_{(1t, 1s)}$ depends only on the pair $([x_{0s}, y_{1s}], [x_{1t}, y_{0t}])$.

More formally, to prove the claim for n and \mathcal{M} , we use the induction hypothesis for $n - 1$ with the measure $(\mathcal{M} \times \mathcal{M})$. Every vertex $v \in Q_{n-1}$ corresponds to two adjacent

vertices $0v$ and $1v$. To use the induction hypothesis, for $u \in X(Q_{n-1})$, define

$$f'_u : \Omega^2 \rightarrow \mathbb{R}$$

$$f'_u : [x, y] \mapsto f_{0u}(x)w_{(0u,1u)}(x, y)g_{1u}(y).$$

For $v \in Y(Q_{n-1})$, define

$$g'_v : \Omega^2 \rightarrow \mathbb{R}$$

$$g'_v : [x, y] \mapsto f_{1v}(x)w_{(1v,0v)}(x, y)g_{0v}(y).$$

and for $e = (u, v) \in E(Q_{n-1})$,

$$w'_e : (\Omega^2) \times (\Omega^2) \rightarrow \mathbb{R}$$

$$w'_e : ([x, y], [x', y']) \mapsto w_{(0u,0v)}(x, y')w_{(1v,1u)}(x', y).$$

Then

$$\begin{aligned} \text{L.H.S. of (5.5)} &= \text{R.H.S. of (5.7)} \\ &= \int \left| \prod_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} f'_u([x_{0u}, y_{1u}])g'_v([x_{1v}, y_{0v}]) \right. \\ &\quad \left. \prod_{e=(s,t) \in E(Q_{n-1})} w'_e([x_{0s}, y_{1s}], [x_{1t}, y_{0t}]) \right|. \end{aligned} \quad (5.8)$$

Then we apply the induction hypothesis to the right-hand side of (5.8) and obtain

$$\begin{aligned} \text{R.H.S. of (5.8)} &\leq \prod_{e=(a,b) \in E(Q_{n-1})} \left(\int \left| \prod_{u \in X(Q_{n-1}), v \in Y(Q_{n-1})} f'_u([x_{0u}, y_{1u}])g'_v([x_{1v}, y_{0v}]) \right. \right. \\ &\quad \left. \left. \prod_{(s,t) \in E(Q_{n-1})} w'_{(a,b)}([x_{0s}, y_{1s}], [x_{1t}, y_{0t}]) \right| \right)^{1/|E(Q_{n-1})|} \\ &= \prod_{e=(a,b) \in E(Q_{n-1})} \left(\int R_{\psi_e} \right)^{1/|E(Q_{n-1})|}, \end{aligned}$$

where for $e = (a, b)$

$$\begin{aligned}
\psi_e(0u) &= 0a & \forall u \in X(Q_{n-1}) \\
\psi_e(1u) &= 1a & \forall u \in X(Q_{n-1}) \\
\psi_e(0v) &= 0b & \forall v \in Y(Q_{n-1}) \\
\psi_e(1v) &= 1b & \forall v \in Y(Q_{n-1})
\end{aligned} \tag{5.9}$$

Combining this with (5.8) we obtain

$$\text{L.H.S. of (5.5)} \leq \prod_{e=(a,b) \in E(Q_{n-1})} \left(\int R_{\psi_e} \right)^{1/|E(Q_{n-1})|}. \tag{5.10}$$

Step 2: In this step we obtain a different bound for the left-hand side of (5.5). In Step 1, for every $v \in Q_{n-1}$, we grouped the two vertices $0v, 1v$ and the edge between them as one vertex (see (5.7)) and this reduced Q_n to Q_{n-1} . In this step we reduce Q_n to Q_2 . For every vertex $s \in \{00, 11\} = X(Q_2)$, define

$$f_s'' = \left(\prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} f_{su} g_{sv} \right) \left(\prod_{(u,v) \in E(Q_{n-2})} w_{(su,sv)} \right),$$

for every $t \in \{01, 10\} = Y(Q_2)$, define

$$g_t'' = \left(\prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} f_{tv} g_{tu} \right) \left(\prod_{(u,v) \in E(Q_{n-2})} w_{(tv,tu)} \right),$$

and for every edge $e = (s, t) \in E(Q_2)$,

$$w_e'' = \prod_{u \in X(Q_{n-2}), v \in Y(Q_{n-2})} w_{(su,tu)} w_{(tv,sv)}.$$

Note that the product in the left-hand side of (5.5) is equal to

$$\left(\prod_{s \in X(Q_2), t \in Y(Q_2)} f_s'' g_t'' \right) \left(\prod_{(s,t) \in E(Q_2)} w_{(s,t)}'' \right).$$

We can apply Lemma 5.1.11 with proper index sets to these functions. We get

$$\text{L.H.S of (5.5)} \leq \prod_{e=(s,t) \in E(Q_2)} \left(\int R_{\rho_e} \right)^{1/4}, \tag{5.11}$$

where for $e = (s, t)$

$$\begin{aligned}\rho_e(s'v) &= sv & \forall s' \in X(Q_2), v \in V(Q_{n-2}) \\ \rho_e(t'v) &= tv & \forall t' \in Y(Q_2), v \in V(Q_{n-2})\end{aligned}\tag{5.12}$$

Step 3: In this step we combine Steps 1 and 2. Note that in (5.10), the product R_{ψ_e} has the same form as the product in the left-hand side of (5.5). Thus we can apply Step 2 to $\int R_{\psi_e}$. For $e \in E(Q_{n-1})$ we get

$$\int R_{\psi_e} \leq \prod_{e'=(s,t) \in E(Q_2)} \left(\int R_{\rho_{e'} \circ \psi_e} \right)^{1/4}.\tag{5.13}$$

Combining this with (5.10) we obtain

$$\text{L.H.S. of (5.5)} \leq \prod_{e \in E(Q_{n-1})} \left(\prod_{e' \in E(Q_2)} \left(\int R_{\rho_{e'} \circ \psi_e} \right)^{1/4|E(Q_{n-1})|} \right)\tag{5.14}$$

Step 4: Now for some integer $k > 0$ we repeatedly apply Step 3, and by (5.14) we get,

$$\text{L.H.S. of (5.5)} \leq \prod_{e_1, \dots, e_k \in E(Q_{n-1})} \left(\prod_{e'_1, \dots, e'_k \in E(Q_2)} \left(\int R_{\rho_{e'_1} \circ \psi_{e_1} \circ \dots \circ \rho_{e'_k} \circ \psi_{e_k}} \right)^{4^{-k}|E(Q_{n-1})|^{-k}} \right).\tag{5.15}$$

Let us first assume that $\|w_e\|_\infty, \|f_u\|_\infty, \|g_v\|_\infty < C$ for some constant $C > 0$. We shall deal with the general case later. Note first that for every arbitrary $\phi : V(Q_n) \rightarrow V(Q_n)$, we have $\int R_\phi < L$ for some large number L which depends on C, f_u 's, g_v 's, and w_e 's but does not depend on ϕ . Notice that for $e = (a, b) \in E(Q_{n-1})$

$$\rho_{(00,01)} \circ \psi_e = \phi_{(0a,0b)},\tag{5.16}$$

and

$$\rho_{(11,10)} \circ \psi_e = \phi_{(1a,1b)},\tag{5.17}$$

where $\phi_{(0a,0b)}$ and $\phi_{(1a,1b)}$ are defined as in Remark 5.1.12.

Next note that for every $\tilde{e} \in E(Q_n)$, $e \in E(Q_{n-1})$, and $e' \in E(Q_2)$, we have $\rho_{e'} \circ \psi_e \circ \phi_{\tilde{e}} = \phi_{\tilde{e}}$. Then from (5.16) and (5.17) we can conclude that whenever there exists

$1 \leq i \leq k$ such that $e'_i \in \{(00, 01), (11, 10)\}$, then $\rho_{e'_1} \circ \psi_{e_1} \circ \dots \circ \rho_{e'_k} \circ \psi_{e_k} = \phi_e$ for some $e \in E(Q_n)$. Thus from (5.15), there exists numbers $p_e \geq 0$ such that

$$\sum_{e \in E(Q_n)} p_e = 1 - 2^{-k},$$

and

$$\text{L.H.S. of (5.5)} \leq \prod_{e \in E(Q_n)} \left(\int R_{\phi_e} \right)^{p_e} L^{2^{-k}}. \quad (5.18)$$

Since Q_n is edge transitive, by applying the bound (5.18) to different permutations of the edges and taking the geometric average, we finally conclude that

$$\text{L.H.S. of (5.5)} \leq L^{2^{-k}} \prod_{e \in E(Q_n)} \left(\int R_{\phi_e} \right)^{(1-2^{-k})/|E(Q_n)|}. \quad (5.19)$$

By tending k to infinity, (5.19) reduces to (5.6).

Step 5: Now consider the general case where $\|f_u\|_\infty$, $\|g_v\|_\infty$, and $\|w_e\|_\infty$ need not be bounded. Fix $C > 0$ and let $f'_u := \max(f_u, C)$, $g'_v := \max(g_v, C)$ and $w'_e := \max(w_e, C)$. We know that Claim 5.1.10 holds for these functions. By tending C to infinity the dominated convergence theorem implies the claim for the general case as well. ■

Chapter 6

The Erdős-Simonovits-Sidorenko Conjecture

In this chapter we apply the results of Chapter 5 to a conjecture from extremal graph theory.

6.1 The conjecture

Given a graph H , let $\text{ex}(n, H)$ be the maximum number of edges in a graph on n vertices which does not contain a copy of H as a subgraph. Determining the values of $\text{ex}(n, H)$ for different graphs H is one of the most important problems in extremal graph theory. These problems are usually referred to as Turán problems, or forbidden subgraph problems. Pál Turán [61] determined the exact values of $\text{ex}(n, K_r)$, for every positive integer r , and in his memory, $\text{ex}(n, H)$ is called the Turán number of H .

For integers $n, r > 0$, the Turán graph $T(n, r)$ is a graph formed by partitioning a set of n vertices into r subsets, with sizes as equal as possible, and connecting two vertices by an edge whenever they belong to different subsets. The graph will have $(n \bmod r)$ subsets of size $\lceil n/r \rceil$, and $r - (n \bmod r)$ subsets of size $\lfloor n/r \rfloor$. Note that $T(n, r)$ does not contain a copy of K_{r+1} . According to Turán's theorem, the Turán graph has the maximum possible

number of edges among all K_{r+1} -free graphs, or equivalently $\text{ex}(n, K_{r+1})$ is equal to the number of edges of $T(n, r)$.

Determining the exact Turán number of H is often very difficult, and this problem is open even in the simple case of the cycle of length 4. However, if one allows a small error, then the following theorem of Erdős and Stone [15] determines $\text{ex}(n, H)$:

Theorem 6.1.1 (Erdős, Stone [15]) *Let H be a graph. Then*

$$\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2),$$

where $\chi(H)$ is the minimum number of colors required to color the vertices of H so that no two adjacent vertices of H have the same color.

Note that the Turán graph $T(n, \chi(H) - 1)$ has $\left(1 - \frac{1}{\chi(H) - 1}\right) \binom{n}{2} + o(n^2)$ edges and it does not contain a copy of H .

For bipartite graphs H , the Turán graph $T(n, \chi(H) - 1)$ has no edges and so the situation is different. Theorem 6.1.1 gives only the limited information that an H -free graph on n vertices must have $o(n^2)$ edges. Since every bipartite graph H is a subgraph of a complete bipartite graph, a quantitative version of this fact follows from a theorem due to Kövári, T. Sós, and Turán:

Theorem 6.1.2 (Kövári, T. Sós, Turán [36]) *For $1 \leq r \leq s$,*

$$\text{ex}(n, K_{r,s}) \leq n^{2-1/r}.$$

It is conjectured in [16] that the bound in Theorem 6.1.2 is of the right order of magnitude. This conjecture is verified in [23] for $r \leq 2$, and in [6] for $r = 3$ and $s = 3$.

The situation for bipartite graphs is considerably more complicated than for non-bipartite graphs. For cycles of even length, in an unpublished note Erdős (see [14]) proved that $\text{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$. For the cycle of length 4, Kövári, T. Sós and Turán [36], Erdős, Rényi and V. T. Sós [12] and W. G. Brown [6] proved that

$$\text{ex}(n, C_4) = \frac{1}{2}n^{4/3} + o(n^{3/2}).$$

But for example, for the cycle of length 6, the exact constant is unknown, and the best known bounds are due to Füredi, Naor, and Verstraëte [24] who showed that

$$0.5338n^{4/3} \leq \text{ex}(n, C_6) \leq 0.6272n^{4/3}.$$

Apart from cycles of even length, there are very few nontrivial bipartite graphs H for which the right order of magnitude of $\text{ex}(n, H)$ is known. For many bipartite graphs H , our knowledge about $\text{ex}(n, H)$ is limited to poor bounds.

Erdős and Simonovits [14] studied the number of copies of H in graphs with more than $\text{ex}(n, H)$ edges. Trivially every such graph has at least one copy of H , but they noted that if the number of edges is more than $\text{ex}(n, H)$ then it must contain many copies of H . A simple count shows that the number of copies of H in a graph with n vertices is at most $n^{|V(H)|}$. The following conjecture is due to Erdős and Simonovits.

Conjecture 6.1.3 (Erdős-Simonovits [14]) *Let H be a bipartite graph. There exist constants $\alpha, c, c' > 0$ such that for every graph G on n vertices, if $|E(G)| \geq cn^{2-\alpha}$ then G contains at least*

$$c'n^{|V(H)|} \left(\frac{|E(G)|}{n^2} \right)^{E(H)}$$

copies of H .

Remark 6.1.4 Without any lower-bounds for $|E(G)|$, Conjecture 6.1.3 is trivially false, as for example if $|E(G)| \leq \text{ex}(n, H)$, then G might be H -free. Note that by Theorem 6.1.2, for every bipartite graph H , there exists a constant $\alpha > 0$ such that $\text{ex}(n, H) = O(n^{2-\alpha})$. There is a stronger version of Conjecture 6.1.3 in [14] which claims that in the statement of the conjecture, one can take $\alpha \in (0, 1)$ to be any number satisfying $\text{ex}(n, H) = O(n^{2-\alpha})$.

■

Let us explain the background of Conjecture 6.1.3. In extremal graph theory, the extremal graphs tend to be either very structured, or very chaotic in the sense that they look similar to random graphs. The motivation behind Conjecture 6.1.3 is that for every

bipartite graph H , it seems that among all graphs with fixed number of vertices and edges, graphs having roughly the minimum number of copies of H tend to look random. Let us consider the Erdős-Rényi random graphs. For an integer $n > 0$ and $0 \leq p \leq 1$, let $G := G(n, p)$ be the Erdős-Rényi random graph on n vertices where each edge is present independently with probability p . Note that the complete graph K_n contains exactly $\frac{|V(H)|!}{|\text{Aut}(H)|} \binom{n}{|V(H)|}$ copies of H , where $\text{Aut}(H)$ is the set of automorphisms of H . Each one of these copies is present in G with probability $p^{|E(H)|}$. Hence the expected number of copies of H in G is $\frac{|V(H)|!}{|\text{Aut}(H)|} p^{|E(H)|} \binom{n}{|V(H)|} = \Theta(n^{|V(H)|} p^{|E(H)|})$. Trivially $\mathbb{E}|E(G)| = p \binom{n}{2} = \Theta(pn^2)$. It is standard that for sufficiently large p , with very high probability $|E(G)|$, and the number of copies of H in G are both concentrated around their expected values (see [41]). Taking $p = n^{-\alpha}$, we have $\mathbb{E}|E(G)| = \Theta(n^{2-\alpha})$ and the expected number of copies of H in G is $\Theta(n^{|V(H)|} n^{-\alpha|E(H)|})$. This shows that Conjecture 6.1.3 is sharp if true.

Recall from Section 2.2.2 that $h_H^{\text{inj}}(G)$ is the number of *injective* homomorphisms from H to G . The number of copies of H inside G is equal to $\frac{1}{|\text{Aut}(H)|} h_H^{\text{inj}}(G)$. We also defined a normalized version of $h_H^{\text{inj}}(G)$: namely $t_H^{\text{inj}}(G)$ is the probability that a random *injective* mapping from $V(H)$ to $V(G)$ defines a homomorphism. Hence

$$t_H^{\text{inj}}(G) = \frac{h_H^{\text{inj}}(G)}{|V(G)|(|V(G)| - 1) \dots (|V(G)| - |V(H)| + 1)}.$$

Using these notations, Conjecture 6.1.3 says that under the assumptions of the conjecture

$$t_H^{\text{inj}}(G) \geq c' t_{K_2}^{\text{inj}}(G)^{|E(H)|}. \quad (6.1)$$

Note that the constant c' in (6.1) might be different from the constant c' in Conjecture 6.1.3. However, they are both positive constants depending only on H . The following conjecture is due to Sidorenko.

Conjecture 6.1.5 (Sidorenko's conjecture [53, 54]) *For every graph G , and every bipartite graph H , we have*

$$t_H(G) \geq t_{K_2}(G)^{|E(H)|}. \quad (6.2)$$

Sidorenko formulated Conjecture 6.1.5 and observed that it is equivalent to Conjectures 6.1.3:

First we show that the Erdős-Simonovits conjecture implies Sidorenko's conjecture. Consider a graph H , and suppose that (6.1) holds for every graph G with $|E(G)| \geq cn^{2-\alpha}$. To get a contradiction, suppose that (6.2) fails for a graph G , i.e.

$$t_H(G) < t_{K_2}(G)^{|E(H)|}$$

We will take G and amplify it by taking its tensor powers. Then we take a proper blowup to obtain a counter-example to Conjecture 6.1.3. Blowups and tensor products of graphs are defined in Section 2.2.3. Recall that in Section 2.2.3 for a positive integer k , we defined

$$G^{\otimes k} := \underbrace{G \otimes \dots \otimes G}_{k \text{ copies}}.$$

Lemma 2.2.5 shows that $t_K(G^{\otimes k}) = t_K(G)^k$, for every positive integer k and every graph K . Hence by taking k to be sufficiently large, we obtain a graph $G_1 := G^{\otimes k}$ such that

$$t_H(G_1) < \frac{c'}{2} t_{K_2}(G_1)^{|E(H)|}.$$

For a positive integer m , let G_2 be the m -blowup of G_1 . Lemma 2.2.4 shows that $t_K(G_2) = t_K(G_1)$, for every graph K . Hence

$$t_H(G_2) < \frac{c'}{2} t_{K_2}(G_2)^{|E(H)|}. \quad (6.3)$$

Lemma 2.2.3 shows that for every graph K ,

$$|t_K(G_2) - t_K^{\text{inj}}(G_2)| \leq \frac{1}{|V(G_2)|} \binom{|V(K)|}{2} = \frac{1}{m|V(G_1)|} \binom{|V(K)|}{2} = O(1/m).$$

Hence it follows from (6.3) that

$$\begin{aligned} t_H^{\text{inj}}(G_2) &\leq t_H(G_2) + O(1/m) < \frac{c'}{2} t_{K_2}(G_2)^{|E(H)|} + O(1/m) \\ &< \frac{c'}{2} \left(t_{K_2}^{\text{inj}}(G_2) + O(1/m) \right)^{|E(H)|} + O(1/m) = \frac{c'}{2} t_{K_2}^{\text{inj}}(G_2)^{|E(H)|} + O(1/m), \end{aligned}$$

and so for sufficiently large m ,

$$t_H^{\text{inj}}(G_2) < c' t_{K_2}^{\text{inj}}(G_2)^{|E(H)|}.$$

Furthermore, since $|V(G_2)| = m|V(G_1)|$ and $|E(G_2)| = m^2|E(G_1)|$, for sufficiently large m , $|E(G_2)| \geq c|V(G_2)|^{2-\alpha}$. We obtained a graph G_2 with $|E(G_2)| \geq c|V(G_2)|^{2-\alpha}$ which does not satisfy

$$t_H^{\text{inj}}(G_2) \leq c' t_{K_2}^{\text{inj}}(G_2)^{|E(H)|},$$

and this contradicts our assumption.

Now let us show that Sidorenko's conjecture implies the Erdős-Simonovits conjecture. If Sidorenko's conjecture is true, then it follows from (6.2) and Lemma 2.2.3 that for a graph G with n vertices

$$t_H^{\text{inj}}(G) \geq \left(t_{K_2}^{\text{inj}}(G) - \frac{1}{n} \right)^{|E(H)|} - \frac{1}{n} \binom{|V(H)|}{2} = t_{K_2}^{\text{inj}}(G)^{|E(H)|} - O(1/n).$$

This shows that there exists a constant c such that if $t_{K_2}^{\text{inj}}(G) \geq cn^{-1/|E(H)|}$, then

$$t_H^{\text{inj}}(G) \geq \frac{1}{2} t_{K_2}^{\text{inj}}(G)^{|E(H)|}.$$

Since $t_{K_2}^{\text{inj}}(G) = \frac{|E(G)|}{\binom{n}{2}} \geq \frac{|E(G)|}{n^2}$, we conclude that if $|E(G)| \geq cn^{2-\frac{1}{|E(H)|}}$, then

$$t_{K_2}^{\text{inj}}(G)^{|E(H)|} \geq \frac{1}{2} t_{K_2}^{\text{inj}}(G)^{|E(H)|},$$

which shows that Conjecture 6.1.3 holds with $\alpha = \frac{1}{|E(H)|}$. We summarize

Conjecture 6.1.3 \Leftrightarrow Conjecture 6.1.5

Remark 6.1.6 The trick of tensoring has another interesting consequence. Consider a bipartite graph H . Suppose that for a universal constant $c > 0$, we could show that $t_H(G) \geq ct_{K_2}(G)^{|E(H)|}$, for all graphs G . Then for every positive integer k , we have

$$t_H(G)^k = t_H(G^{\otimes k}) \geq ct_{K_2}(G^{\otimes k})^{|E(H)|} = ct_{K_2}(G)^{k|E(H)|}.$$

In this case, by tending k to infinity, we can conclude $t_H(G) \geq t_{K_2}(G)^{|E(H)|}$. ■

Recall that graphons are symmetric measurable functions from $[0, 1]^2$ to $[0, 1]$. In Section 2.2.4, we defined $t_H(w)$ for graphons w as

$$t_H(w) := \mathbb{E} \prod_{uv \in E(H)} w(x_u, x_v),$$

where $\{x_u\}_{u \in V(H)}$ are independent random variables taking values uniformly in $[0, 1]$.

We also corresponded a graphon w_G to every graph G so that $t_K(w_G) = t_K(G)$ for every graph K . On the other hand, Theorem 2.2.7 says that given a graphon w , there exists a sequence of graphs $\{G_i\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} t_K(G_i) = t_K(w)$, for every graph K . These facts show that Sidorenko's conjecture is equivalent to the seemingly stronger statement that for every graphon w , and every bipartite graph H

$$t_H(w) \geq t_{K_2}(w)^{|E(H)|}. \quad (6.4)$$

We can strengthen the statement further: Graphons are *symmetric* measurable functions from $[0, 1]^2$ to $[0, 1]$. It turns out that the condition of symmetry is irrelevant to Conjecture 6.1.5, and the following more general statement is equivalent to Conjectures 6.1.3 and 6.1.5.

Conjecture 6.1.7 (Sidorenko's conjecture reformulated [53, 54]) *For every measurable function $w : [0, 1]^2 \rightarrow \mathbb{R}^+$, and every bipartite graph $H = (X, Y; E)$, we have*

$$\left(\mathbb{E} \prod_{(u,v) \in E} w(x_u, y_v) \right)^{1/|E|} \geq \mathbb{E} w(x, y). \quad (6.5)$$

Note that for a symmetric $w : [0, 1]^2 \rightarrow [0, 1]$, by raising both sides of (6.5) to the power $|E|$, we see that it is equivalent to (6.4). Let us explain why Conjectures 6.1.5 and 6.1.7 are equivalent. First note that (6.4) is homogeneous with respect to scaling, and this shows that changing the range of w in (6.4) from the interval $[0, 1]$ to the larger set of \mathbb{R}^+ leads to an equivalent statement. This explains why in Conjecture 6.1.7, the range of w is \mathbb{R}^+ .

It remains to justify the removal of the symmetry condition. We use the trick of symmetrization which was already used once in Chapter 5. Consider a measurable $w : [0, 1]^2 \rightarrow \mathbb{R}^+$ which is not necessarily symmetric. We will symmetrize w to obtain a symmetric measurable function $w' : [0, 1]^2 \rightarrow \mathbb{R}^+$. Figure 6.1 shows that intuitively how w' is defined according to w .

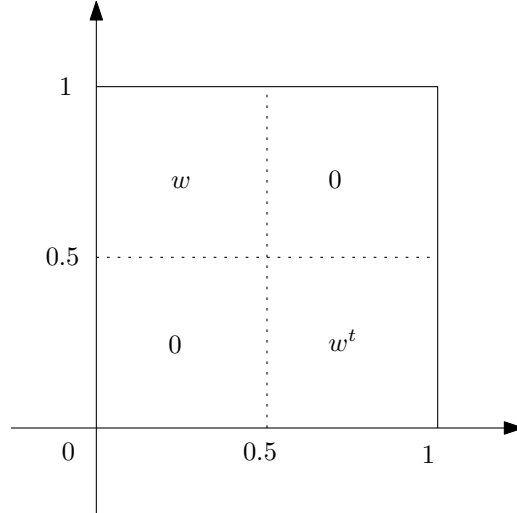


Figure 6.1: This figure shows how w' is defined according to w . Here w^t is the transpose of w , defined by $w^t(x, y) := w(y, x)$.

More formally $w' : [0, 1]^2 \rightarrow \mathbb{R}^+$ is defined as

$$w'(x) := \begin{cases} w(2x_1, 2x_2 - 1) & x_1 \in [0, 1/2], x_2 \in (1/2, 1] \\ w(2x_2, 2x_1 - 1) & x_2 \in [0, 1/2], x_1 \in (1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

Note that w' is measurable and symmetric, and hence assuming Conjecture 6.1.5, by (6.4) we have

$$\left(\mathbb{E} \prod_{(u,v) \in E} w'(x_u, y_v) \right)^{1/|E|} \geq \mathbb{E} w'(x, y). \quad (6.6)$$

Trivially

$$\mathbb{E} w'(x, y) = \frac{1}{2} \mathbb{E} w(x, y). \quad (6.7)$$

Let H_1, \dots, H_k be the connected components of H . For every $1 \leq i \leq k$, by definition of w' , $\prod_{(u,v) \in E(H_i)} w'(x_u, y_v) \neq 0$, if $x_u \in [0, 1/2]$ and $y_v \in (1/2, 1]$ for all $u, v \in V(H_i)$ or $x_u \in (1/2, 1]$ and $y_v \in [0, 1/2]$ for all $u, v \in V(H_i)$. Each one of these two events happens with probability $2^{-|V(H_i)|}$, and furthermore by definition of w' , conditioned on one of those events, the expected value of $\prod_{(u,v) \in E(H_i)} w'(x_u, y_v)$ is equal to $\mathbb{E} \prod_{(u,v) \in E(H_i)} w(x_u, y_v)$. Hence

$$\mathbb{E} \prod_{(u,v) \in E(H_i)} w'(x_u, y_v) = 2^{1-|V(H_i)|} \mathbb{E} \prod_{(u,v) \in E(H_i)} w(x_u, y_v).$$

Then

$$\begin{aligned} \mathbb{E} \prod_{(u,v) \in E} w'(x_u, y_v) &= \mathbb{E} \prod_{i=1}^k \prod_{(u,v) \in E(H_i)} w'(x_u, y_v) \\ &= \prod_{i=1}^k \left(\mathbb{E} \prod_{(u,v) \in E(H_i)} w'(x_u, y_v) \right) \\ &= \prod_{i=1}^k \left(2^{1-|V(H_i)|} \mathbb{E} \prod_{(u,v) \in E(H_i)} w(x_u, y_v) \right) \\ &= 2^{k-|V(H)|} \mathbb{E} \prod_{(u,v) \in E(H)} w(x_u, y_v) \\ &\geq \mathbb{E} \prod_{(u,v) \in E(H)} w(x_u, y_v). \end{aligned}$$

This together with (6.6) and (6.7) shows that if Conjecture 6.1.5 is true, then

$$\left(\mathbb{E} \prod_{(u,v) \in E} w(x_u, y_v) \right)^{1/|E|} \geq \left(\mathbb{E} \prod_{(u,v) \in E} w'(x_u, y_v) \right)^{1/|E|} \geq \mathbb{E} w'(x, y) = \frac{1}{2} \mathbb{E} w(x, y). \quad (6.8)$$

We have now shown that, assuming Conjecture 6.1.5, (6.8) holds for every measurable $w : [0, 1]^2 \rightarrow \mathbb{R}^+$. Similar to the case of Remark 6.1.6, we can remove the $1/2$ constant from (6.8), and obtain (6.5). We conclude that

$$\boxed{\text{Conjecture 6.1.3} \Leftrightarrow \text{Conjecture 6.1.5} \Leftrightarrow \text{Conjecture 6.1.7}}$$

We will refer to the equivalent Conjectures 6.1.3, 6.1.5, and 6.1.7 as the Erdős-Simonovits-Sidorenko conjecture.

Let A be the adjacency matrix of a graph G on n vertices, and let P_k denote the path of length k , where k is a positive integer. Let $B := A^k$, and note that $B_{i,j}$ is the number of walks of length k in G starting from i and ending at j . In other words $B_{i,j}$ is the number of homomorphisms from a path of length k to G that maps one end of the path to i , and the other end to j . Hence $t_{P_k}(G) = n^{-k-1} \sum_{i,j=1}^n B_{i,j}$ and $t_{K_2}(G) = n^{-2} \sum_{i,j=1}^n A_{i,j}$. This shows that for $H = P_k$, Conjecture 6.1.5 is equivalent to the following theorem of Blakley and Roy [4] proven in 1965.

Theorem 6.1.8 (Blakley-Roy) *Let A be an $n \times n$ symmetric matrix with nonnegative real entries, and k be a positive integer. For $B := A^k$, we have*

$$n^{-k-1} \sum_{i,j=1}^n B_{i,j} \geq \left(n^{-2} \sum_{i,j=1}^n A_{i,j} \right)^{k-1}.$$

Erdős and Simonovits formulated Conjecture 6.1.3 in 1984, and verified it for cycles of even length, trees, and the 3-dimensional cube. In order to verify the conjecture for the 3-dimensional cube, they showed that if the conjecture is valid for a graph H , then it remains valid if one adds two adjacent new vertices u and v to H , and connect u to all the vertices in one part, and v to all the vertices in the other part. Applying this procedure to the cycle of length 6 results in the 3 dimensional cube.

Later, Sidorenko studied this problem further in [53]. He reformulated the Erdős-Simonovits conjecture as Conjecture 6.1.5, and verified it for more classes of graphs. He proved the following theorem.

Theorem 6.1.9 (Sidorenko [53]) *The following statements hold.*

- *If the number of vertices in one of the parts of H is 4 or less (while the size of the other part is arbitrary) then Conjecture 6.1.3 is valid.*
- *If Conjecture 6.1.3 is valid for graphs H_1, \dots, H_k , then it is also valid for the disjoint union of these graphs.*

- If Conjecture 6.1.3 is valid for a graph H , then it remains valid when a new vertex w is added to one of the parts and it is joined to all the vertices in the other part.
- If Conjecture 6.1.3 is valid for a graph H , then it is valid for the graph obtained by joining a new vertex to one vertex of H .

Now let us apply our results to the equivalent Conjectures 6.1.3, 6.1.5, and 6.1.7. Note that in our notation (6.5) says that $\|w\|_H \geq \|w\|_{K_2}$. We prove the following theorem.

Theorem 6.1.10 *Let $H = (X, Y; E)$ be a weakly norming partitioned bipartite graph. Then for every subgraph $K \subseteq H$ and every measurable map $w : [0, 1]^2 \rightarrow \mathbb{R}^+$ we have*

$$\|w\|_H \geq \|w\|_K.$$

Proof. For $e \in E(G)$, define $w_e = w$, and for $e \in E(H) \setminus E(K)$ define $w_e = 1$. Since H is weakly norming, by Theorem 5.1.4 we have

$$\mathbb{E} \prod_{e=(u,v) \in E(H)} w_e(x_u, y_v) \leq \prod_{e \in E(H)} \|w_e\|_{r(H)}.$$

By our choice of w_e we get

$$t_K(w) = \mathbb{E} \prod_{e=(u,v) \in E(H)} w_e(x_u, x_v) \leq \left(\prod_{e \in E(K)} \|w\|_{r(H)} \right) \left(\prod_{e \in E(H) \setminus E(K)} \|1\|_{r(H)} \right) = \|w\|_{r(H)}^{|E(G)|},$$

or equivalently $\|w\|_{r(K)} \leq \|w\|_{r(H)}$. ■

Note that taking $K = K_2$ in Theorem 6.1.10 verifies the Erdős-Simonovits-Sidorenko conjecture for weakly norming graphs. In Theorem 5.1.9 we showed that hypercubes are norming. Hence we have the following corollary.

Corollary 6.1.11 *For every $w : [0, 1]^2 \rightarrow \mathbb{R}^+$, and every two positive integers $n_1 < n_2$ we have*

$$\|w\|_{Q_{n_1}} \geq \|w\|_{Q_{n_2}}.$$

In particular taking $n_1 = 1$ verifies the Erdős-Simonovits-Sidorenko conjecture for the hypercubes.

Note that what Theorem 6.1.10 shows for weakly norming graphs is much stronger than the assertion of the Erdős-Simonovits-Sidorenko conjecture. Given a weakly norming graph H and a subgraph $K \subseteq H$, it shows that

$$t_H(G) \geq t_K(G)^{|E(H)|/|E(K)|}, \quad (6.9)$$

for every graph G . Note that again the Erdős-Rényi random graphs show that (6.9) is sharp. The inequality (6.9) is not valid for every bipartite graph as for example although $P_3 \subseteq P_4$, there exists a graph G with $t_{P_4}(G) < t_{P_3}(G)^{3/2}$. This shows that the direct application of the above approach cannot be used to give a positive answer to the Erdős-Simonovits-Sidorenko conjecture in its full generality.

A similar (but weaker) statement to (6.9) for paths is proven by Erdős and Simonovits in [13]. Consider a graph G . Erdős and Simonovits proved that for positive integers $n \leq m$, we have $t_{P_{2m}}(G) \geq t_{P_{2n}}(G)^{\frac{2m-1}{2n-1}}$. This generalizes the Blakley-Roy Theorem as for $n = 1$ it is equivalent to that theorem. They furthermore conjectured $t_{P_{2m-1}}(G) \geq t_{P_{2n-1}}(G)^{\frac{2m-2}{2n-2}}$. This conjecture would have followed from Theorem 6.1.10, if P_{2m-1} was weakly norming, but Theorem 5.1.6 shows that for $m > 2$, P_{2m-1} is not weakly norming.

We shall further discuss the Erdős-Simonovits-Sidorenko conjecture in Section 7.2, where we introduce some related open problems.

Chapter 7

Conclusion

In this short chapter, we discuss some open problems and make some concluding remarks.

7.1 Norming Graphs

The starting point of this thesis was a question of Lovász asking for which graphs H , $t_H(w)^{1/|E(H)|}$ defines a norm on the space of bounded measurable functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. We called such graphs norming. We also studied two other variations of this notion, namely a graph H is called semi-norming if $t_H(w)^{1/|E(H)|}$ defines a semi-norm, and it is called weakly norming if $t_H(|w|)^{1/|E(H)|}$ defines a norm on the space of bounded measurable functions $w : [0, 1]^2 \rightarrow \mathbb{R}$. As it is observed in Chapter 5, the following implications hold:

$$\boxed{\text{norming} \Rightarrow \text{semi-norming} \Rightarrow \text{weakly norming}}$$

In Theorem 5.1.4 it is shown that H is semi-norming (weakly norming), if and only if a Cauchy-Schwarz type inequality holds. In Theorems 5.1.7 and 5.1.9, we established these Cauchy-Schwarz type inequalities for certain graphs such as complete bipartite graphs and hypercubes. The proofs of Theorems 5.1.7 and 5.1.9 are both based on iterated applications of the Cauchy-Schwarz inequality, and in fact, we are not aware of

an example of a weakly norming graph which cannot be proven to be weakly norming by this technique. In order to be able to apply this approach, the graph H must be very symmetric; for example it must be edge-transitive. (Recall that a graph H is called edge-transitive, if for every two edges e_1 and e_2 of H , there is an isomorphism that maps e_1 to e_2 .) Note that complete bipartite graphs, and hypercubes are edge-transitive. We conjecture the following:

Conjecture 7.1.1 *Every weakly norming graph is edge-transitive.*

On the other hand, it is not true that one can use iterated applications of the Cauchy-Schwarz inequality to establish a Cauchy-Schwarz type inequality for every edge-transitive bipartite graph. Currently Theorem 5.1.6 is the only available tool to us for refuting a graph from being weakly norming. Every edge-transitive bipartite graph satisfies the conditions of this theorem, and we do not know the answer to the following question:

Question 7.1.2 *Is there an edge-transitive bipartite graph that is not weakly norming?*

Let us give an explicit example of an edge-transitive bipartite graph for which we do not know whether it is weakly norming. Recall that the Cartesian product of two graphs G and H is a graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$. Note that the Cartesian product is different from the tensor product. Now $C_6 \times C_6$, the Cartesian product of the cycle of length 6 by itself is edge-transitive and bipartite. But we do not know whether it is weakly norming.

Note that the n -dimensional hypercube is the n -times cartesian product of K_2 by itself. In Theorem 5.1.9 we showed that hypercubes are weakly norming. One might ask the following question:

Question 7.1.3 *Is it true that the Cartesian product of two weakly norming bipartite graphs is again weakly norming?*

As the smallest case we suggest determining whether $K_2 \times C_6$ is weakly norming. Since $K_2 \times C_6$ is not edge-transitive we feel that the answer is negative. So a positive answer to this question will require new techniques, and a negative answer will probably lead to new necessary conditions on the structure of weakly norming graphs and hence an extension of Theorem 5.1.6.

7.2 The Erdős-Simonovits-Sidorenko Conjecture

Recall that the Erdős-Simonovits-Sidorenko conjecture says that for every bipartite graph H , and every graph G , we have $t_H(G) \geq t_{K_2}(G)^{|E(H)|}$. In Theorem 6.1.10, we showed that every weakly norming graph H satisfies

$$t_H(G) \geq t_K(G)^{|E(H)|/|E(K)|},$$

for every subgraph K of H , and every graph G . In particular taking $K = K_2$ shows that the Erdős-Simonovits-Sidorenko conjecture holds for weakly norming graphs. We ask the following question.

Question 7.2.1 *Which graphs H satisfy $t_H(G) \geq t_K(G)^{|E(H)|/|E(K)|}$, for every subgraph $K \subseteq H$ and every graph G ?*

As mentioned at the end of Chapter 6, $H = P_4$ does not satisfy the assertion of Question 7.2.1, and so the answer is not simply “all bipartite graphs”.

Consider the graph $H := K_{5,5} - C_{10}$ which is the graph obtained by removing the edges of a complete cycle from $K_{5,5}$. This is the smallest graph for which the Erdős-Simonovits-Sidorenko conjecture is open [54].

Question 7.2.2 *Is $K_{5,5} - C_{10}$ weakly norming?*

By Theorem 6.1.10, if the answer to Question 7.2.2 is positive, then the Erdős-Simonovits-Sidorenko conjecture holds for $K_{5,5} - C_{10}$. However, we believe that this graph is not

weakly norming as it is not edge-transitive. In order to show that $K_{5,5} - C_{10}$ is not weakly norming, one might try to show that it does not satisfy the assertion of Question 7.2.1.

7.3 The Hanner inequality

In Chapter 4 we discussed geometric properties of the hypergraph norms and determined their moduli of smoothness and convexity. In certain cases we proved a stronger result. Namely in Theorem 4.1.15 we established the order $|H|$ -Hanner inequality for the L_H spaces, when H is a non-factorizable semi-norming hypergraph pair which is either of Type II, or of Type I with an even integer parameter.

Let $H = (\alpha, \beta)$ be a non-factorizable semi-norming hypergraph pair of Type I with parameter $s \geq 2$. In Conjecture 4.1.16 we conjectured that every L_H space satisfies the $|H|$ -Hanner inequality.

Our knowledge about the Hanner inequality is very limited. Hanner [33] showed that for $1 < p < \infty$ the L_p spaces satisfy the p -Hanner inequality. The inequality is known to hold for some Sobolev spaces [37]. Even for the trace norms, one of the most studied class of norm functions, the situation is not totally understood. Ball *et al* [1] verified the p -Hanner inequality for the trace norm S_p when $1 < p \leq 4/3$ and the dual case of $4 \leq p < \infty$. They conjectured that this is true for every $1 < p < \infty$. The proof of [1] is based heavily on the spectral interpretation of these norms, and it seems very unlikely that it can be used to establish the Hanner inequality for other normed spaces. We believe that Conjecture 4.1.16 is a good starting point, as hypergraph norms have simple descriptions compared to most of the known normed spaces. A positive answer to this conjecture might shed some new light on Hanner's inequality, and provide new tools for establishing the inequality for other normed spaces.

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