Appendix A Envelope operations

In this study, ground motion envelopes are defined as the maximum amplitude of ground motion over a 1-second window of a time history. This definition is consistent with the maximum-amplitude-over-1-second datastream that comes in closest to real-time to the network central processing station at Caltech from the CISN digital stations. The parameterization of these ground motion envelopes introduced in this study characterizes the envelope of a seismogram as a combination of the envelopes of ambient noise, the P-wave, and the S-wave (and later-arriving phases). These envelopes combine according to the rule described by Eqn. 2.1, which is

$$E_{observed}(t) = \sqrt{E_P^2(t) + E_S^2(t) + E_{ambient}^2} + \epsilon$$

where

$$\begin{aligned} E_{observed}(t) &= \text{envelope of observed ground motion} \\ E_P(t) &= \text{envelope of P-wave} \\ E_S(t) &= \text{envelope of S-wave and later-arriving phases} \\ E_{ambient} &= \text{ambient noise at the site} \\ \epsilon &= \text{difference between predicted and observed envelope} \end{aligned}$$

This appendix describes several ways to justify Eqn. 2.1.

A.1 Stochastic process approach

Envelope functions played a role in the early efforts of engineering seimsologists to model accelerograms. In general, acceleration time histories were simulated by modulating a stationary random process designed to have the desired frequency content of observed ground motions with an envelope function describing the time-varying intensity of the time history.

Following the procedure (and notation) used by Jennings, Housner, and Tsai (1968), let $\{x(t)\}$ be a normally distributed stationary random process with mean of zero and variance of unity and with units of acceleration. Let $E(t)$ be the positive semi-definite envelope function describing the desired time-varying intensity of this process. A non-stationary process $\{z(t)\}$ can be generated by modulating $\{x(t)\}$ with $E(t)$:

$$\{z(t)\} = E(t)\{x(t)\} \tag{A.1}$$

This non-stationary process $\{z(t)\}$ will have mean of zero and a time-dependent variance $E^2(t)$ defined in terms of the envelope function. If we sum two similarly defined non-stationary processes $\{z_1(t)\}$ and $\{z_2(t)\}$ with stationary components $\{x_1(t)\}$, $\{x_2(t)\}$ and envelope functions $E_1(t)$, $E_2(t)$, then

$$\begin{aligned}
\{z_{tot}\} &= \{z_1(t)\} + \{z_2(t)\} && \text{(A.2)}\\
&= E_1(t)\{x_1(t)\} + E_2(t)\{x_2(t)\} && \text{(A.3)}\\
\text{with variance} \quad & E_{tot}^2(t) = E_1^2(t) + E_2^2(t) && \text{(A.4)}\\
\text{thus} \quad & E_{tot}(t) = \sqrt{E_1^2(t) + E_2^2(t)} && \text{(A.5)}
\end{aligned}$$

This is consistent with how we combine the P-wave, S-wave, and ambient noise envelopes in Eqn. 2.1.

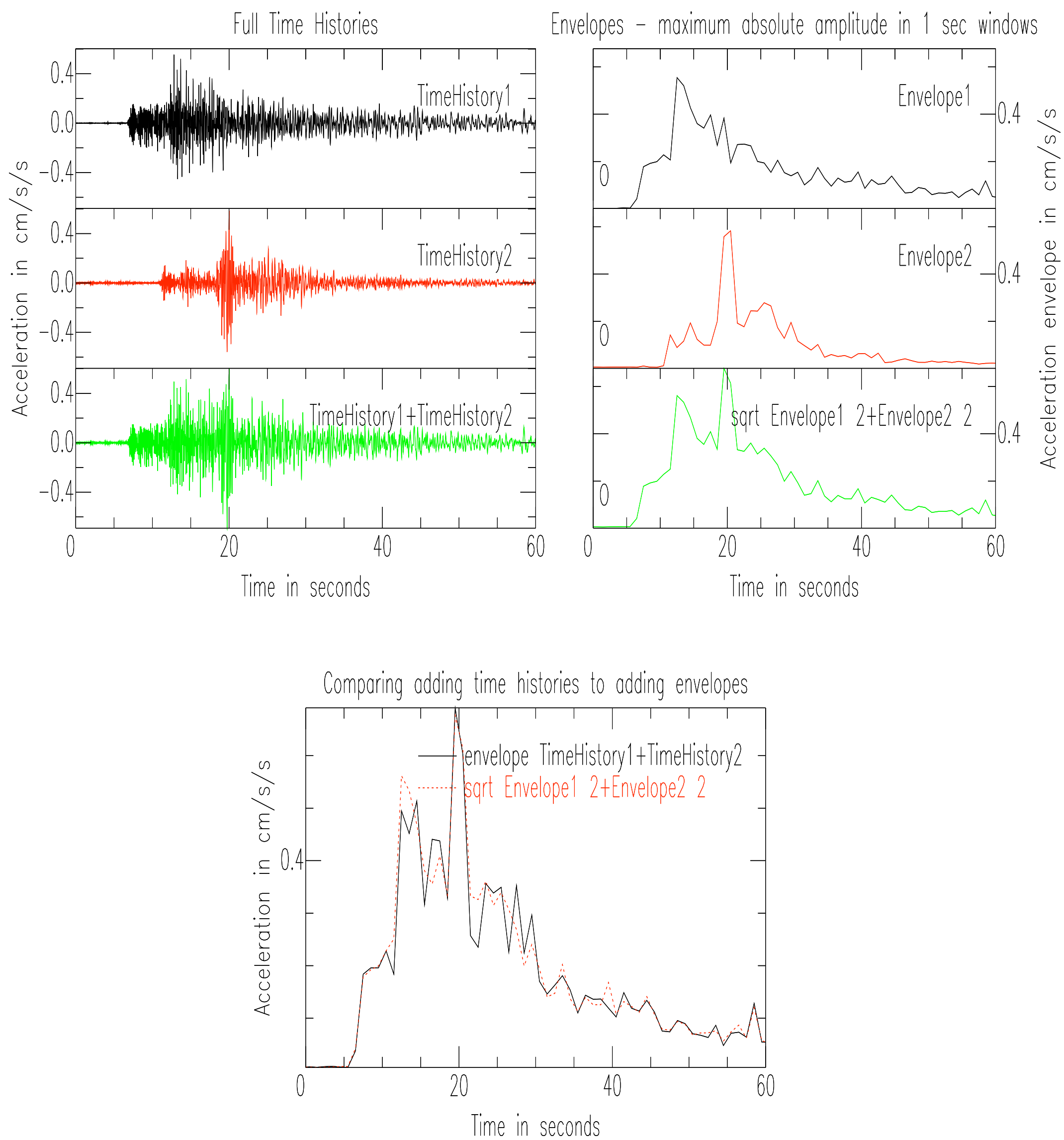

Figure A.1: Adding Envelopes (stochastic process approach): Top left panel: the first two plots show 2 full (100 samples per second) time histories, the third plot shows the sum of these two time histories. Top right panel: the first two plots show envelopes (maximum absolute value over 1 second window) of the time histories to the left, the third plot shows the combined envelope summing according to Eqn. 2.1. Bottom panel: this plot compares the envelope of the summed time histories (bottom plot of top left panel) with the sum of the envelopes (bottom plot of top right panel). While some discrepancies occur (due to differences in when the downsampling is performed), the envelope of the sum is general agreement with the sum of the envelopes.

A.2 Hilbert transform approach

From Bracewell (2000), the Hilbert transform of a real-valued function $f(t)$ is defined by

$$
\begin{aligned}
F_{Hi}(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t')\,dt'}{t'-t} && \text{(A.6)}\\
&= -\frac{1}{\pi t} * f(t) && \text{(A.7)}\\
& \quad \text{where '*' denotes the convolution operator} && \text{(A.8)}
\end{aligned}
$$

[Note: the convolution of two functions $f(t)$ and $g(t)$ in the time domain is the integral $h(t)$ defined as

$$h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(t')g(t-t')\,dt' \qquad \text{(A.9)}$$

Let $H(\omega)$, $F(\omega)$, and $G(\omega)$ be the Fourier transforms of $h(t)$, $f(t)$, and $g(t)$, respectively. Recall that the relationship between a function $f(t)$ and its Fourier transform $F(\omega)$ is given by

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt && \text{(Fourier transform)} && \text{(A.10)}\\
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}dt && \text{(Inverse Fourier transform)} && \text{(A.11)}
\end{aligned}
$$

Convolution in the time-domain is (conveniently) equivalent to multiplication in the frequency domain. That is,

$$H(\omega) = F(\omega)G(\omega) \qquad \text{(A.12)}$$

Thus, an alternate way to find $h(t)$, the convolution of $f(t)$ and $g(t)$, is to take their Fourier transforms, multiply in the frequency domain, and take the inverse Fourier transform.]

Consider a real function $f(t)$. We can associate with this real function a complex function

$$f(t) - iF_{Hi}(t)$$

whose real part is $f(t)$ and whose complex part is its Hilbert transform. Such a complex function is sometimes called an analytic signal. The envelope of a function $f(t)$, which we denote as $E_f(t)$, is defined as the amplitude of its analytic signal.

$$E_f(t) = |f(t) - iF_{Hi}(t)| = \sqrt{f^2(t) + F_{Hi}^2(t)} \tag{A.13}$$

$$\text{or} \qquad E_f^2(t) = f^2(t) + F_{Hi}^2(t) \tag{A.14}$$

Using this formal definition of envelopes, let us examine how to combine envelopes of different functions. Consider two independent, real-valued functions $f(t)$ and $g(t)$. From Eqn. A.14, if F_{Hi} and G_{Hi} denote the Hilbert transforms of $f(t)$ and $g(t)$, repsectively, then the envelope of their sum, $f(t) + g(t)$, is

$$\begin{aligned}
E_{f+g}^2(t) &= (f+g)^2 + (F_{Hi} + G_{Hi})^2 && \text{(A.15)}\\
&= (f^2 + 2fg + g^2) + \left(\frac{-1}{2\pi}\right) * (f^2 + 2fg + g^2) && \text{(A.16)}\\
&= \left[f^2 + \left(\frac{-1}{\pi t}\right)^2 * f^2\right] + \left[g^2 + \left(\frac{-1}{\pi t}\right)^2 * g^2\right] && \text{(A.17)}\\
&= [f^2 + F_{Hi}^2] + [g^2 + G_{Hi}^2] && \text{(A.18)}\\
&= E_f^2(t) + E_g^2(t) && \text{(A.19)}
\end{aligned}$$

and thus

$$E_{f+g}(t) = \sqrt{E_f^2(t) + E_g^2(t)} \tag{A.20}$$

Note that we can go from Eqn. A.16 to Eqn. A.17 because the cross term $2fg$ is zero, since f, g are independent (and hence, orthogonal). Again, this is consistent with how we add P-wave, S-wave, and ambient noise envelopes in Eqn. 2.1, assuming that these signals are independent of each other.

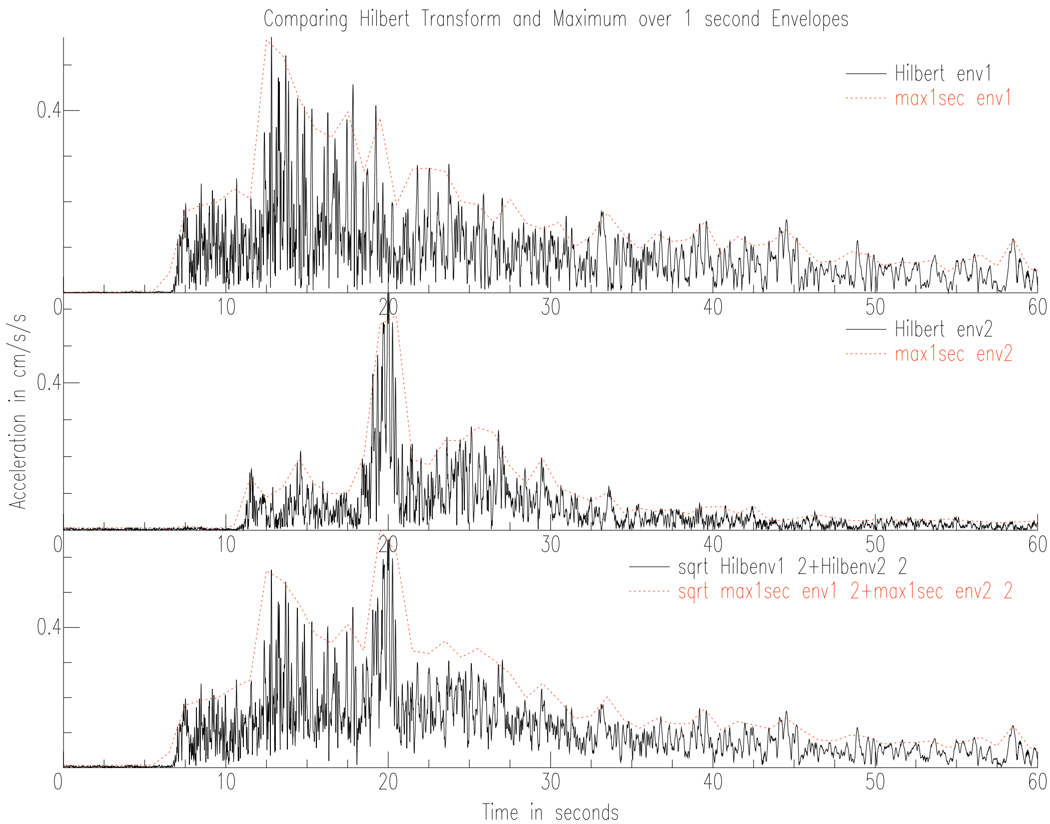

Figure A.2: Adding envelopes (Hilbert transform approach): The top panel shows the Hilbert transform envelope of the full (100 samples per second) Time History 1 in a solid line(Fig. A.1) and the corresponding maximum over 1 second envelope in a dashed line. The middle panel show a similar plot for the full Time History 2. The bottom panel shows the sum (according to Eqn. 2.1) of the Hilbert transform envelopes of Time Histories 1 and 2 in a solid line, and that of the maximum over 1 second envelopes in a dashed line. There is general agreement between the shapes of the Hilbert transform envelopes and the maximum over 1 second envelopes. The discrepancies are likely to be due to the downsampling of the maximum over 1 second envelopes. The Hilbert transform envelope is still a 100 sample per second signal. The envelopes would be in better agreement if a running window was used for the maximum over 1 second datastream.