Semantics of Probabilistic Programs

An Algebraic Approach

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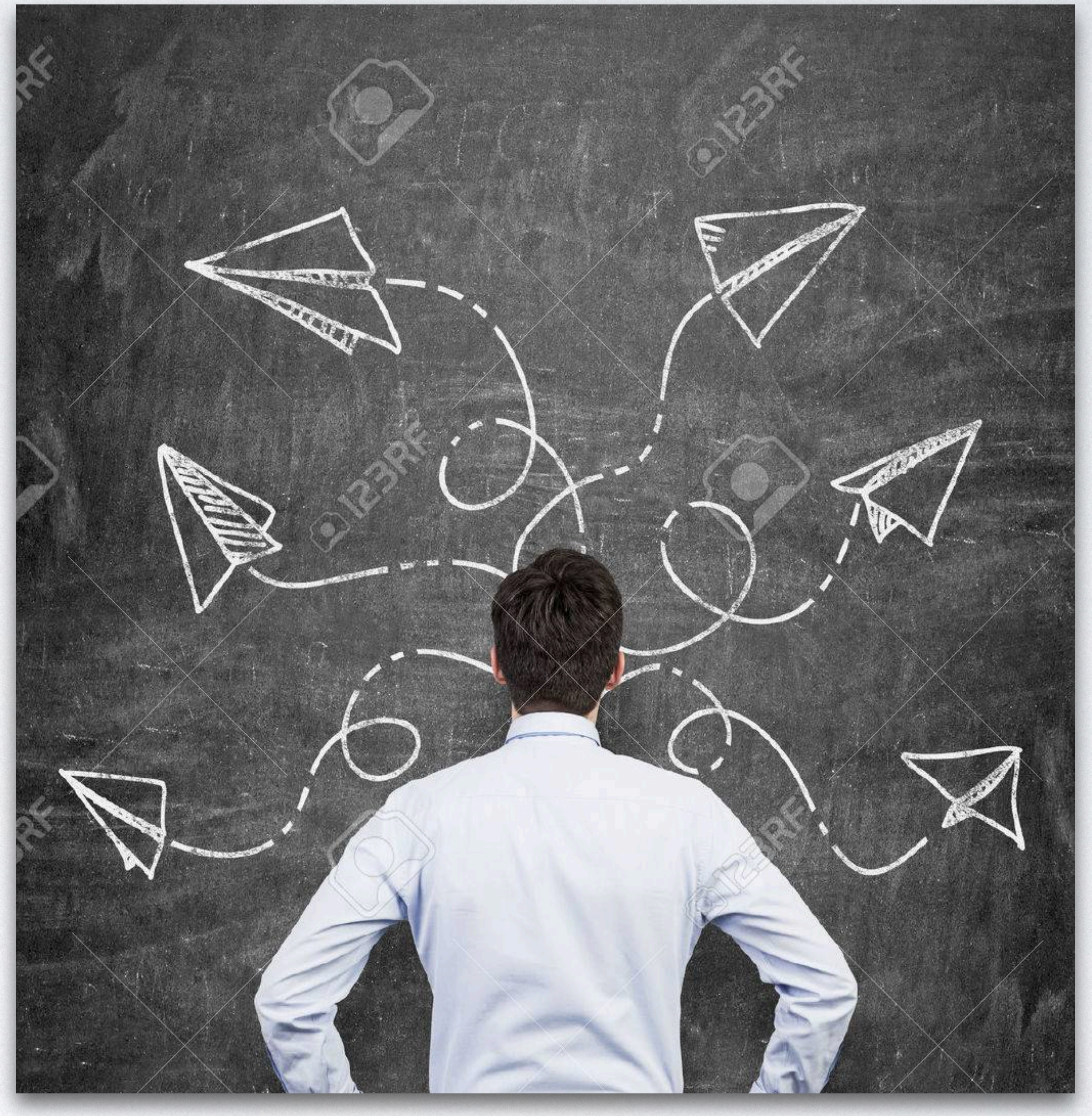
Carnegie Mellon University

Probabilistic Programs



Draw random data from distributions

Condition control-flow at random

Probabilistic Programs

- True randomness
- Distributions on executions

```
b1 ~ Bernoulli(0.5);
b2 ~ Bernoulli(0.7);
while (b1 && b2) do
  if prob(0.6) then
    b1 ~ Bernoulli(0.5)
  else
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  fi;
  tick(1.0)
od;
return (b1, b2)
```
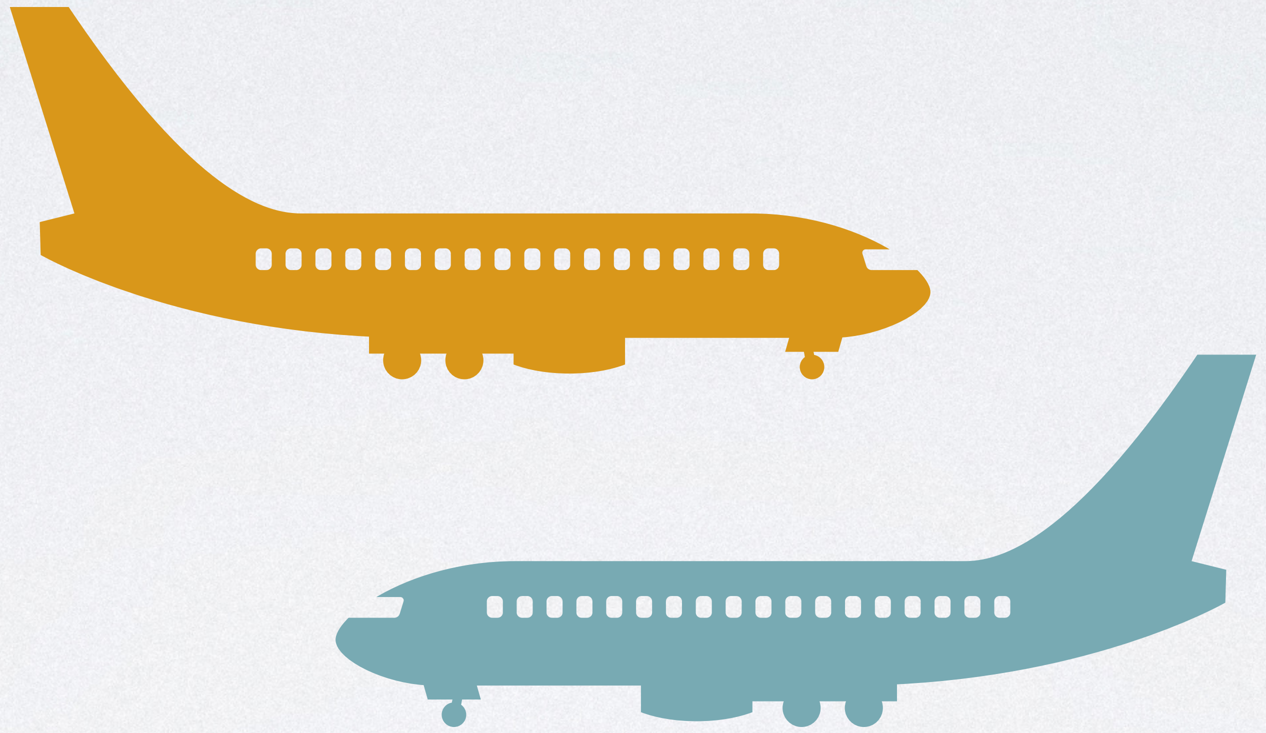
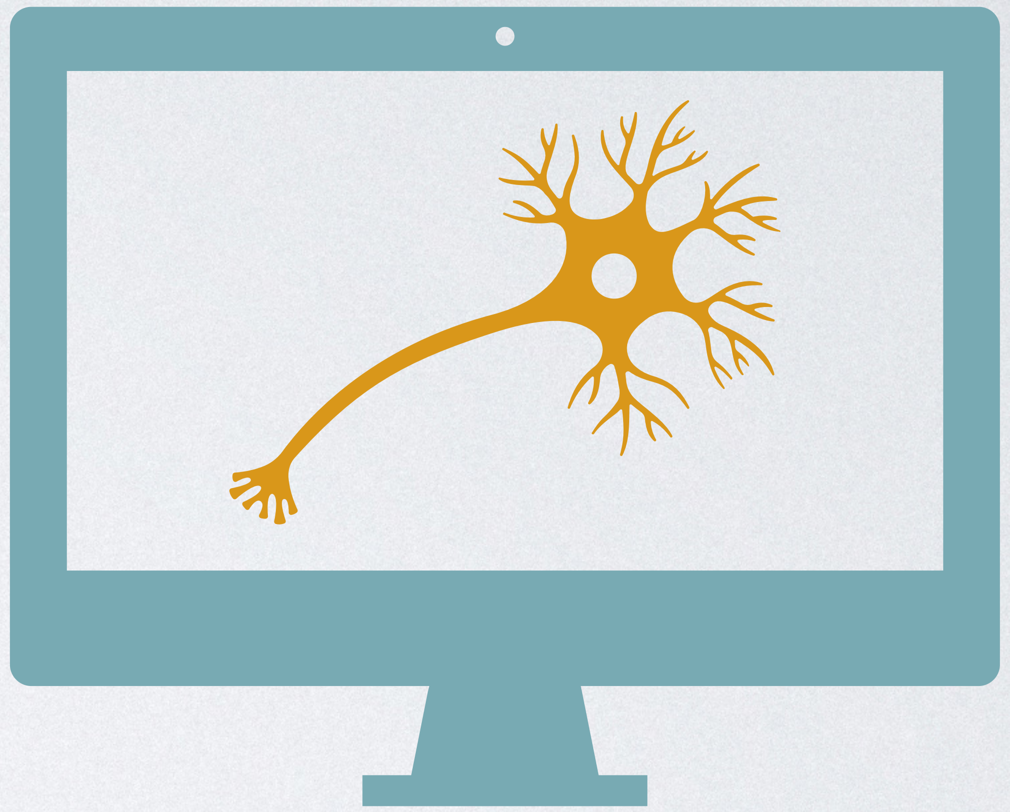

Applications of Prob. Prog.

Randomized Algorithms
(improve efficiency)



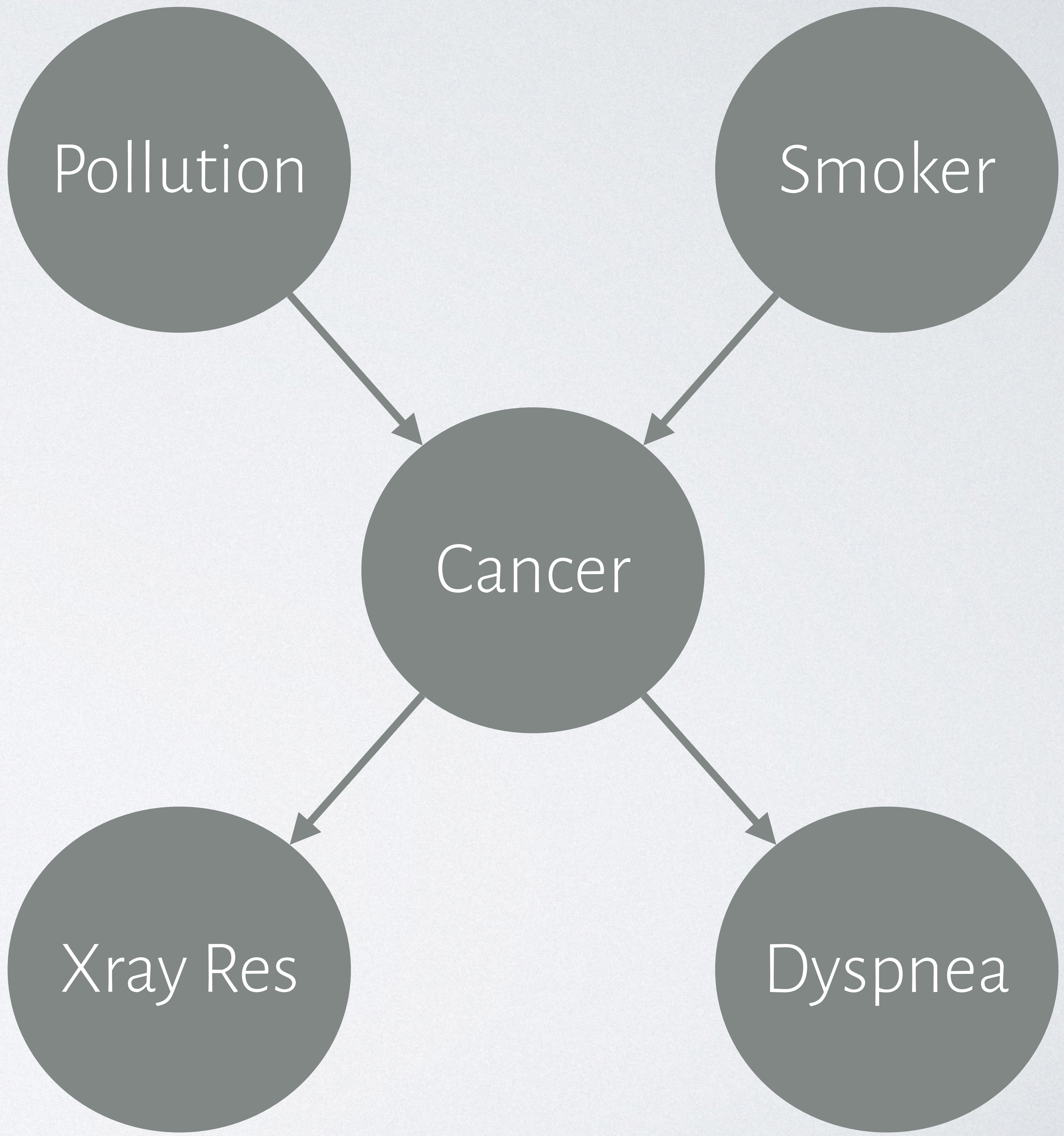
Cyber-Physical Systems
(model uncertainty)



Machine Learning Algorithms
(describe statistical models)

Bayesian Networks

- Conditional distributions
- Query about the posterior



Bayesian Networks

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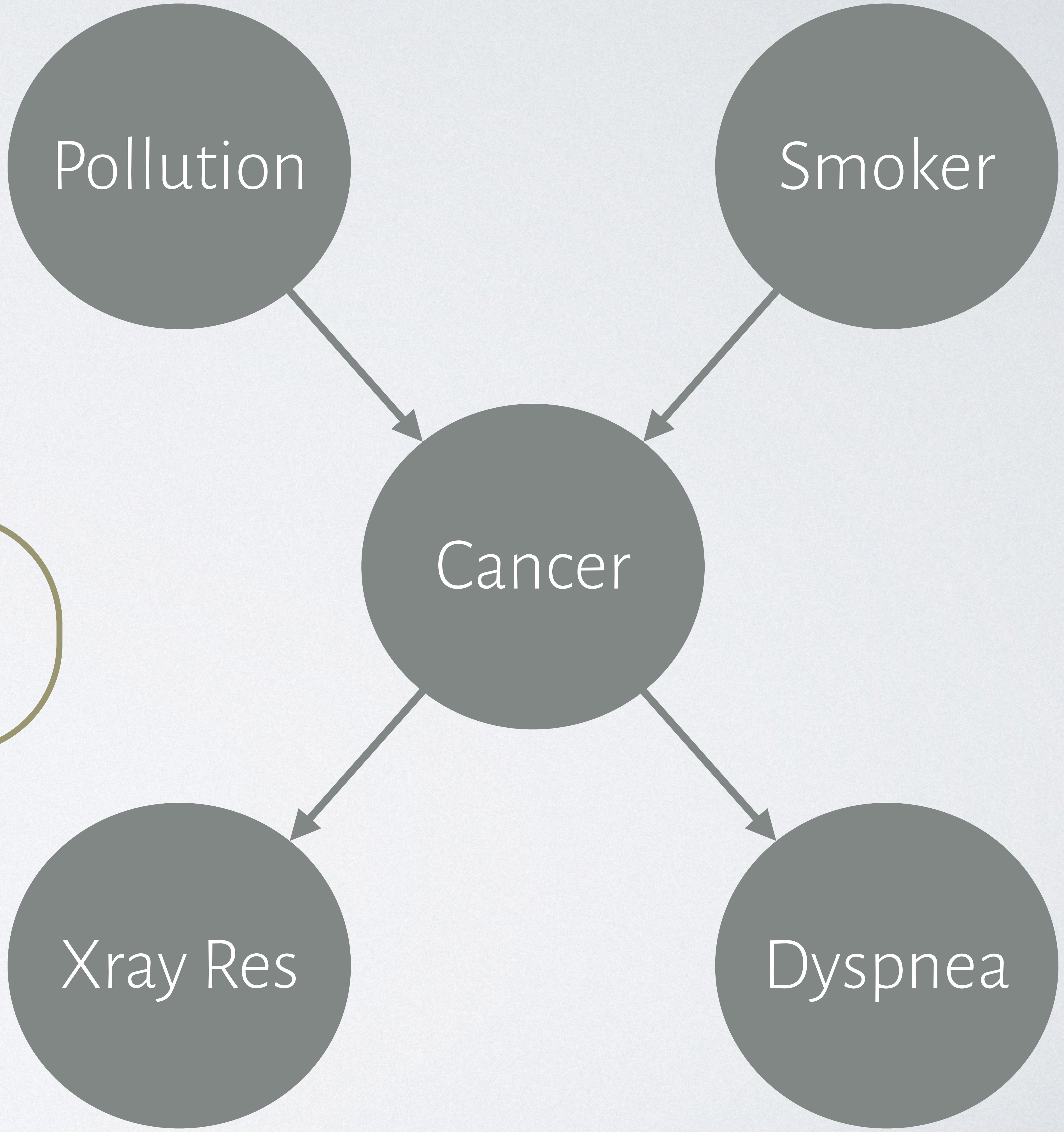
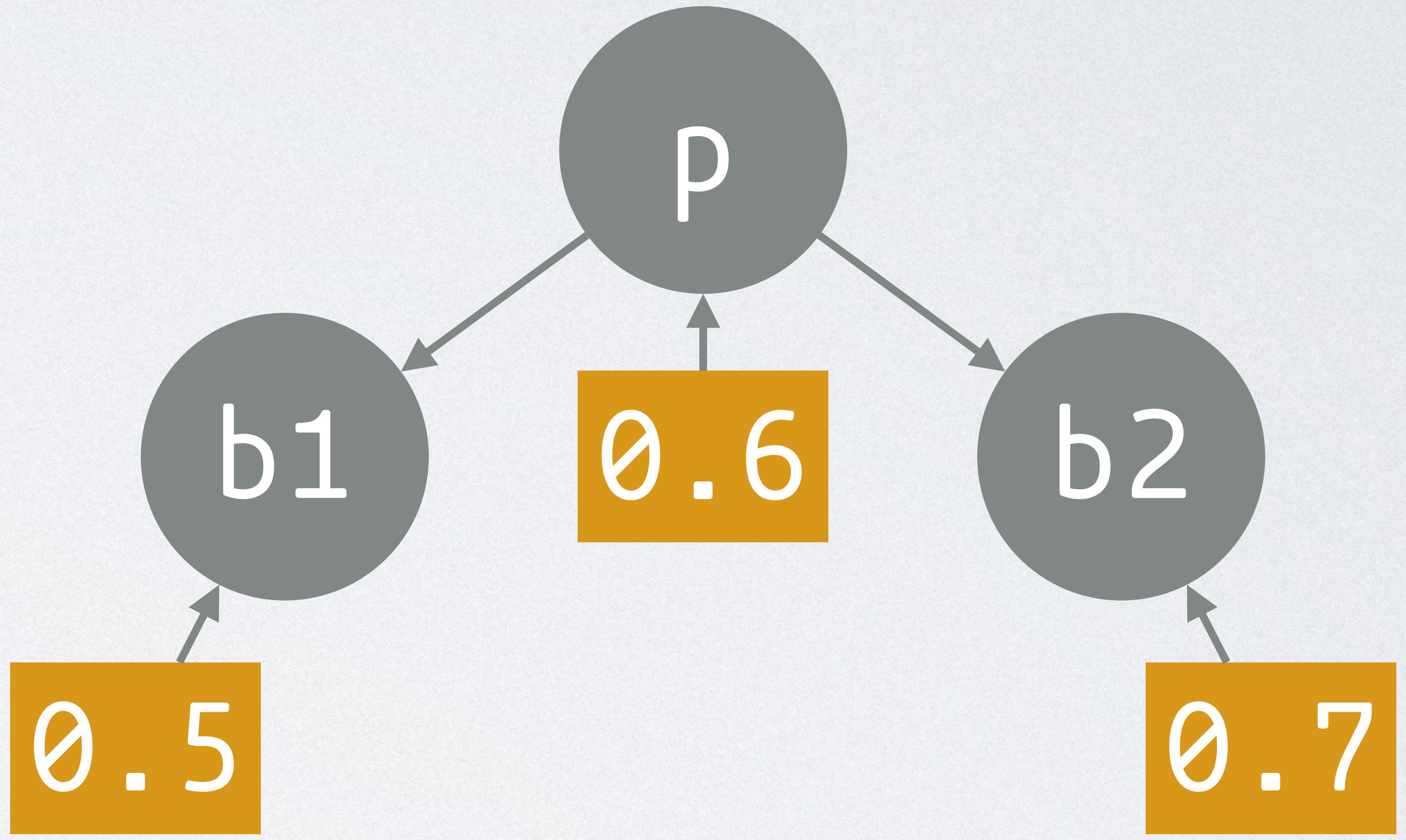
- Query about the posterior

Prob[Cancer | Smoker ∧ Xray Res] = ?



Bayesian Networks as Prob. Prog.



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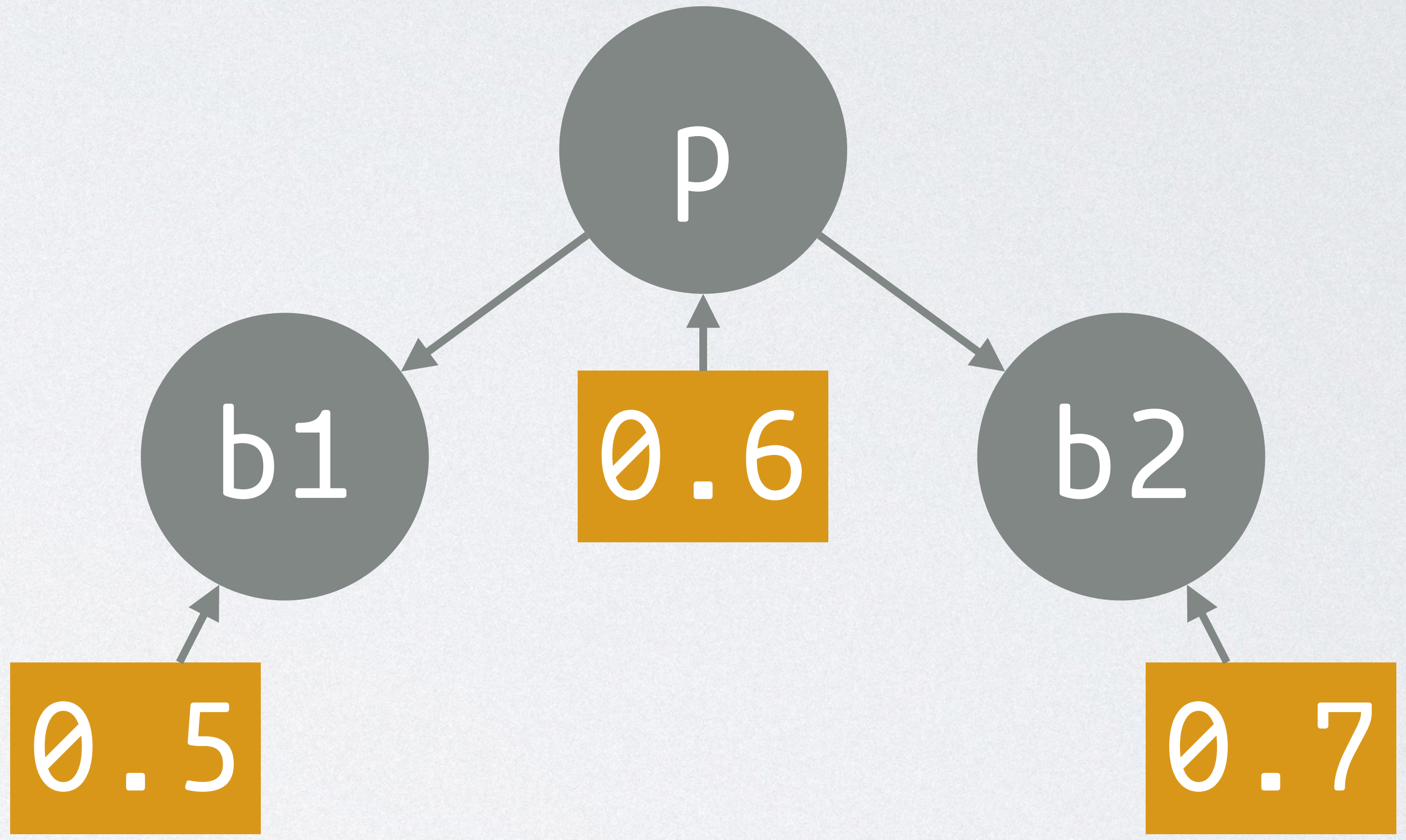

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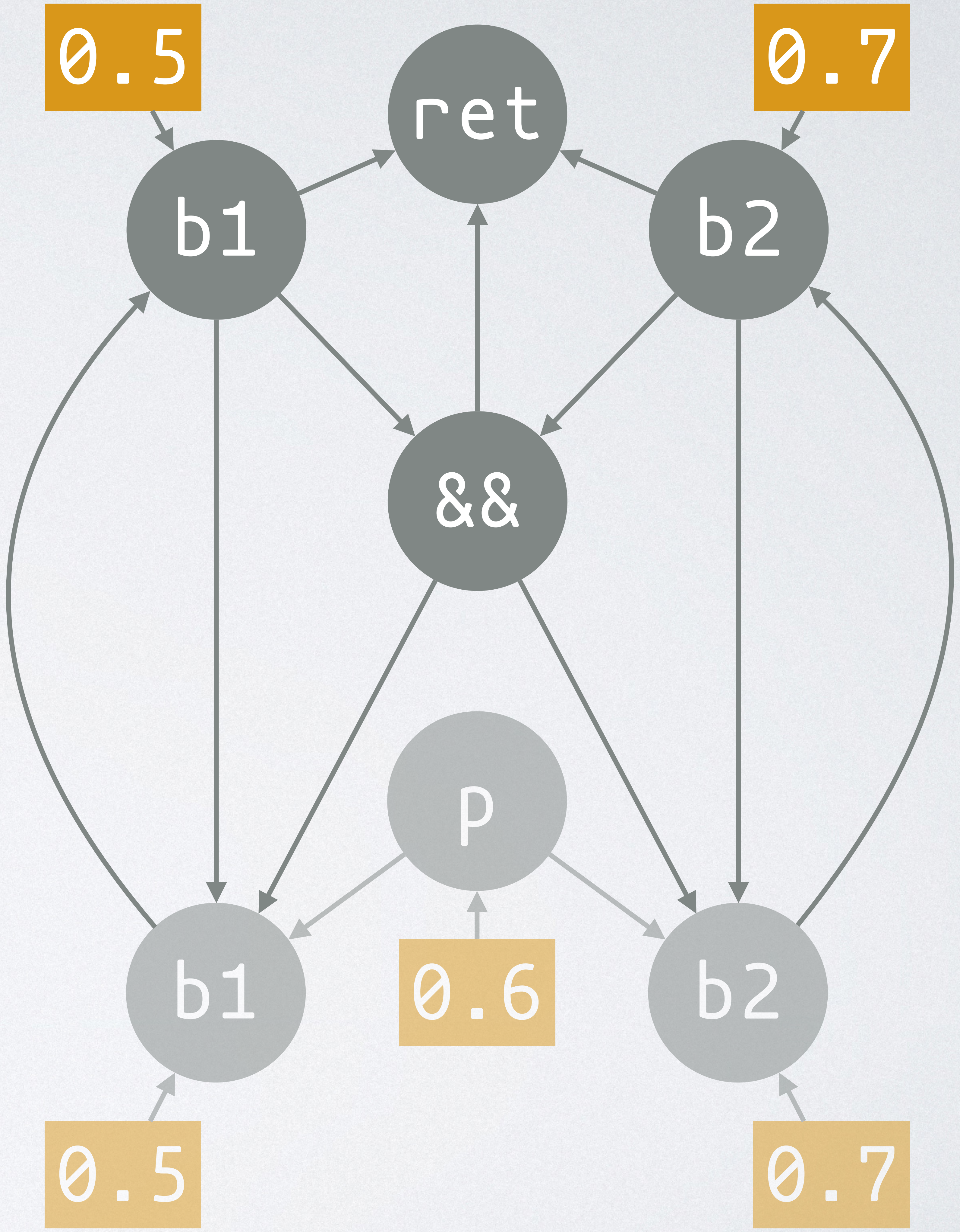

Quantitative Reasoning about Prob. Prog.

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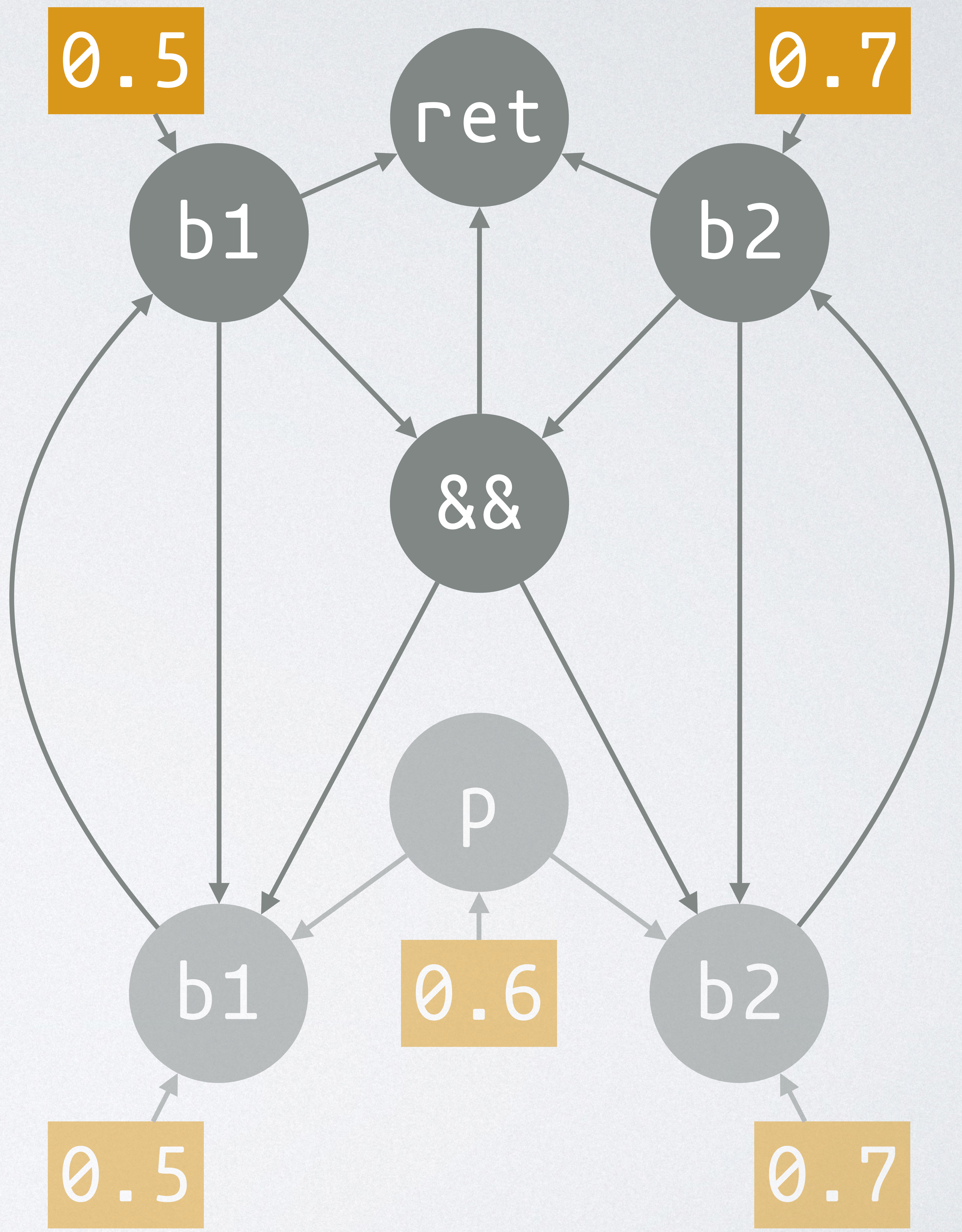

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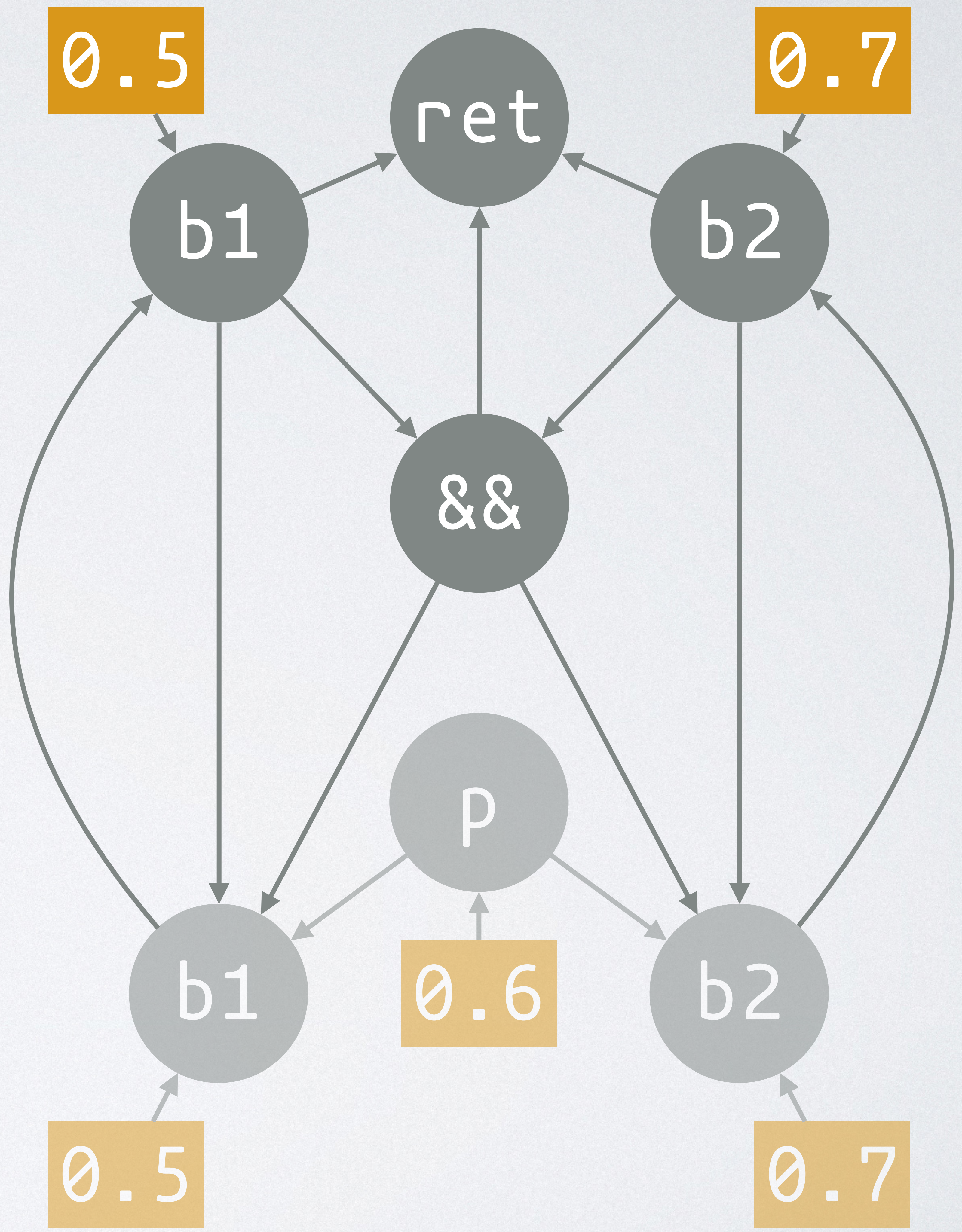

- **Query:** probability that `b1` and `b2` are both `false`?

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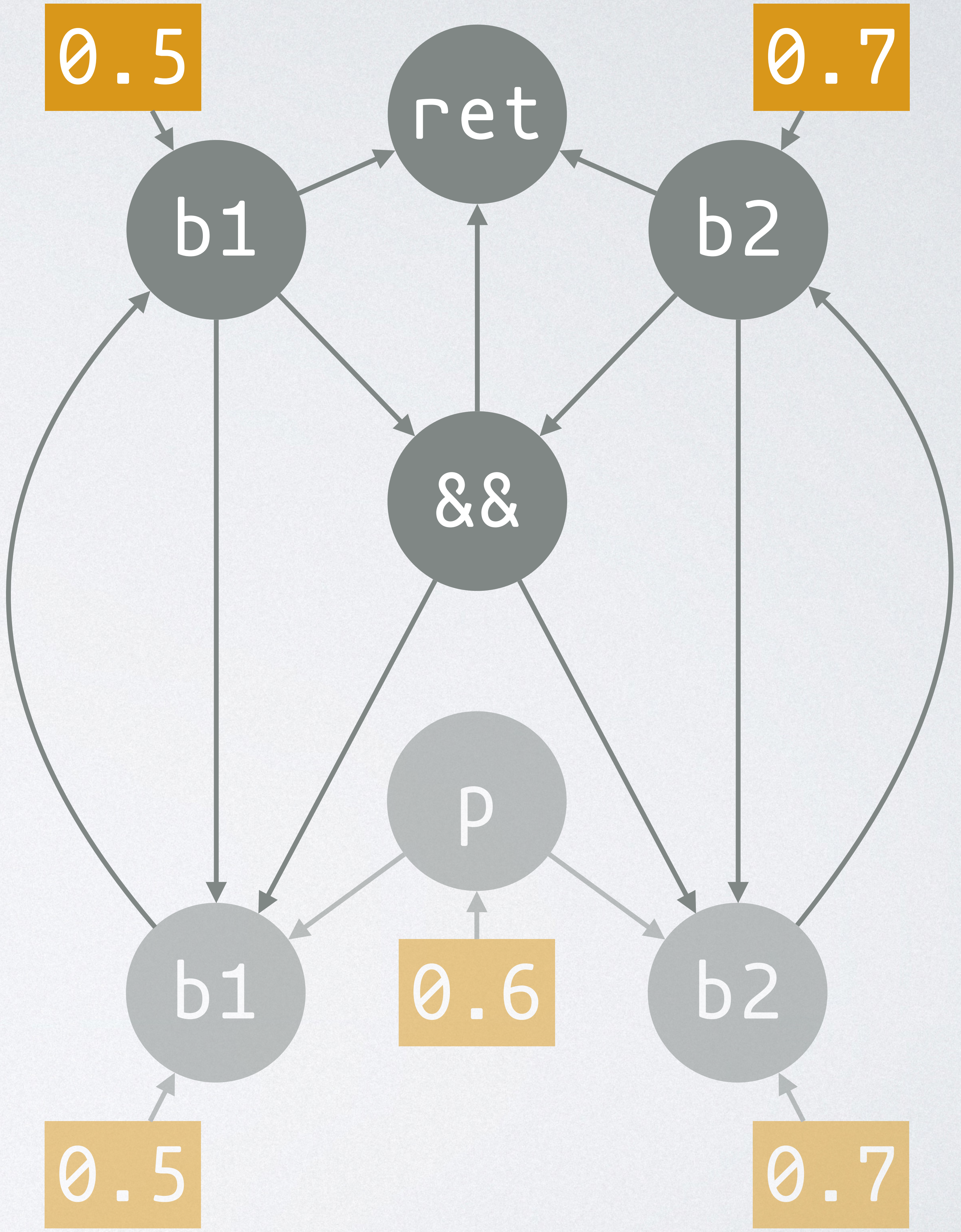
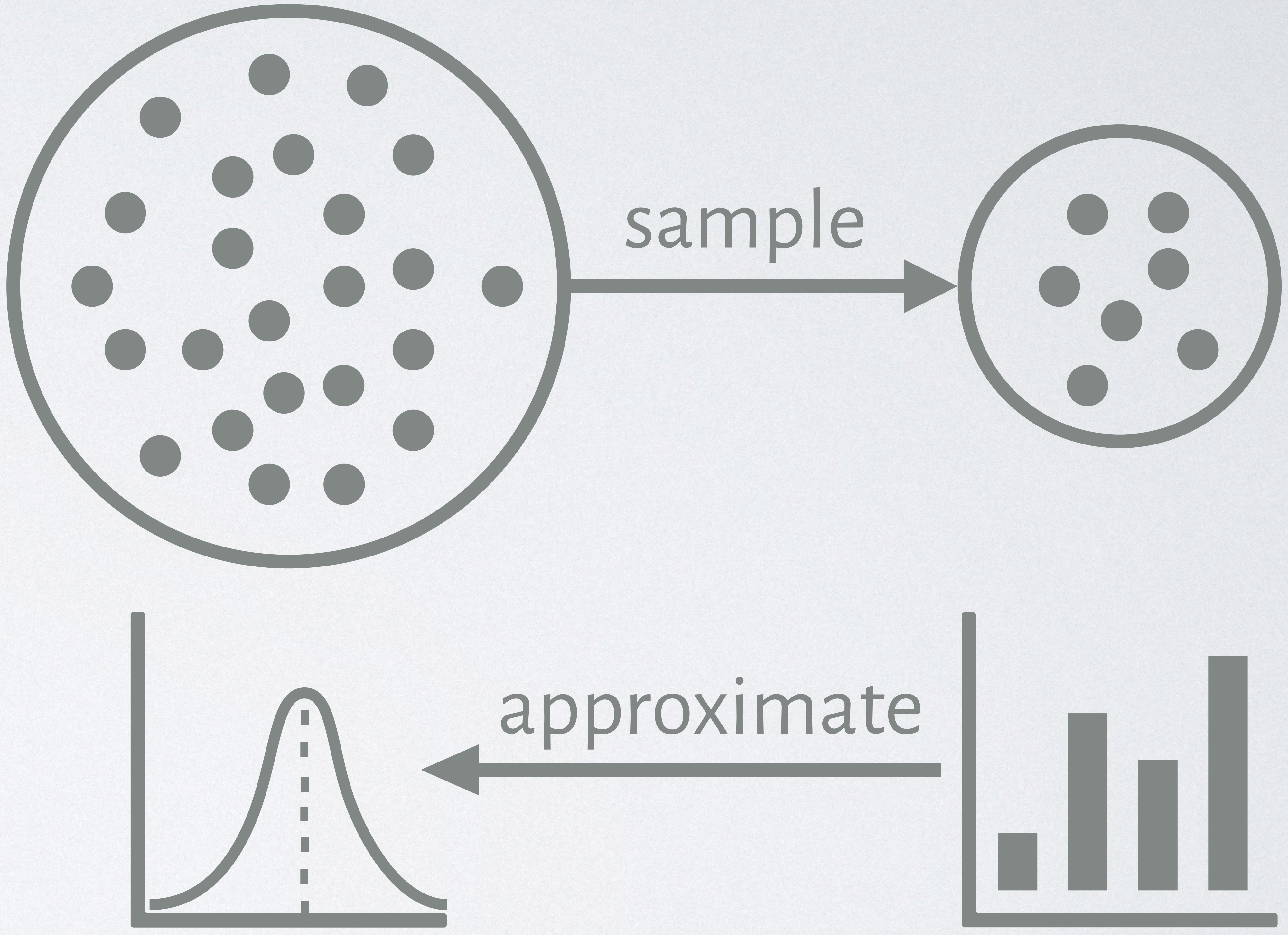
- **Query:** expected termination time?



Sampling-Based Techniques

- Simulation & frequency count
- Flexible & universal
- Potentially unsound & inefficient



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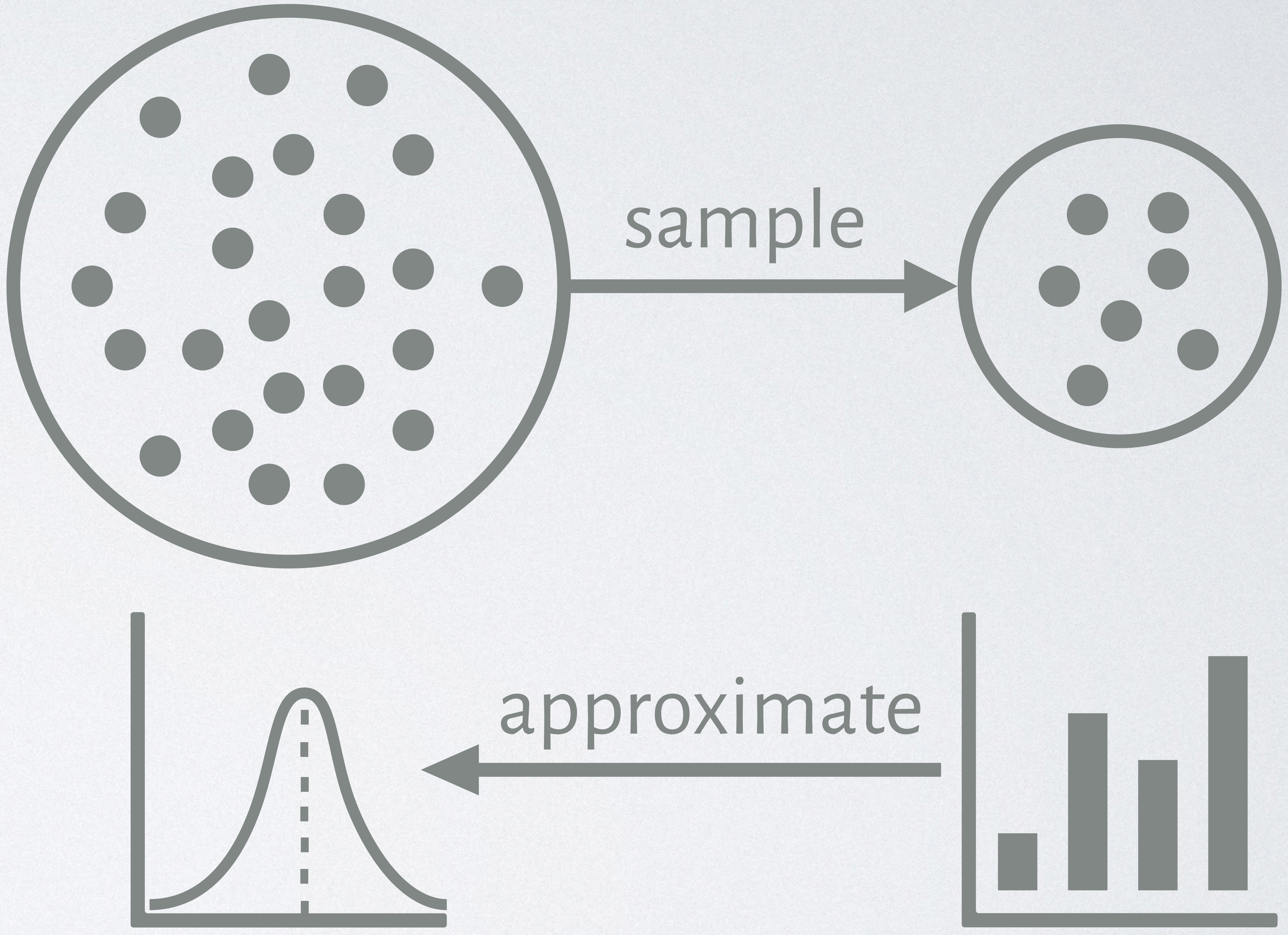
What about static analysis?



SEMANTICS OF PROB. PROG.

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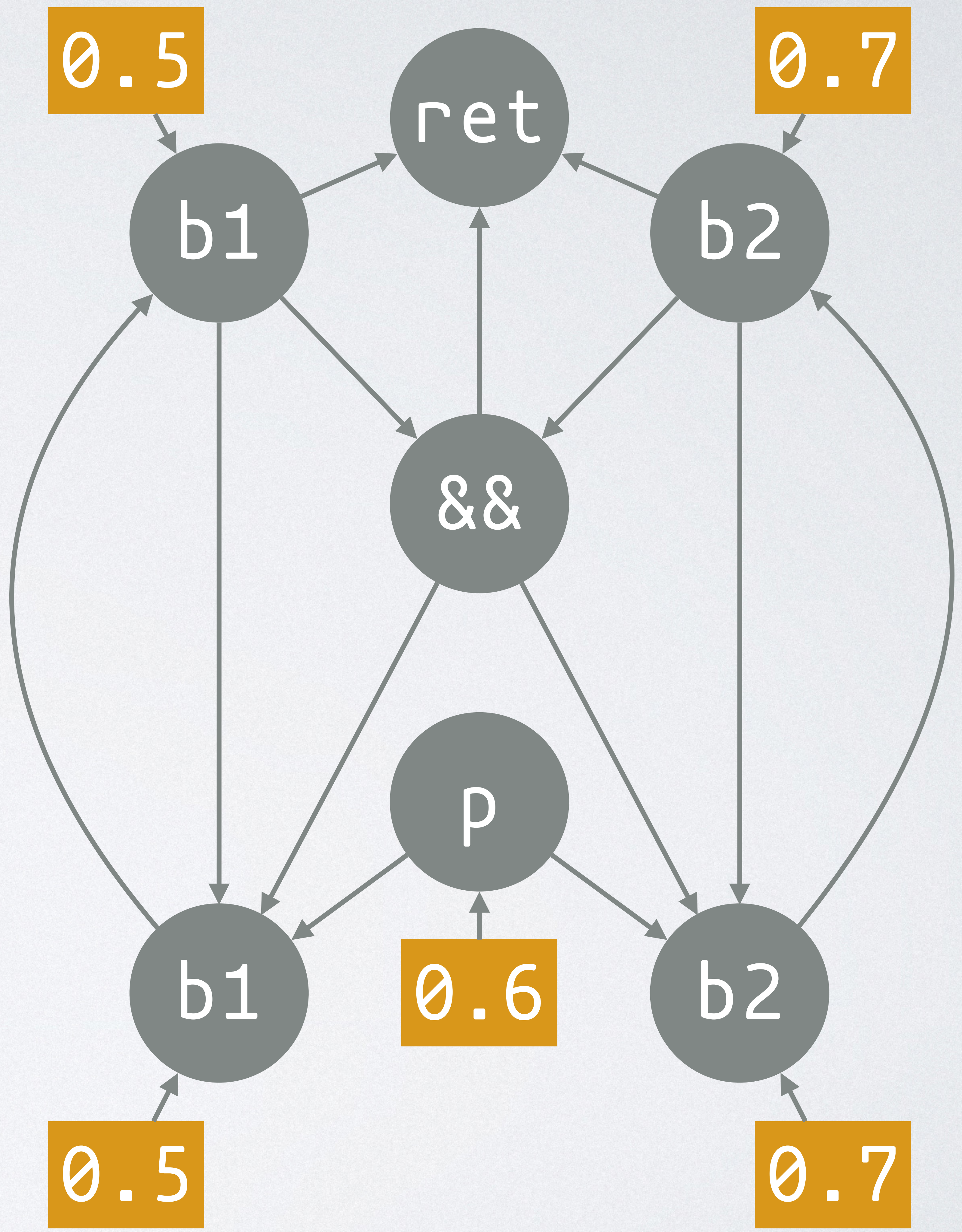
- To develop static analysis, we need to first define a proper semantics



Semantics of Prob. Prog.

[1] J. Borgström, U. D. Lago, A. D. Gordon, and M. Szymczak. A Lambda-Calculus Foundation for Universal Probabilistic Programming. In *ICFP'16*.
[2] F. Ferrer, M. Luis, and H. Hermanns. Probabilistic Termination: Soundness, Completeness, and Compositionality. In *POPL'15*.
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[4] C. Heunen, O. Kammar, S. Staton, and H. Yang. A Convenient Category for Higher-Order Probability Theory. In *LICS'17*.

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Existing Operational Semantics

- Untyped lambda calculus[1]
- Markov chains[2]

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- A general treatment of nondeterminism

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Static Analysis of Prob. Prog.

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Static Analysis of Prob. Prog.

Existing Approaches

- Static analysis of different kinds of program properties for prob. prog.
- Probabilistic Abstract Interpretation (**PAI**)[5]

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STATIC ANALYSIS OF PROB. PROG.

Existing Approaches

- Static analysis of different kinds of program properties for prob. prog.
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What was Missing?

- A unifying framework that covers multiple analyses
- Compositionality and flexibility
- **PAI**'s treatment of nondeterminism sometimes turns out to be *not desirable*

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Algebraic Program Analysis[6]

[6] Z. Kincaid, T. Reps, and J. Cyphert. Algebraic Program Analysis. In *CAV'21*.

Algebraic Program Analysis[6]

- A flexible framework for understanding compositional static analyses

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ALGEBRAIC PROGRAM ANALYSIS[6]

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- Different algebras yield different semantics, e.g., a concrete semantics, or an abstract semantics for static analysis

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Thesis

- **Algebraic** static analysis helps people reason about prob. prog. at compile time in a **compositional** and **flexible** way
- **Markov algebras** provide a natural way to define a **denotational semantics** of prob. prog. with **nondeterminism**
- **An algebraic framework for static analysis** can cover multiple existing analyses and lay the foundation for new analyses

Overview

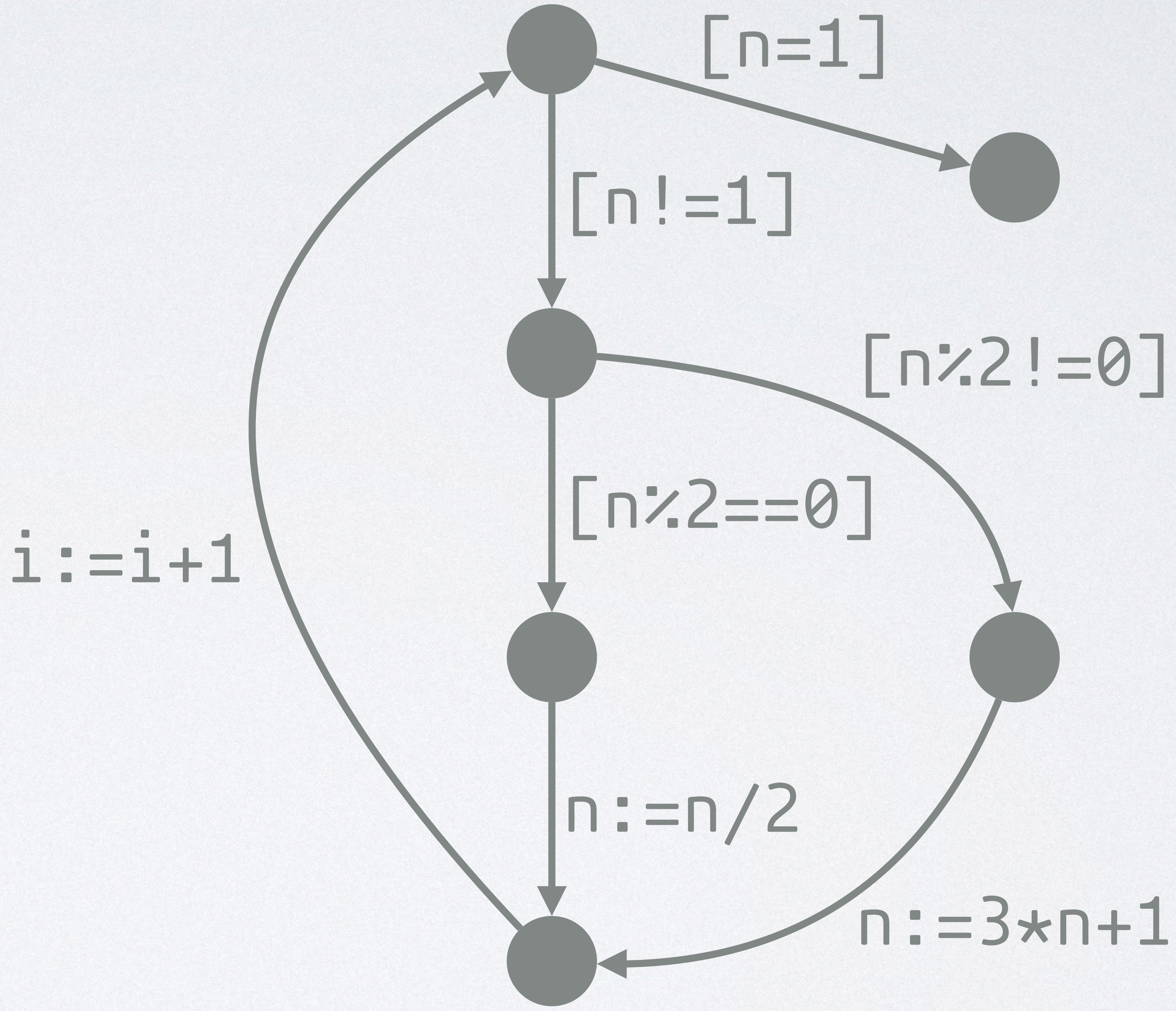
- ☑ Motivation
- ☐ An Algebraic Denotational Semantics
- ☐ Pre-Markov Algebra Framework (PMAF)

An Algebraic Denotational Semantics

Contributions

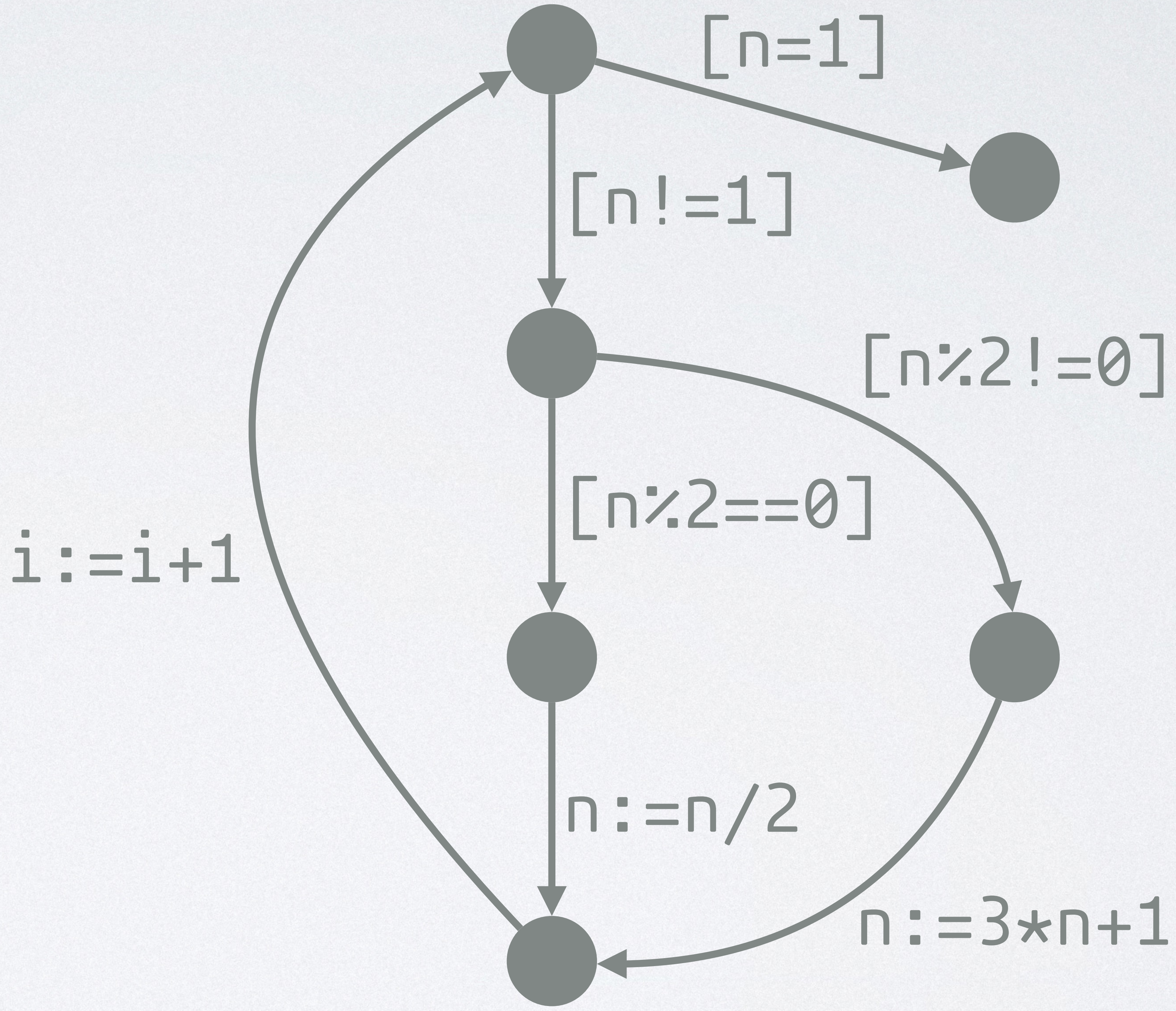
- A hyper-path semantics for low-level prob. prog.
- Markov algebras for understanding prob. prog. with nondeterminism
- A new model for resolving nondeterminism
- Published as **A Denotational Semantics for Low-Level Probabilistic Programs with Nondeterminism** in *MFPS'19*

Path Semantics of Non-Prob. Prog.



Path Semantics of Non-Prob. Prog.

- Control-flow graphs



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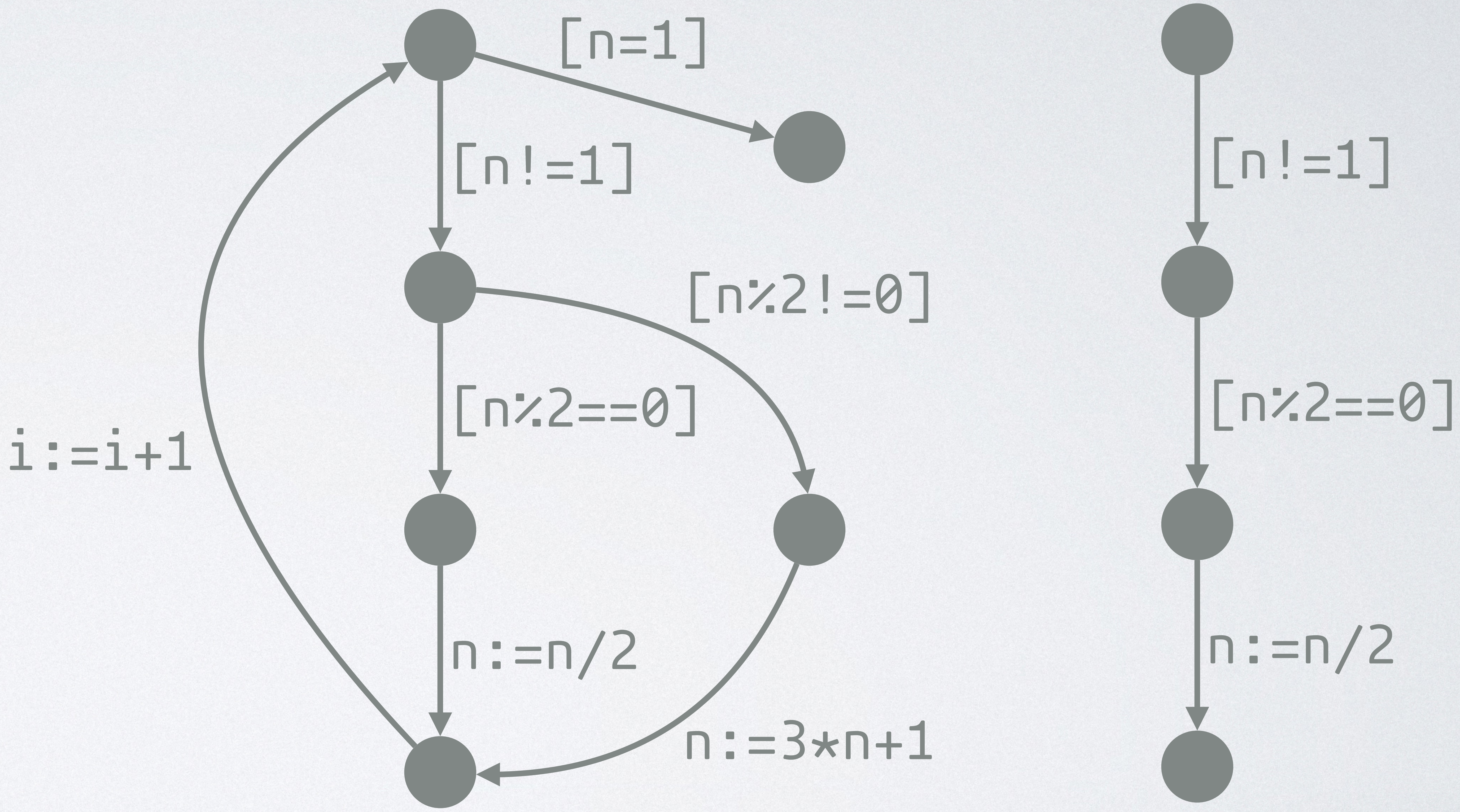
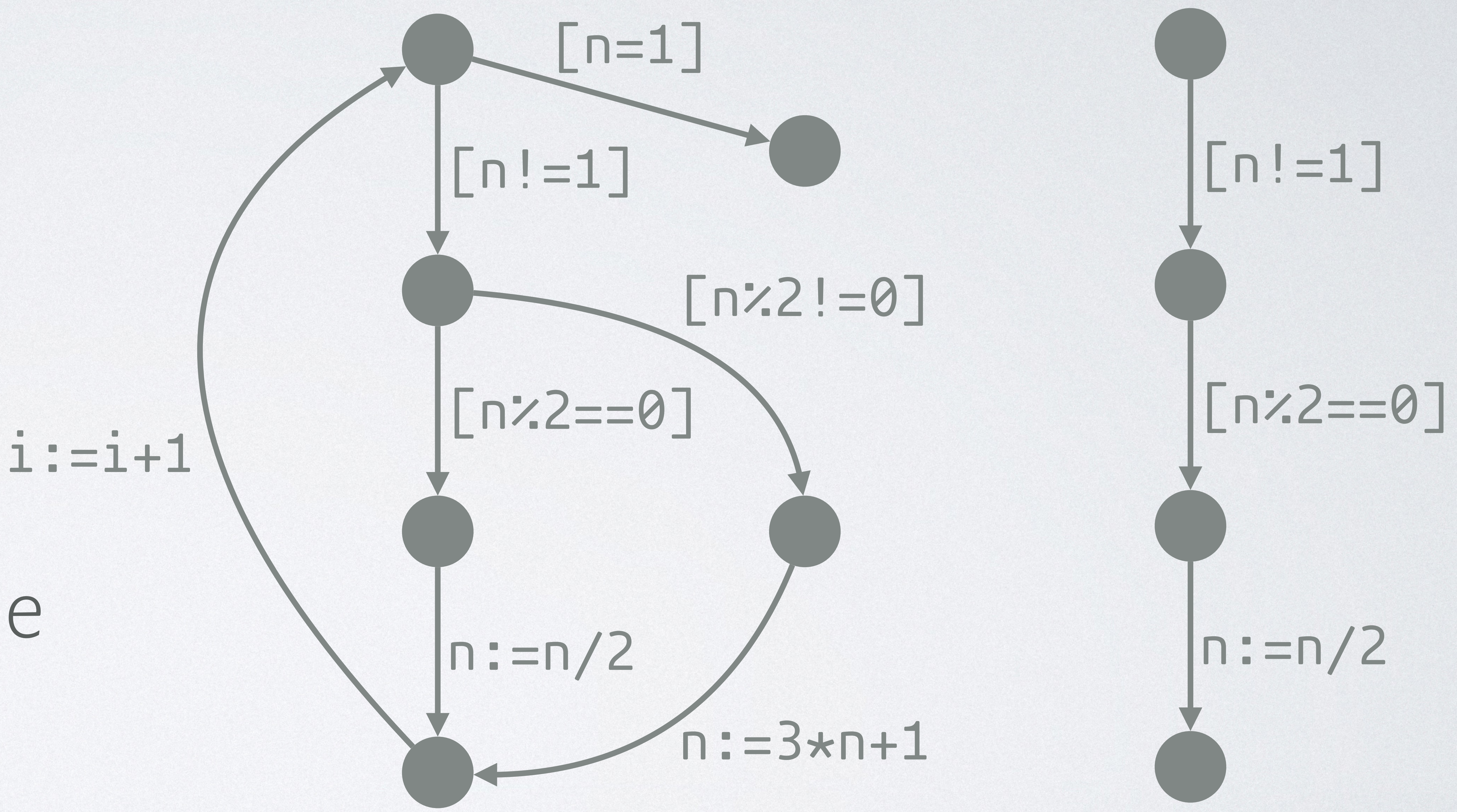
- Control-flow graphs
- Reason about paths



Path Semantics of Non-Prob. Prog.

- Control-flow graphs
- Reason about paths
- Kleene algebras are suitable to describe path semantics



Control-Flow Graphs and Path Semantics

⋆ denotes nondeterministic-choice

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if ⋆ then
  if prob(0.5) then
    x := 0
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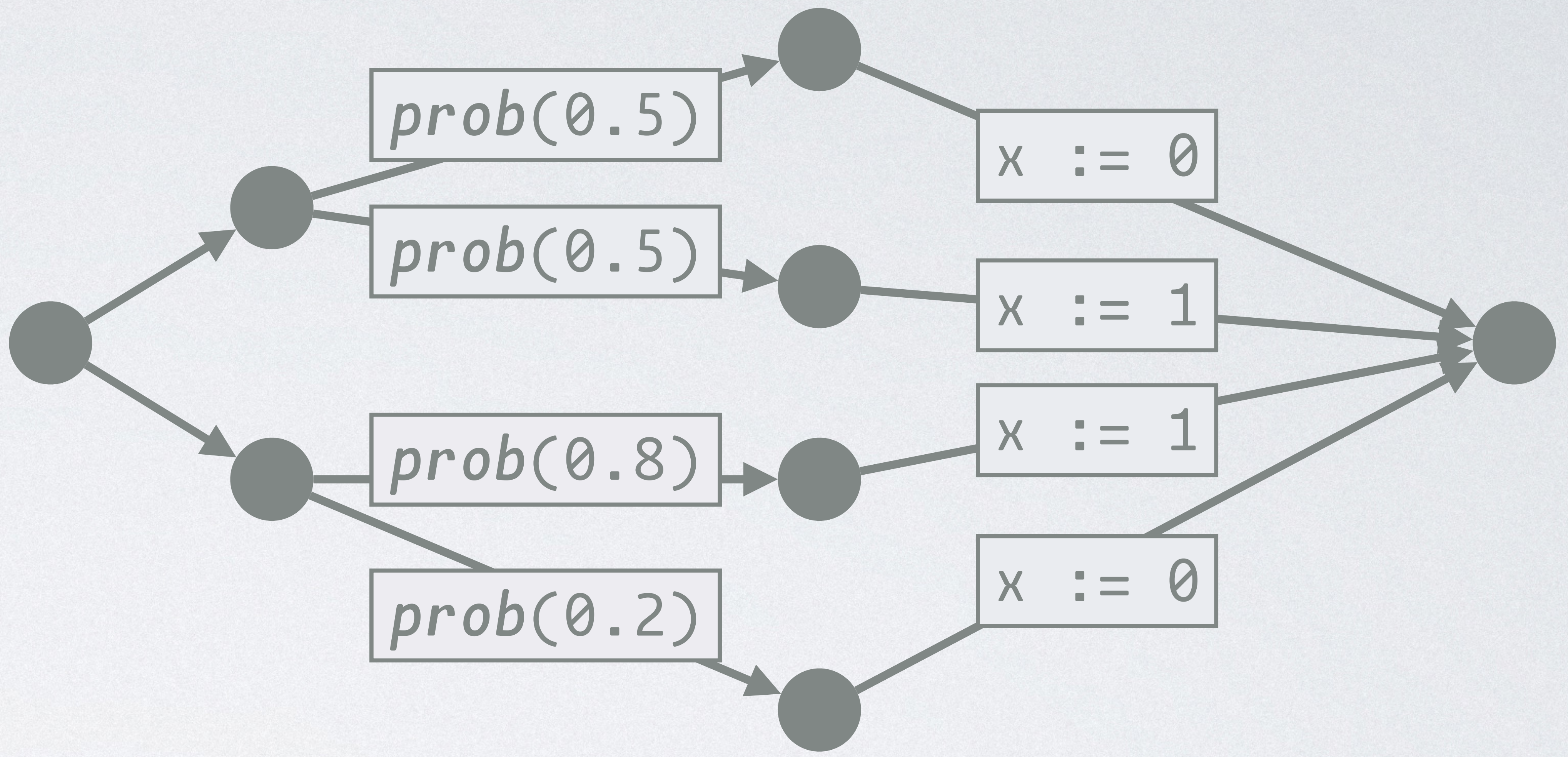

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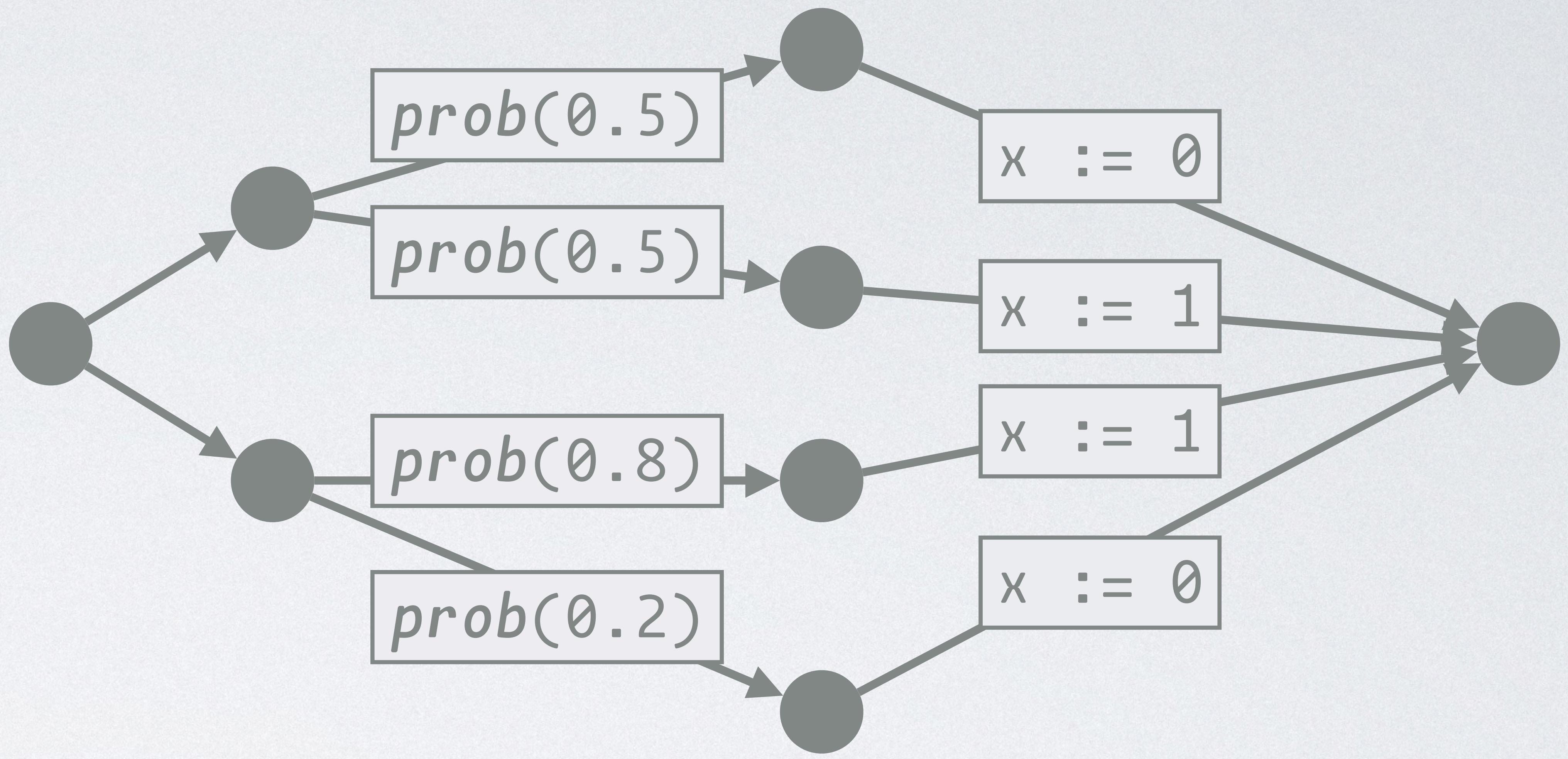
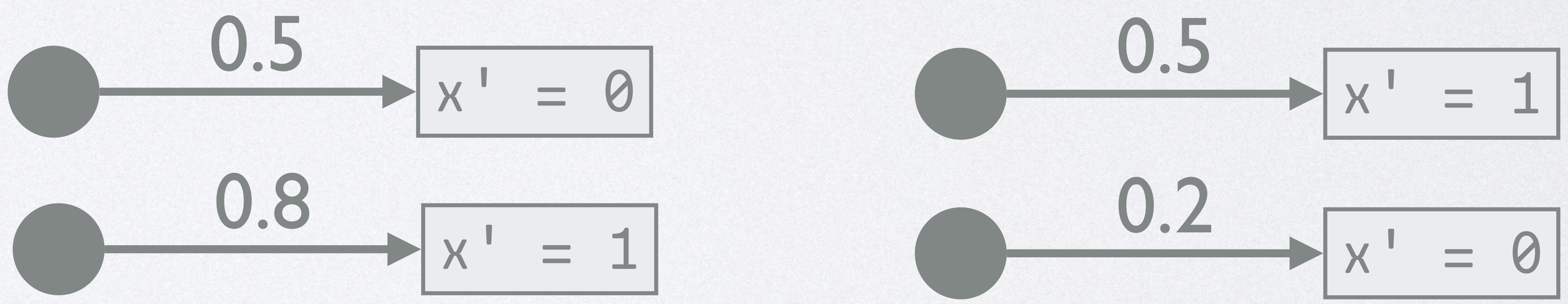


- Paths annotated with probabilities:



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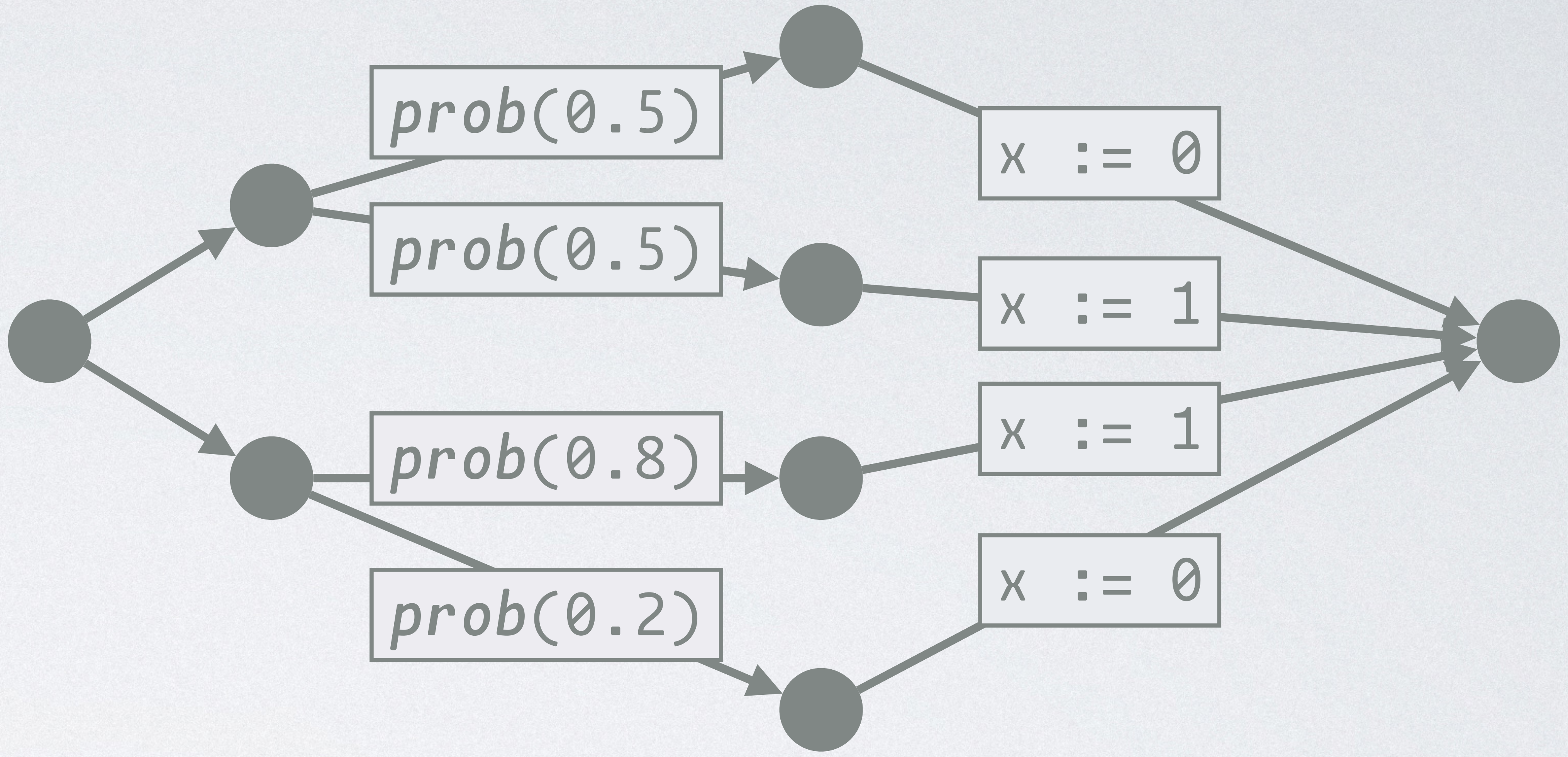

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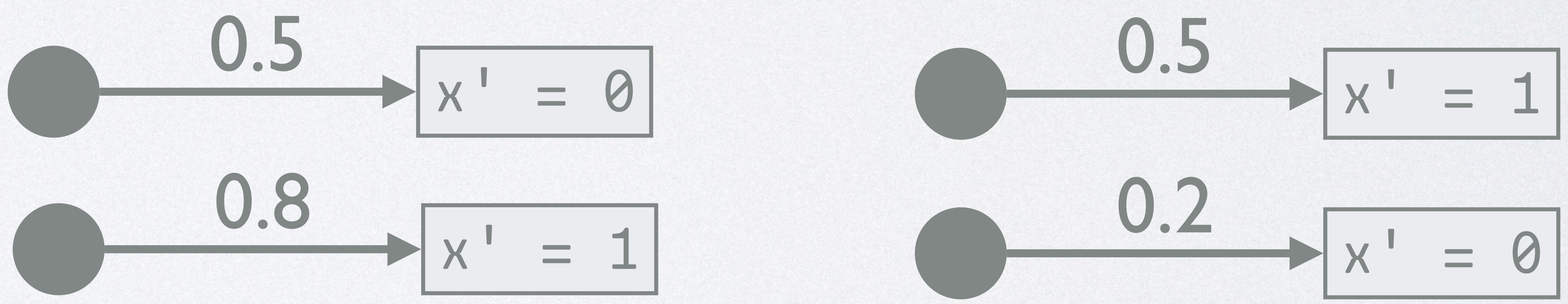

- $\mathbf{Prob}[x' = 1] = 1.3\,?$

Hyper-Path Semantics

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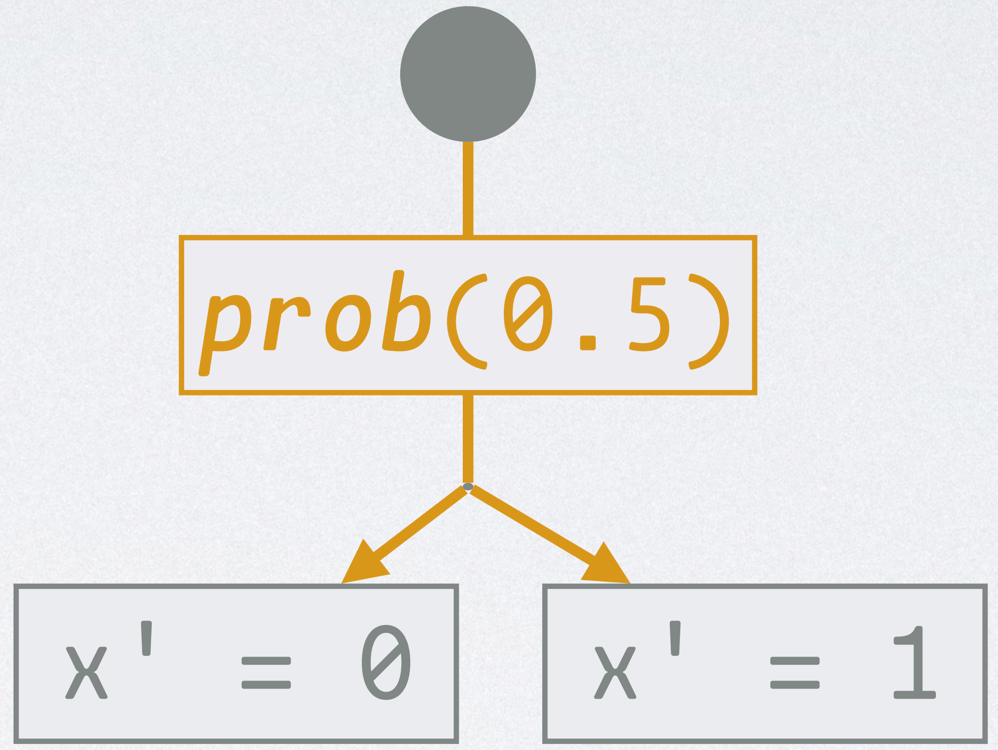
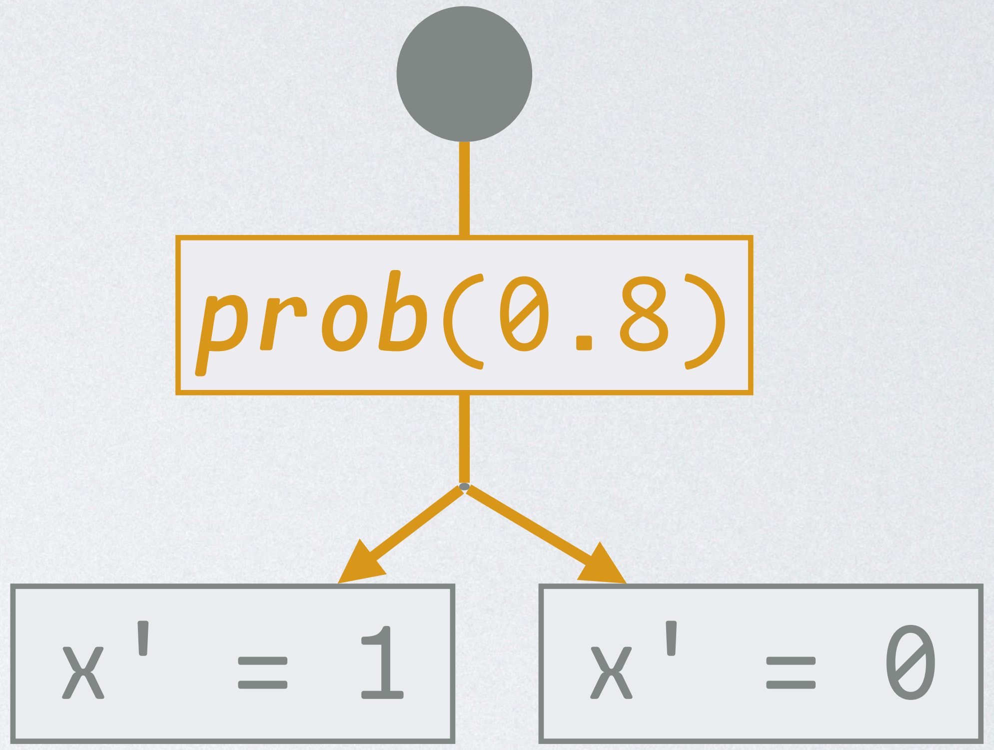
- *prob*() introduces distributions on executions, and * collects such distributions





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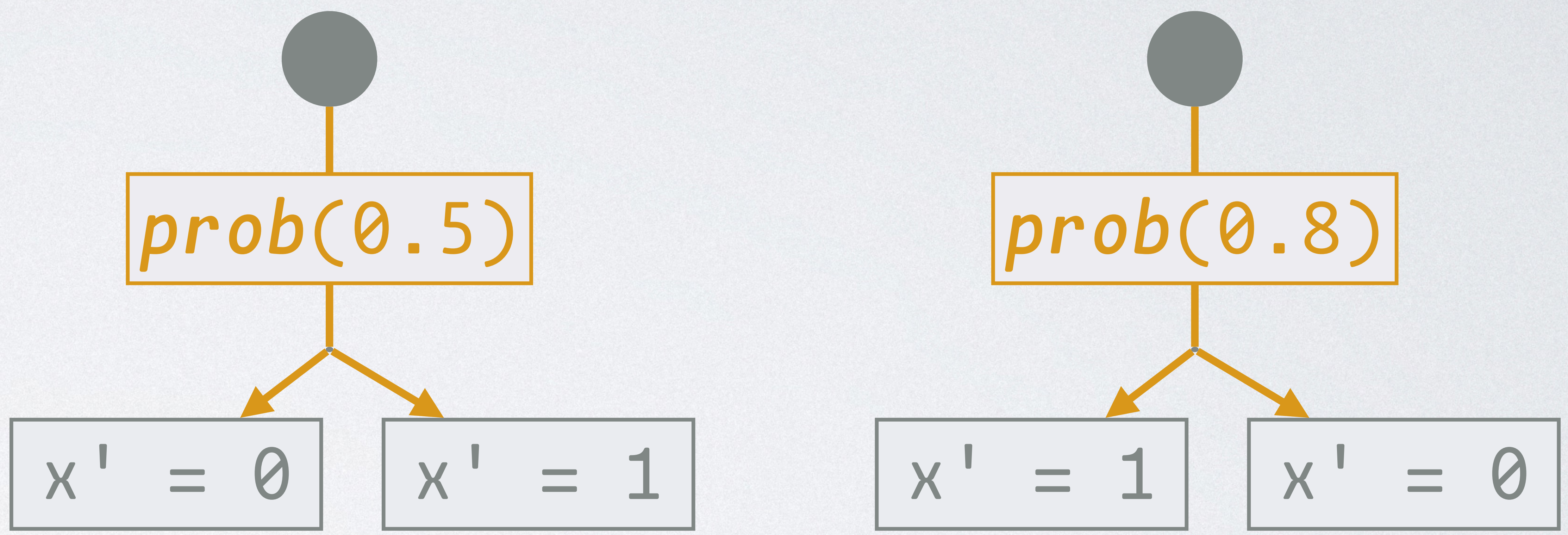

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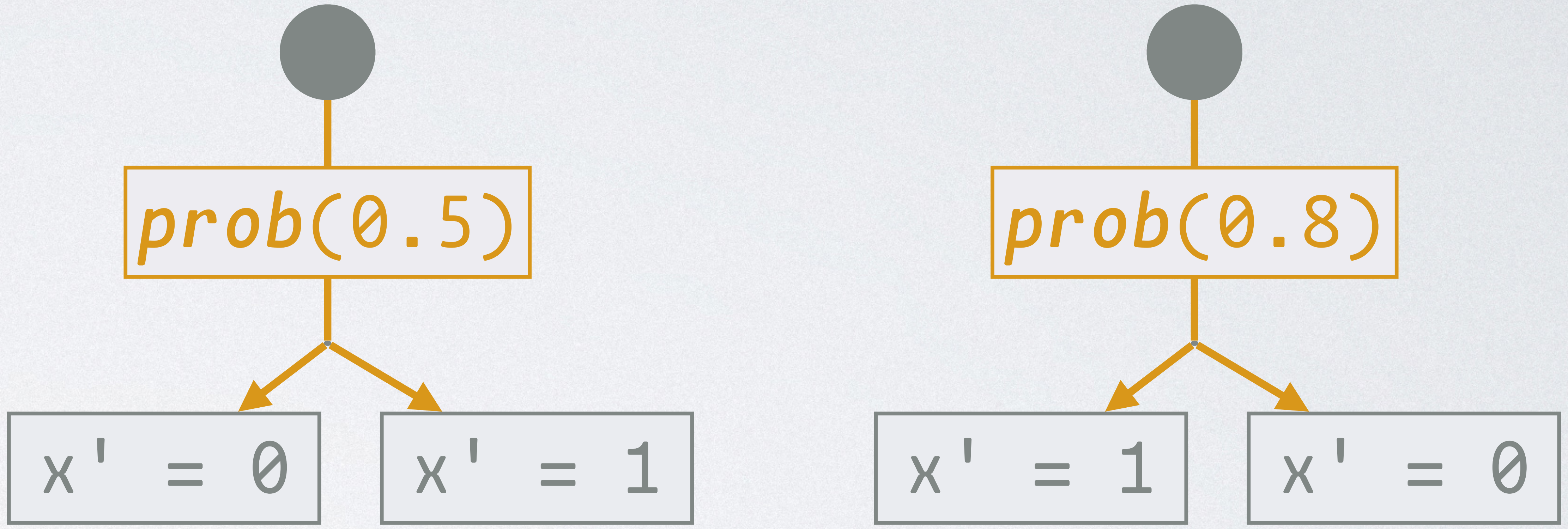

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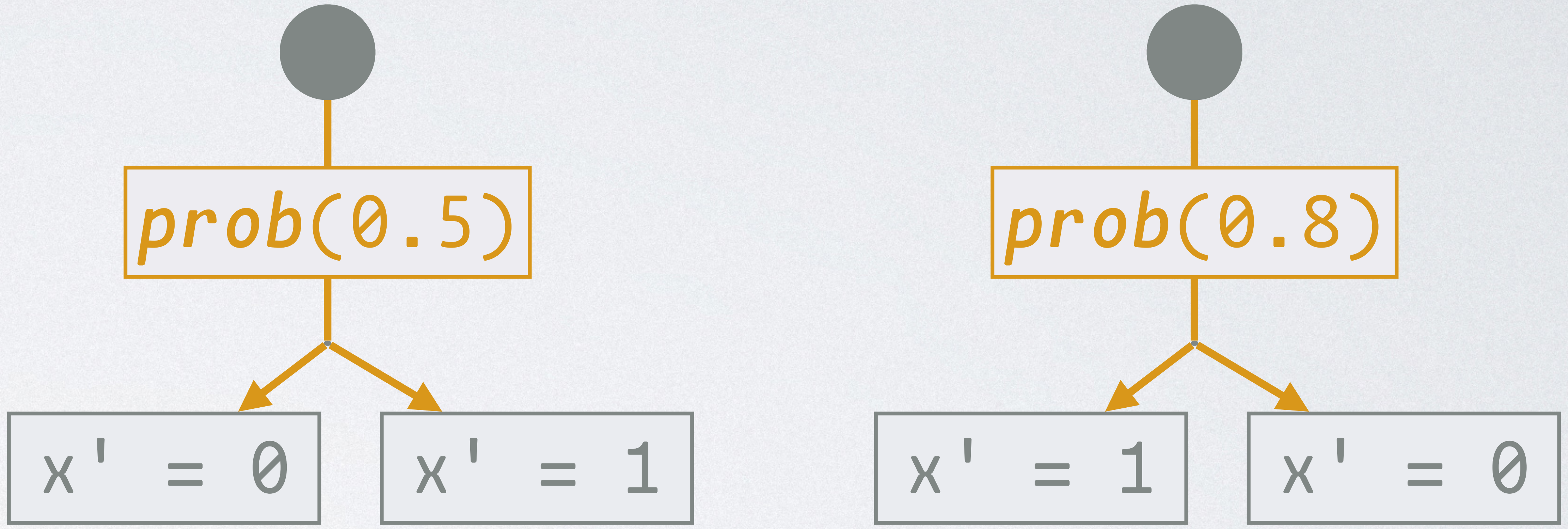

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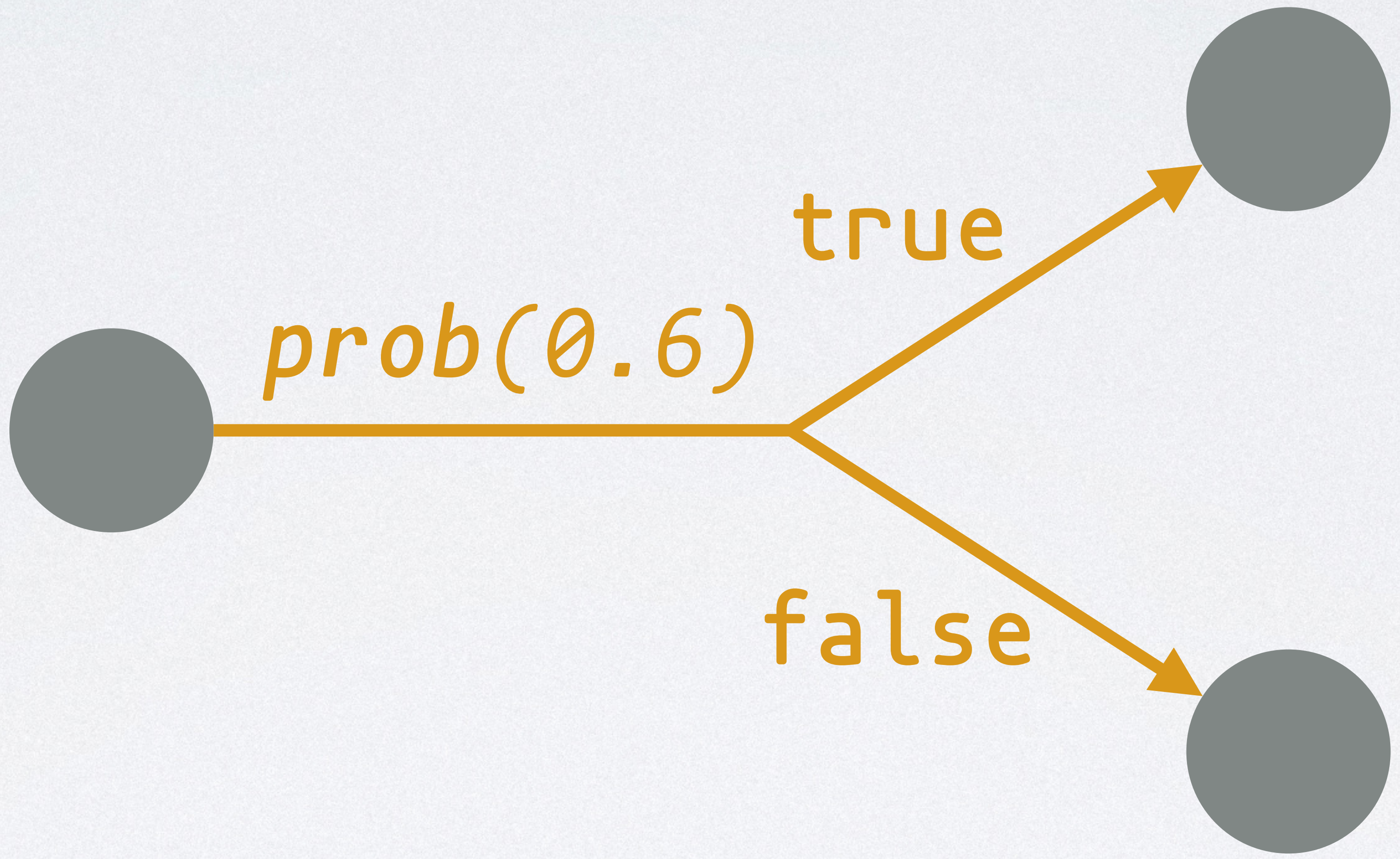
- **Nondeterminism** is modeled by **collections** of hyper-paths
- Kleene algebras are suitable for path semantics, but are ***not*** suitable for hyper-path semantics

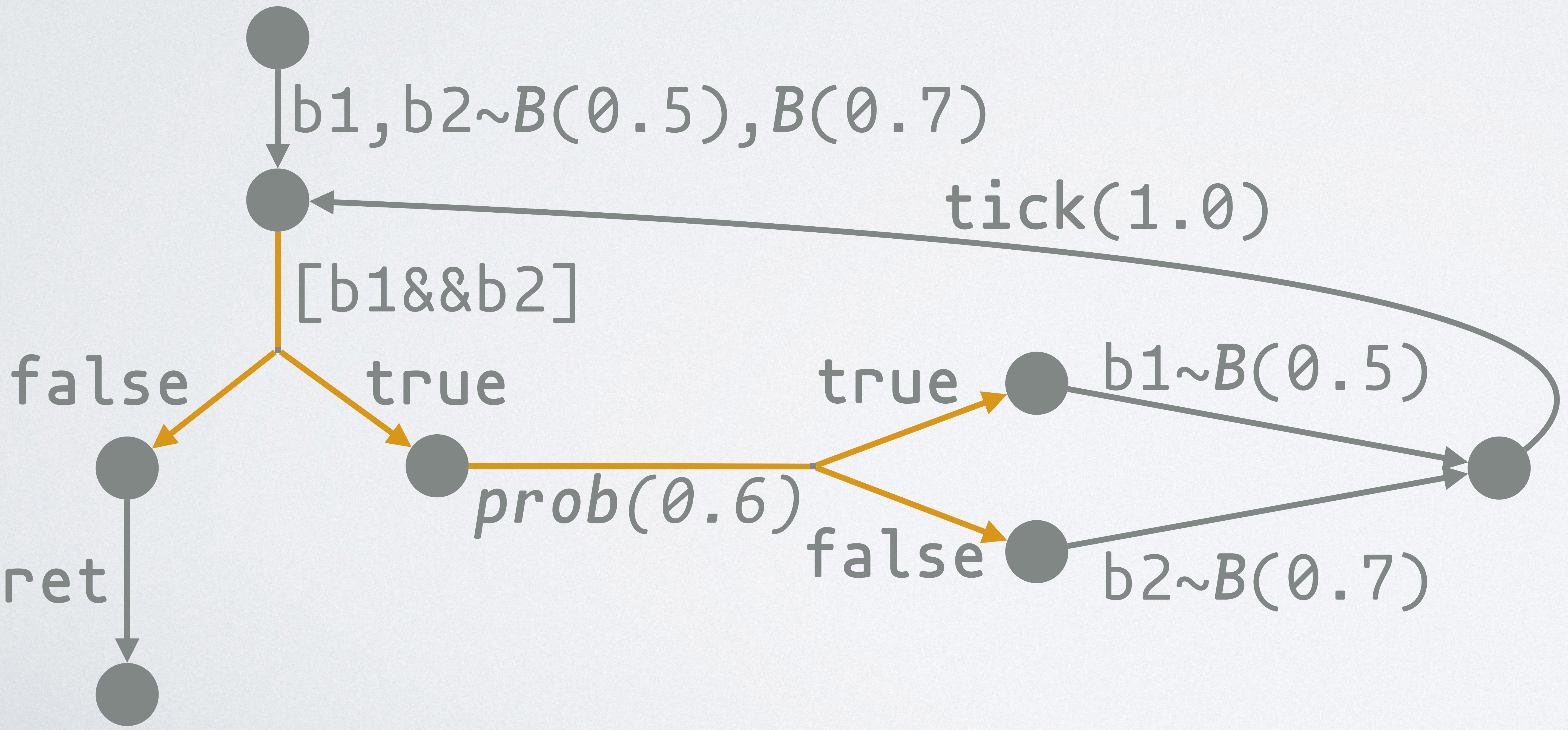
Control-Flow Hyper-Graphs (CFHGs)

A hyper-edge can have multiple destinations



Control-Flow Hyper-Graphs (CFHGs)

- Directed graphs with hyper-edges
- Branching are hyper-edges
- Support nondeterminism, unstructured control flow, and recursion

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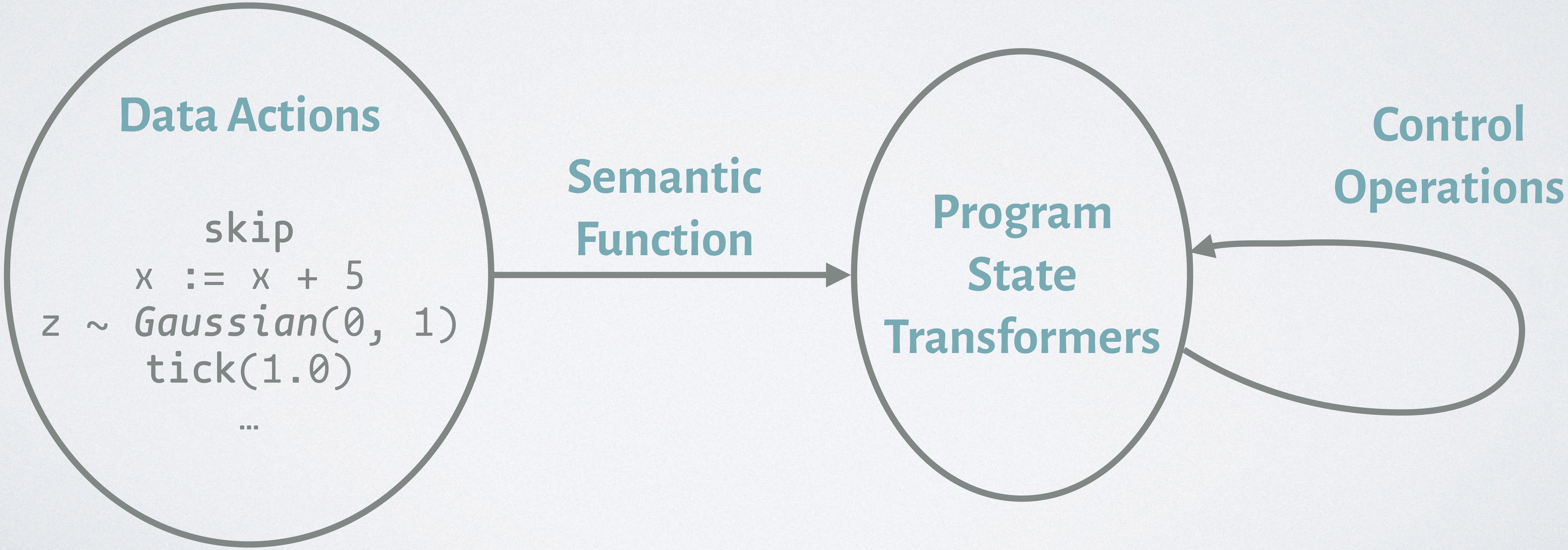
An Algebraic Denotational Semantics

- Perform reasoning in some abstract space of program-state transformers
- The transformers and associated operations obey some algebraic laws



An Algebraic Denotational Semantics

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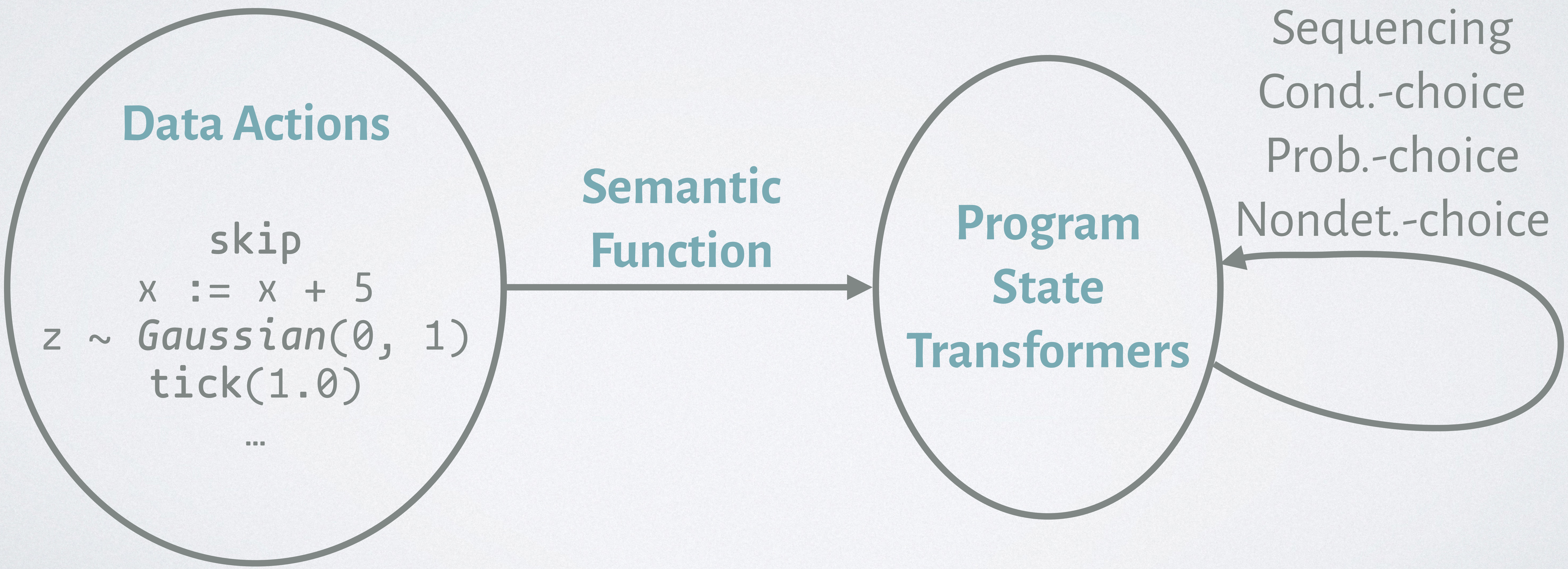

Markov Algebras

- Characterize state transformers and associated operations by algebraic laws

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Sequencing, branching (cond. and prob.), and nondet.-choice

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The bottom element and the identity element

$\bot$ interprets `abort`

1 interprets `skip`

MARKOV ALGEBRAS

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$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

$$a \otimes 1 = 1 \otimes a$$

$$a \ {}_{\varphi}\Diamond\ b = b \ {}_{\neg\varphi}\Diamond\ a$$

$$a \uplus a = a$$

$$\otimes,\ {}_{\varphi}\Diamond,\ \uplus \text{ continuous w.r.t. } \sqsubseteq$$

$$\ldots$$

A New Model for Resolving Nondeterminism

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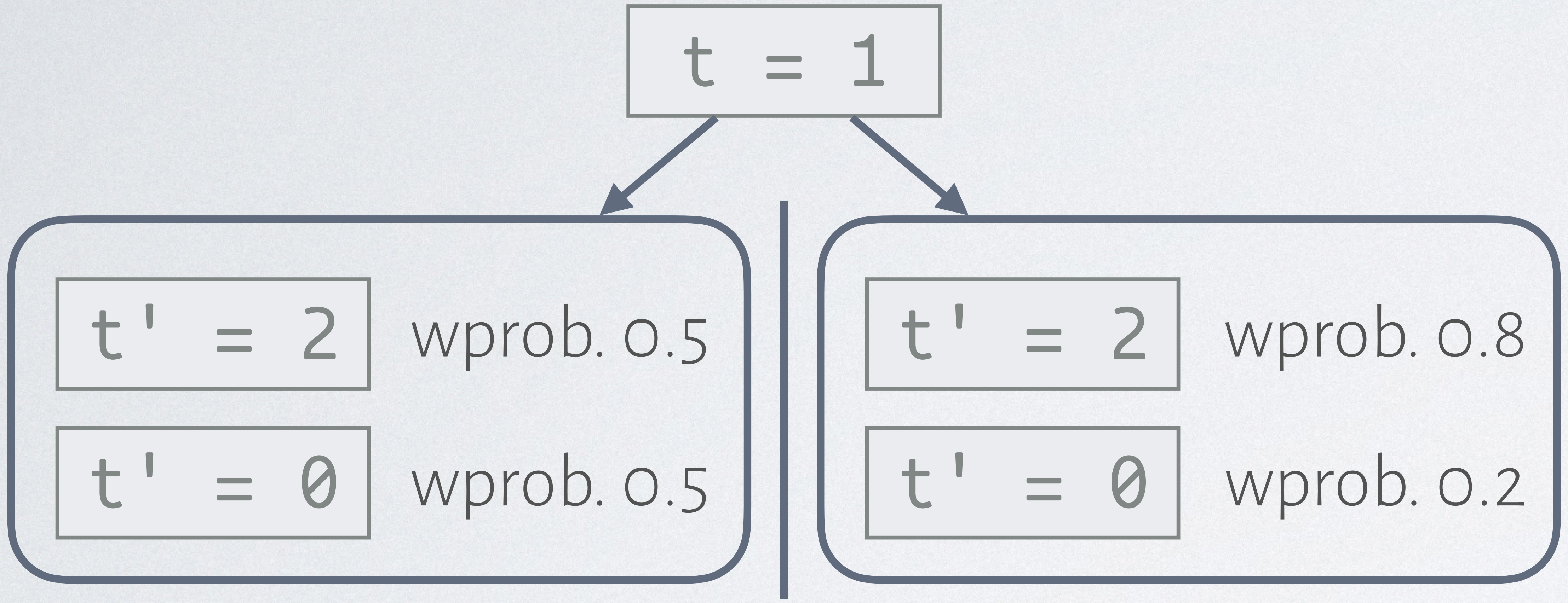

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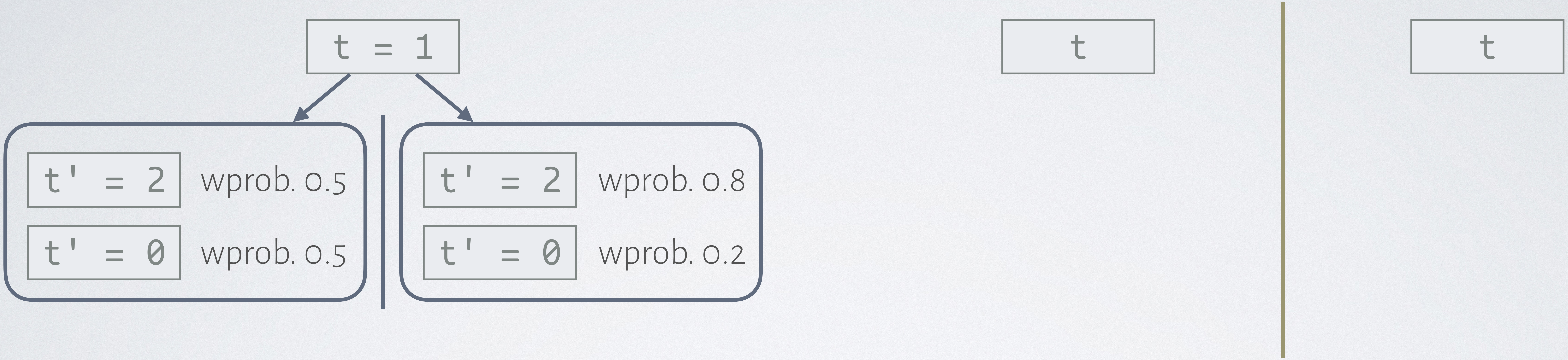

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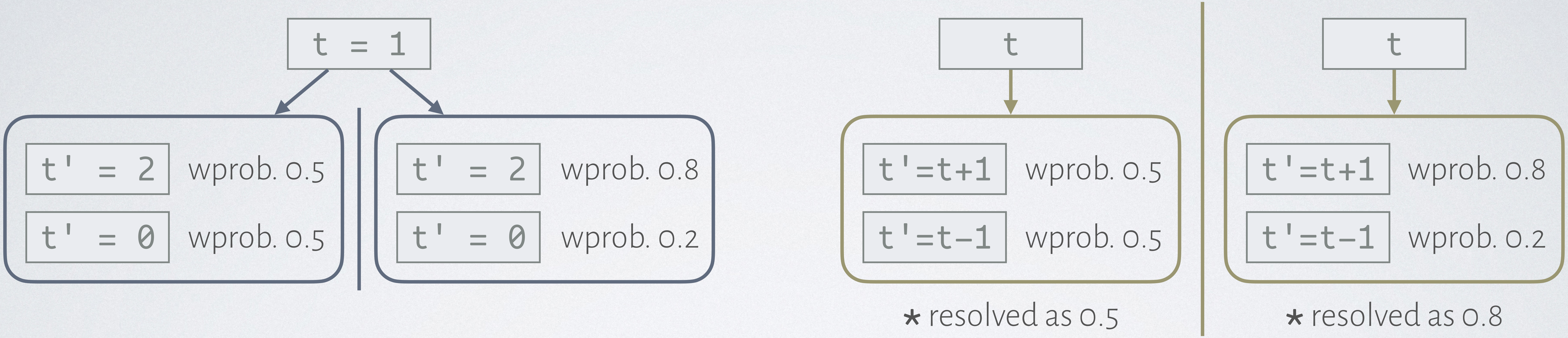

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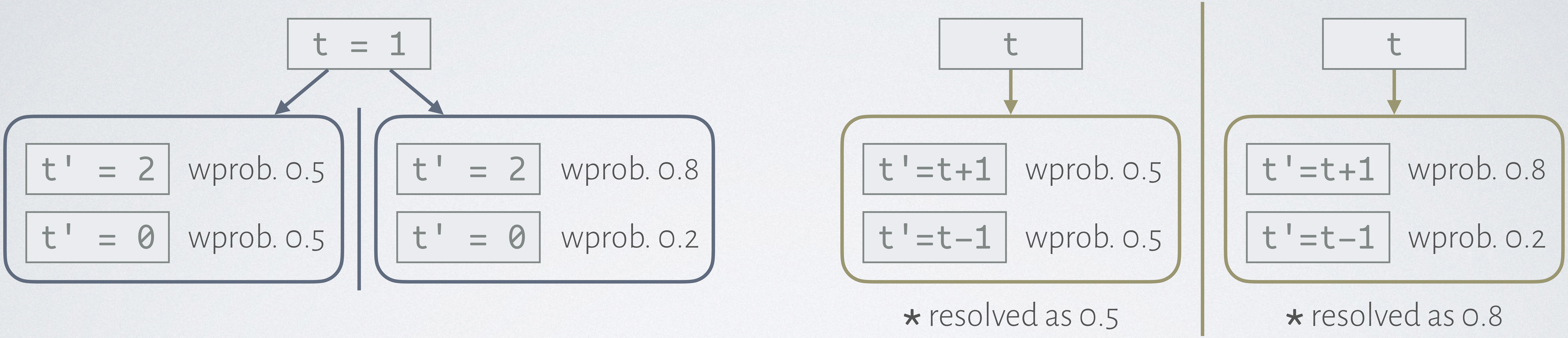

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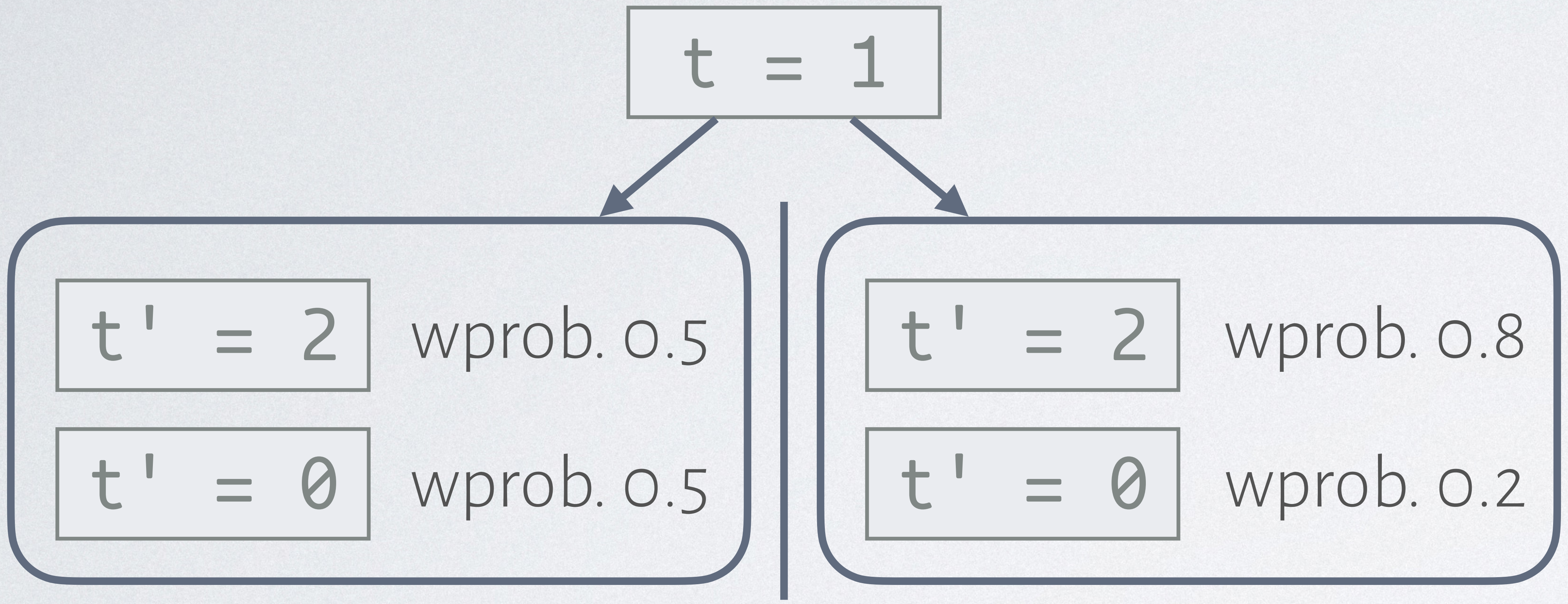

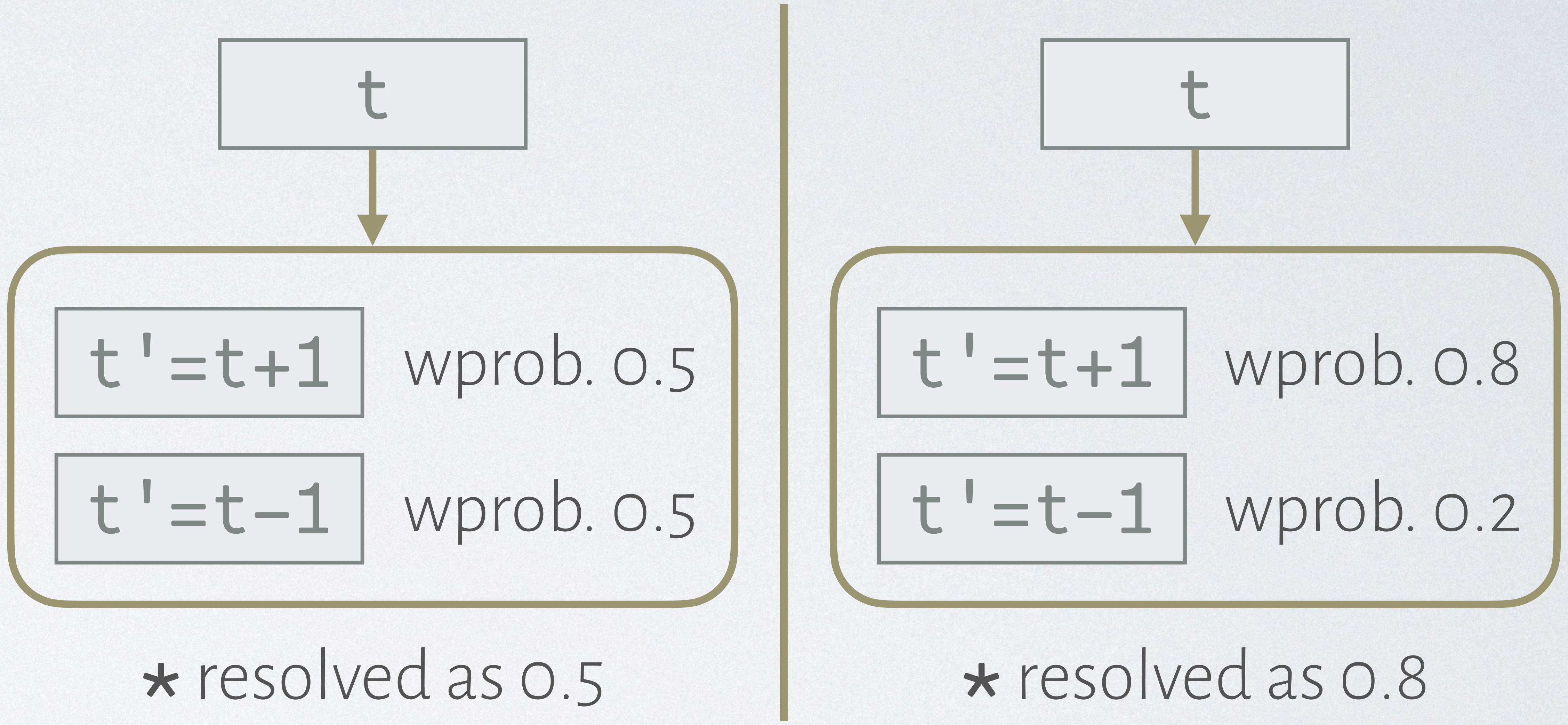

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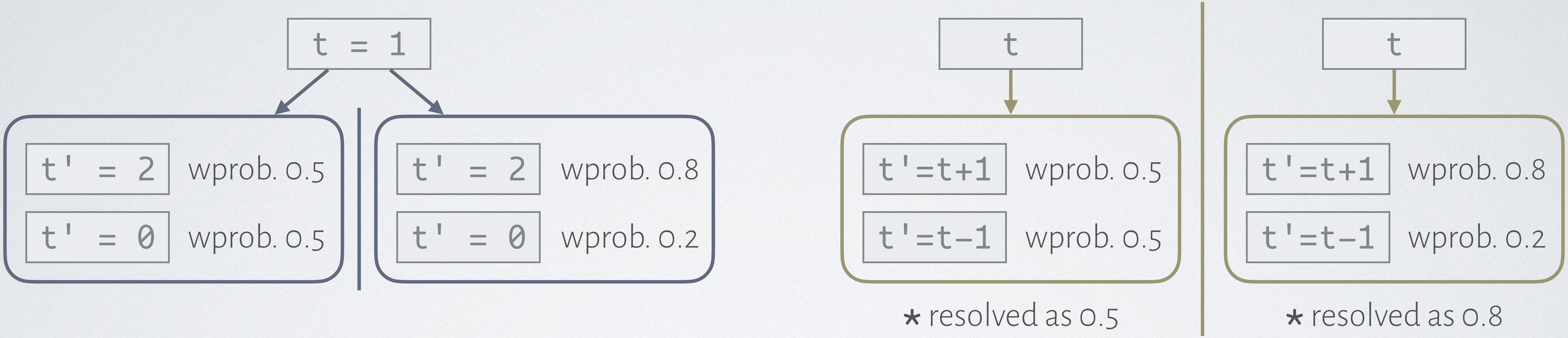

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- Relational reasoning: for any concretization P of the program, for any inputs t_1, t_2, it holds that $\mathbb{E}_{t_1' \sim P(t_1), t_2' \sim P(t_2)}[t_1' - t_2'] = t_1 - t_2$

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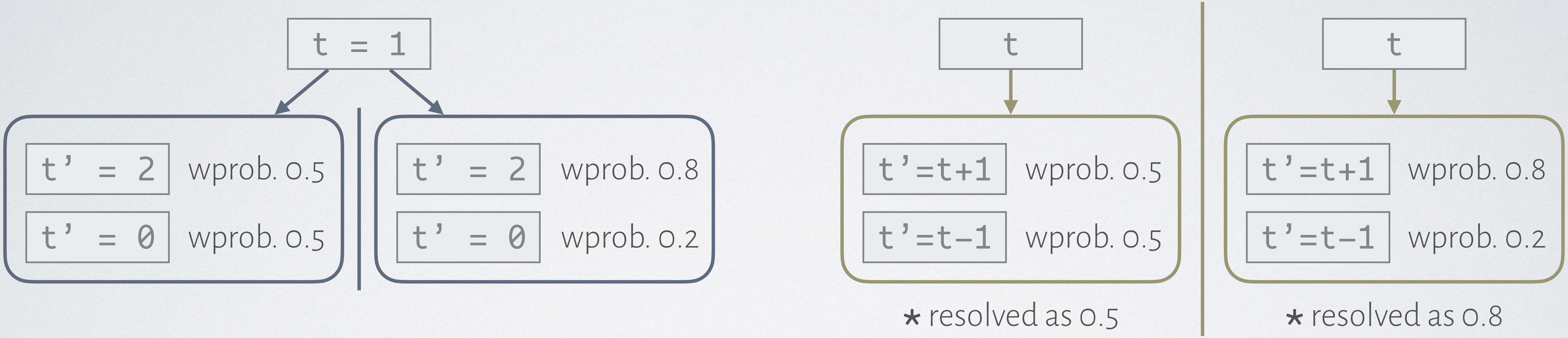

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- We developed a domain-theoretic characterization of the nondeterminism-first resolution

Overview

- ☑ Motivation
- ☑ An Algebraic Denotational Semantics
- ☐ Pre-Markov Algebra Framework (PMAF)

Pre-Markov Algebra Framework (PMAF)

Contributions

- An **algebraic framework** for **interprocedural dataflow analysis** of **first-order probabilistic programs**
- Published as PMAF: An Algebraic Framework for Static Analysis of Probabilistic Programs in *PLDI'18*

PMAF

Recursion
Unstructured control-flow
Divergence
Nondeterminism
…

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- Reachability Prob. Analysis
- Markov Decision Problem

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- Expectation-Invariant Analysis

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⋆ denotes nondeterministic-choice

`tick(q)` increases T by `q`

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whereas our semantics yields $\mathbb{E}[T] = 1.5$

Example Analyses

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b1 ~ Bernoulli(0.5);
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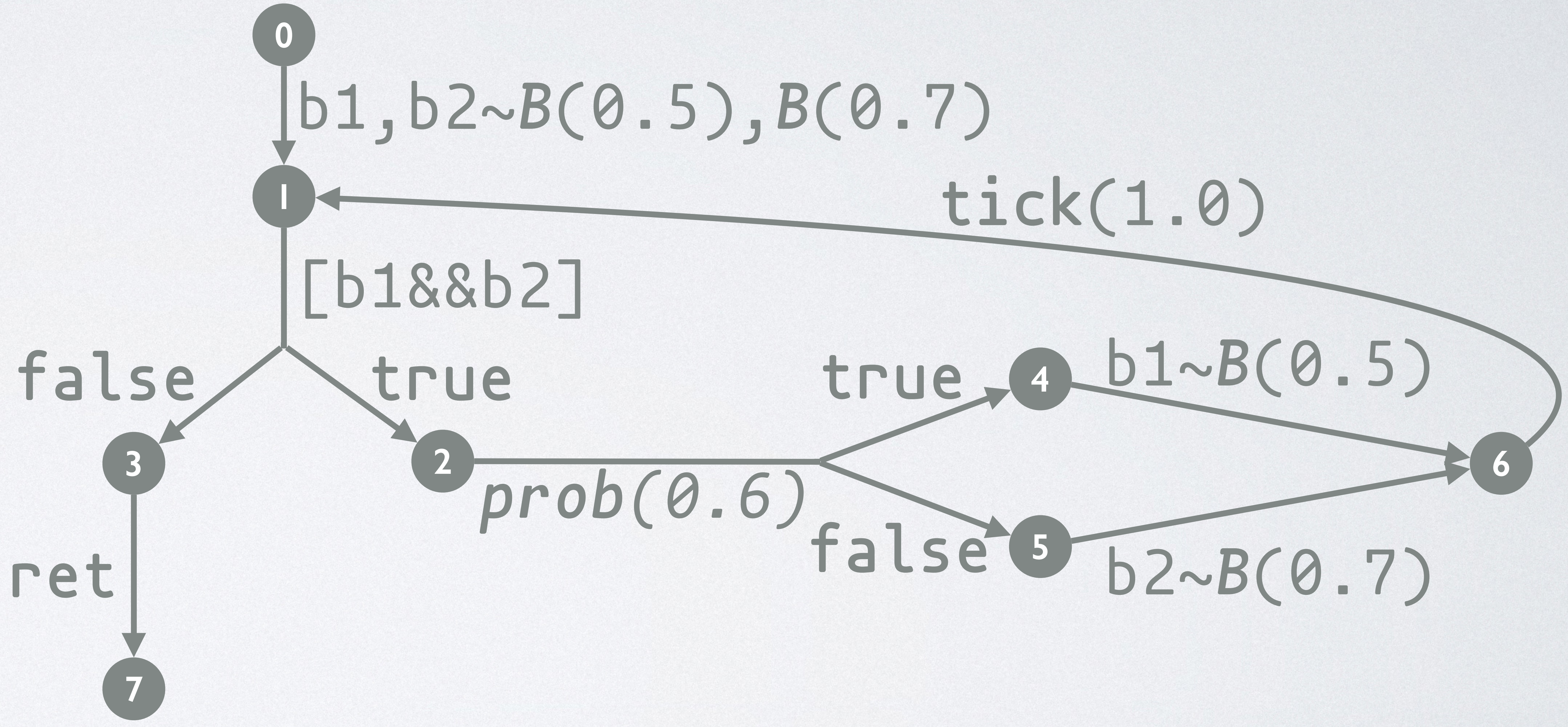
Hyper-Graph Semantics

- The hyper-graph semantics is formulated by an equation system



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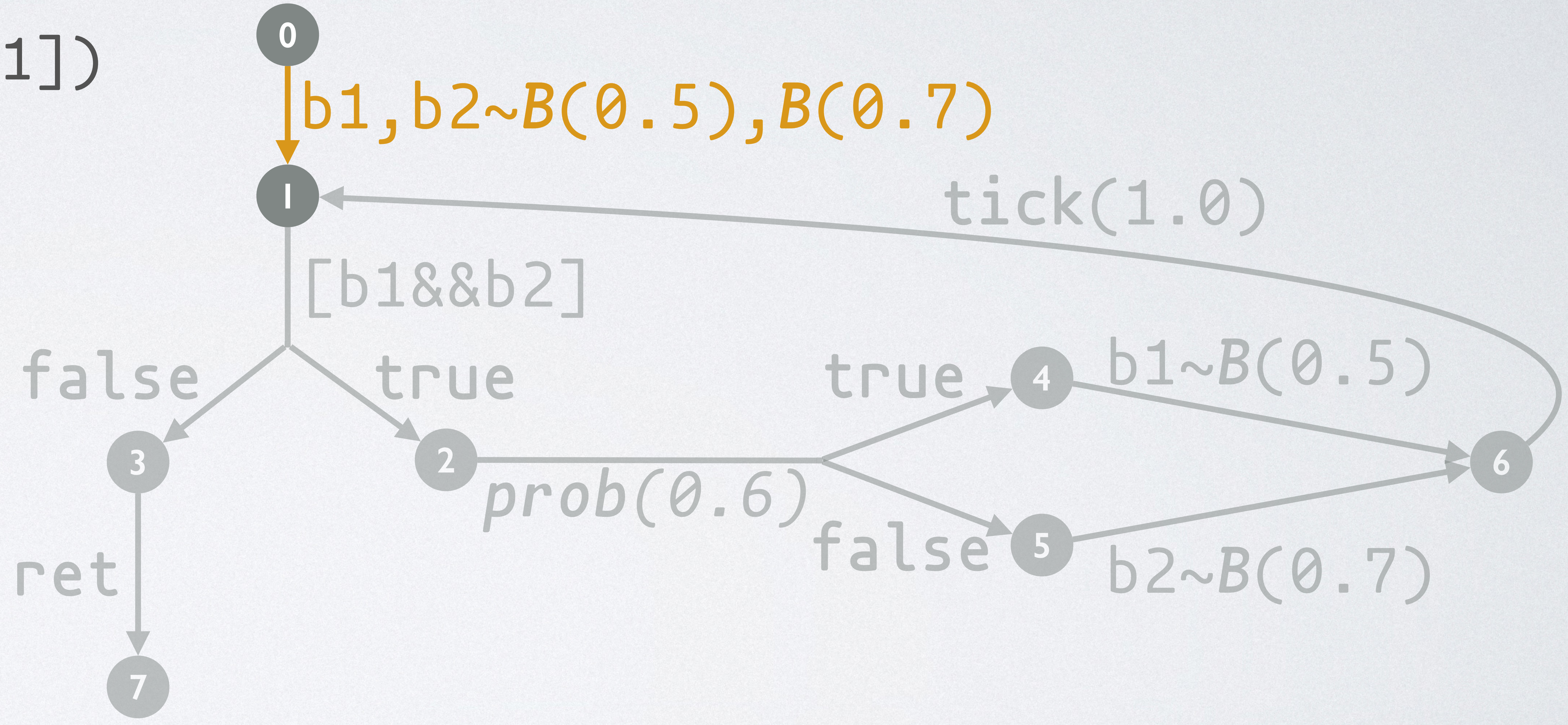

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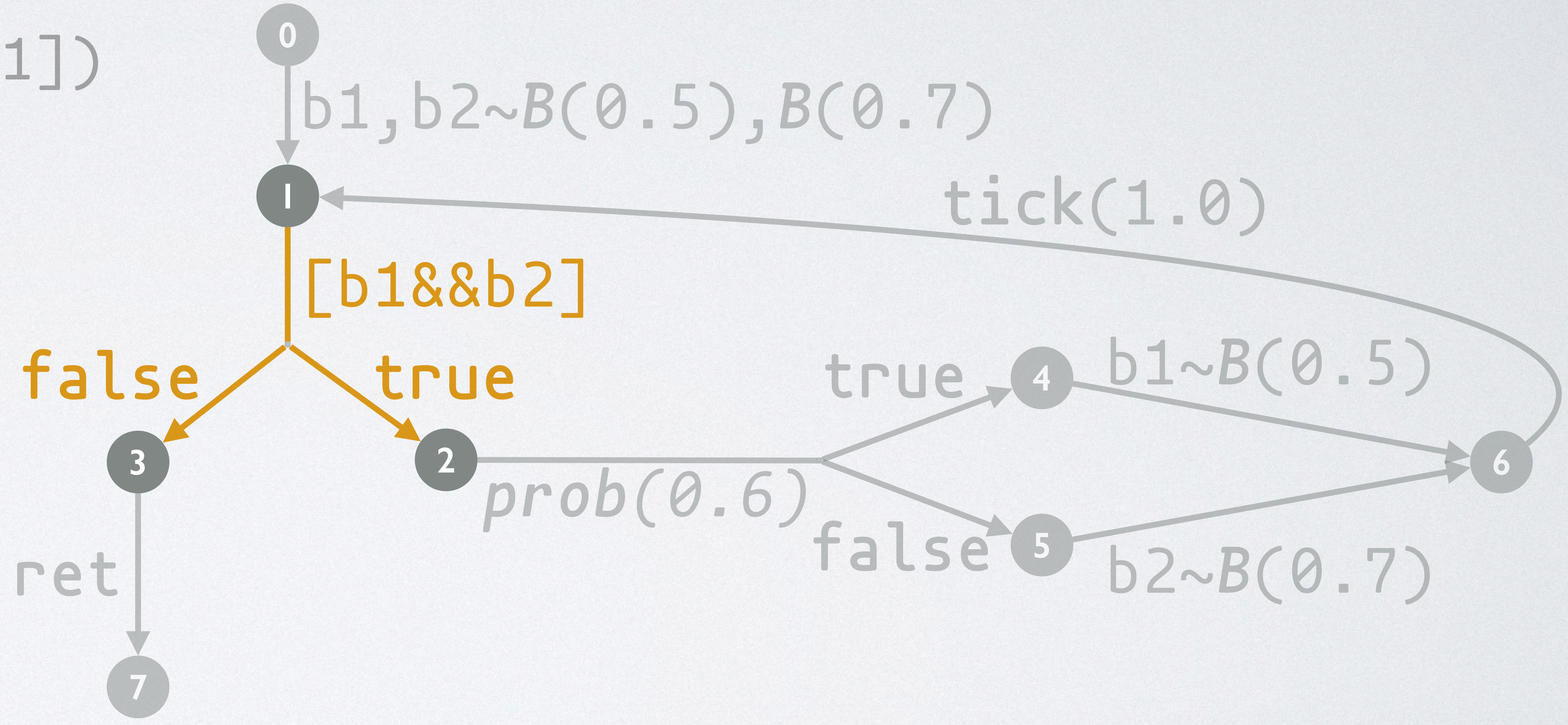

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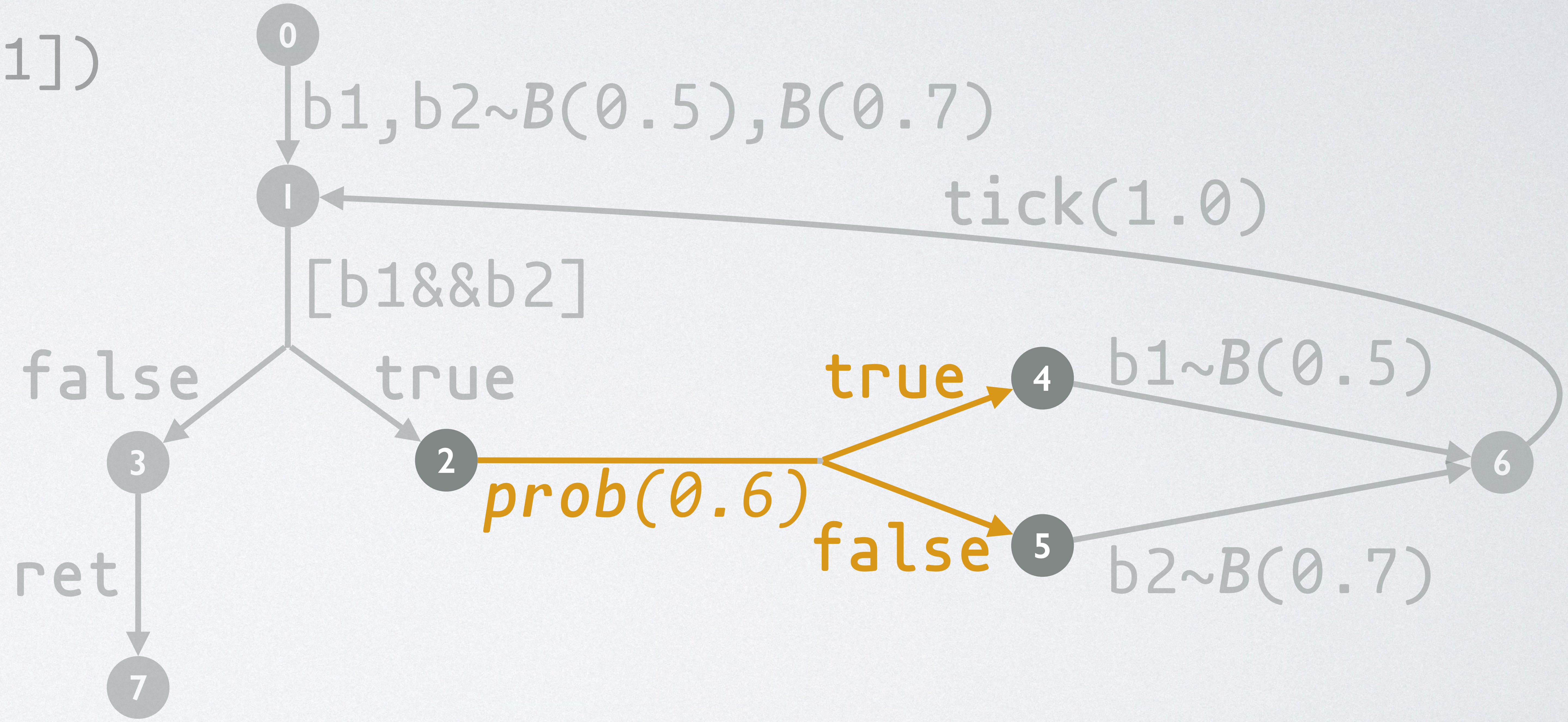

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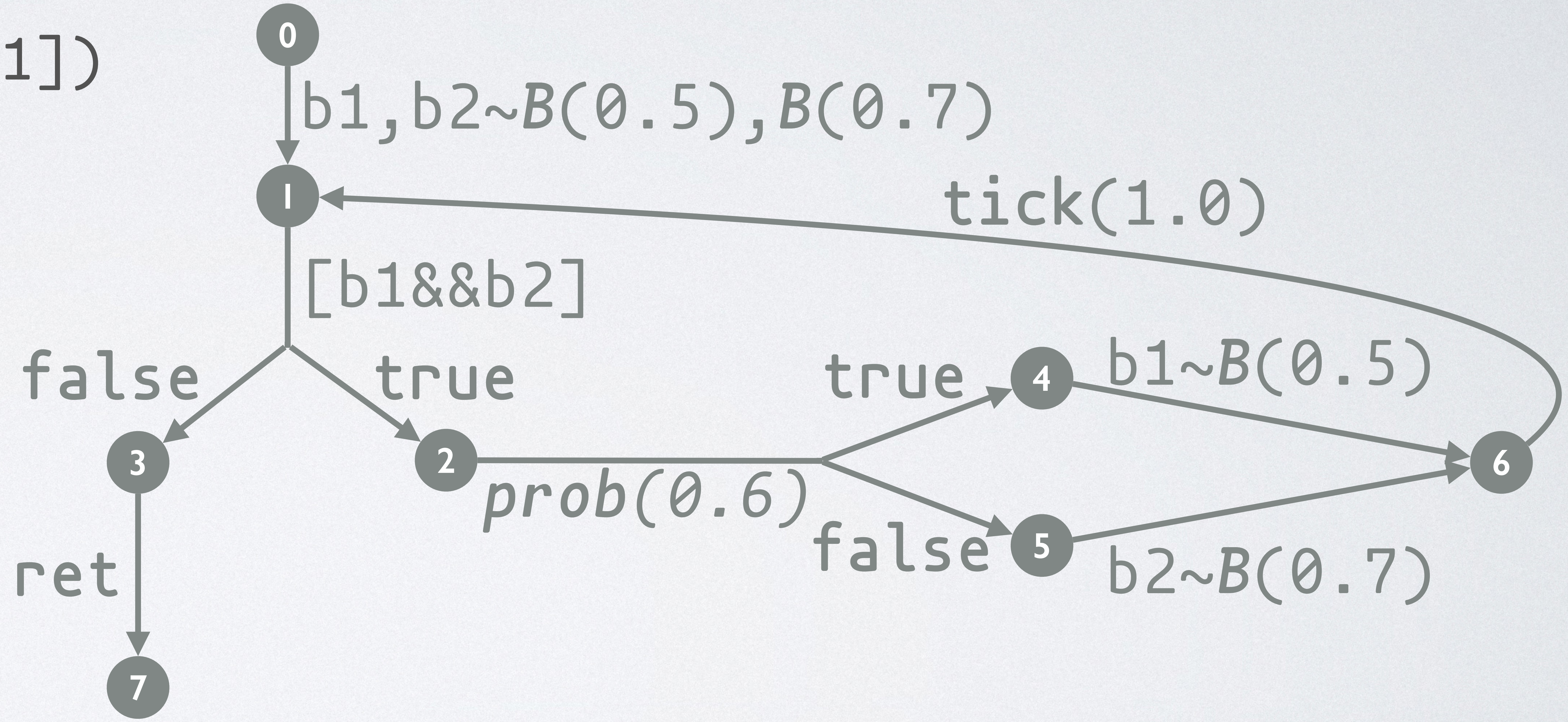

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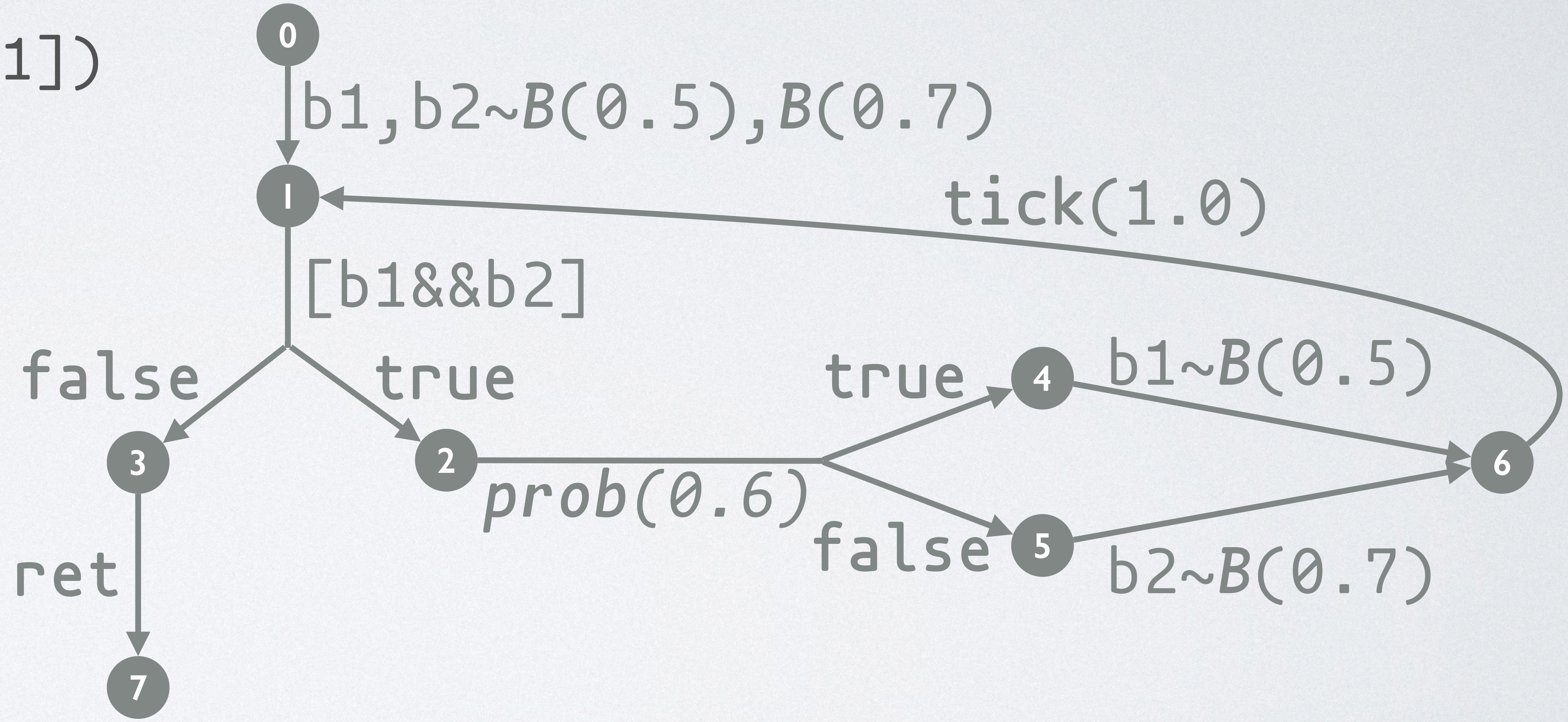

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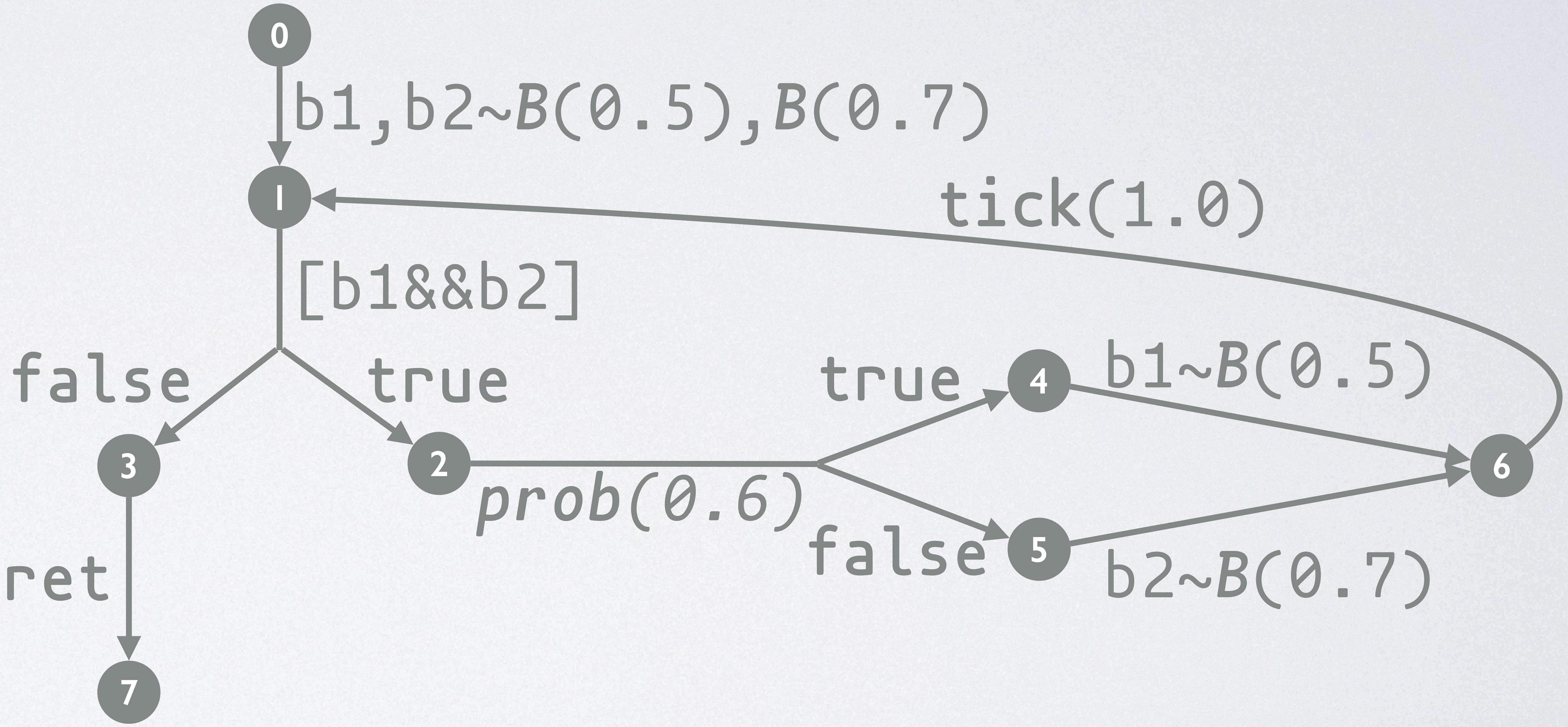

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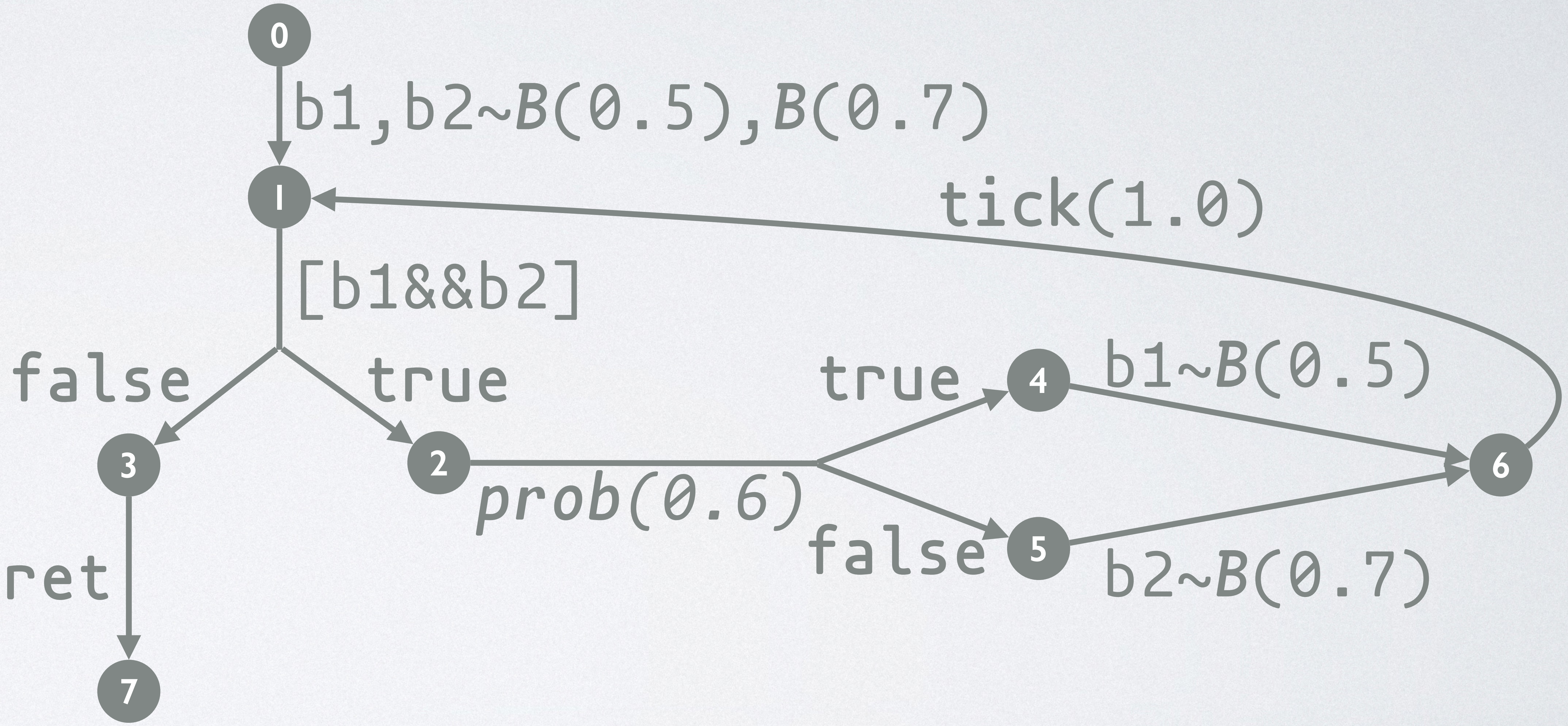
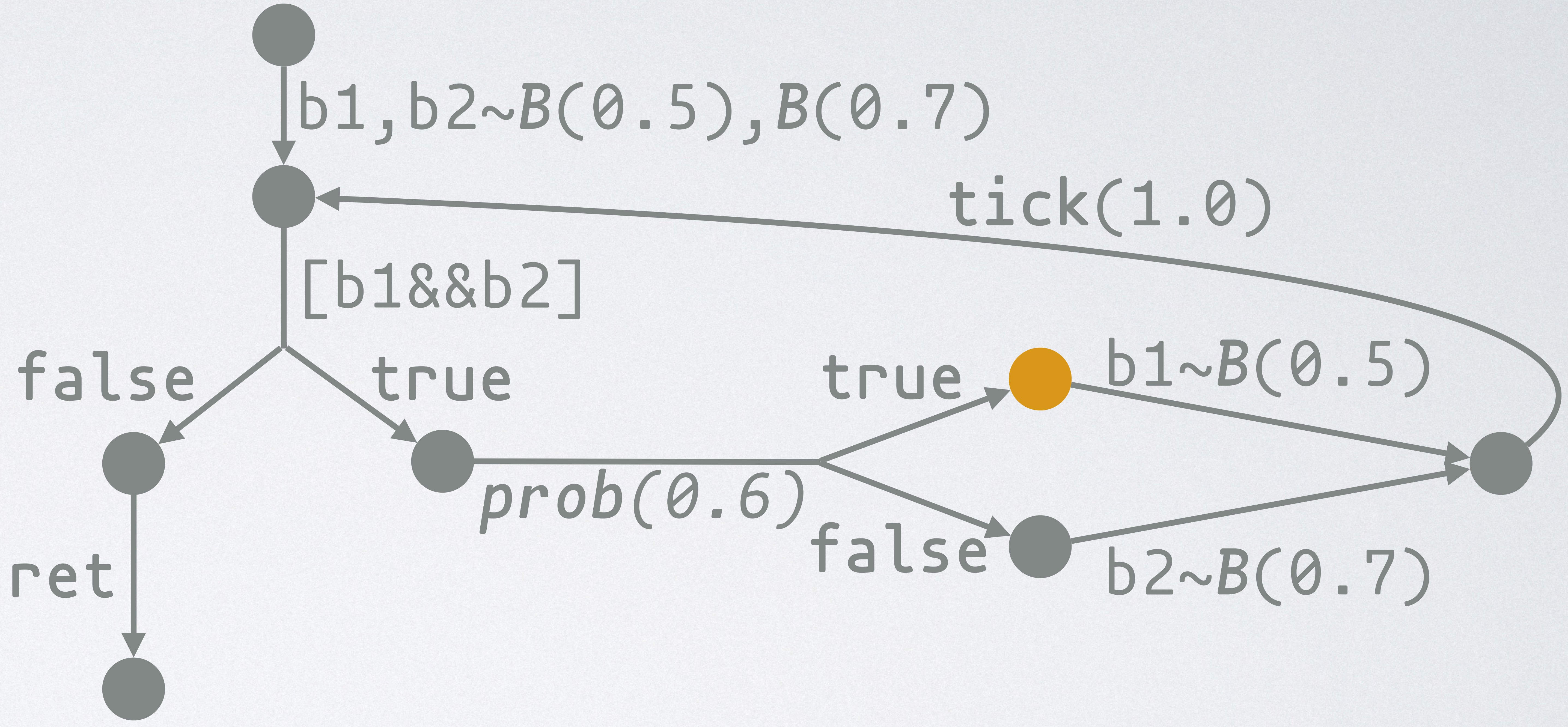
If using **abstract** semantics, we obtain an equation system for **static analysis**



Hyper-Graph Analysis

- **Forward assertions**
- The property of a node is a summary of the computation that continues from the node



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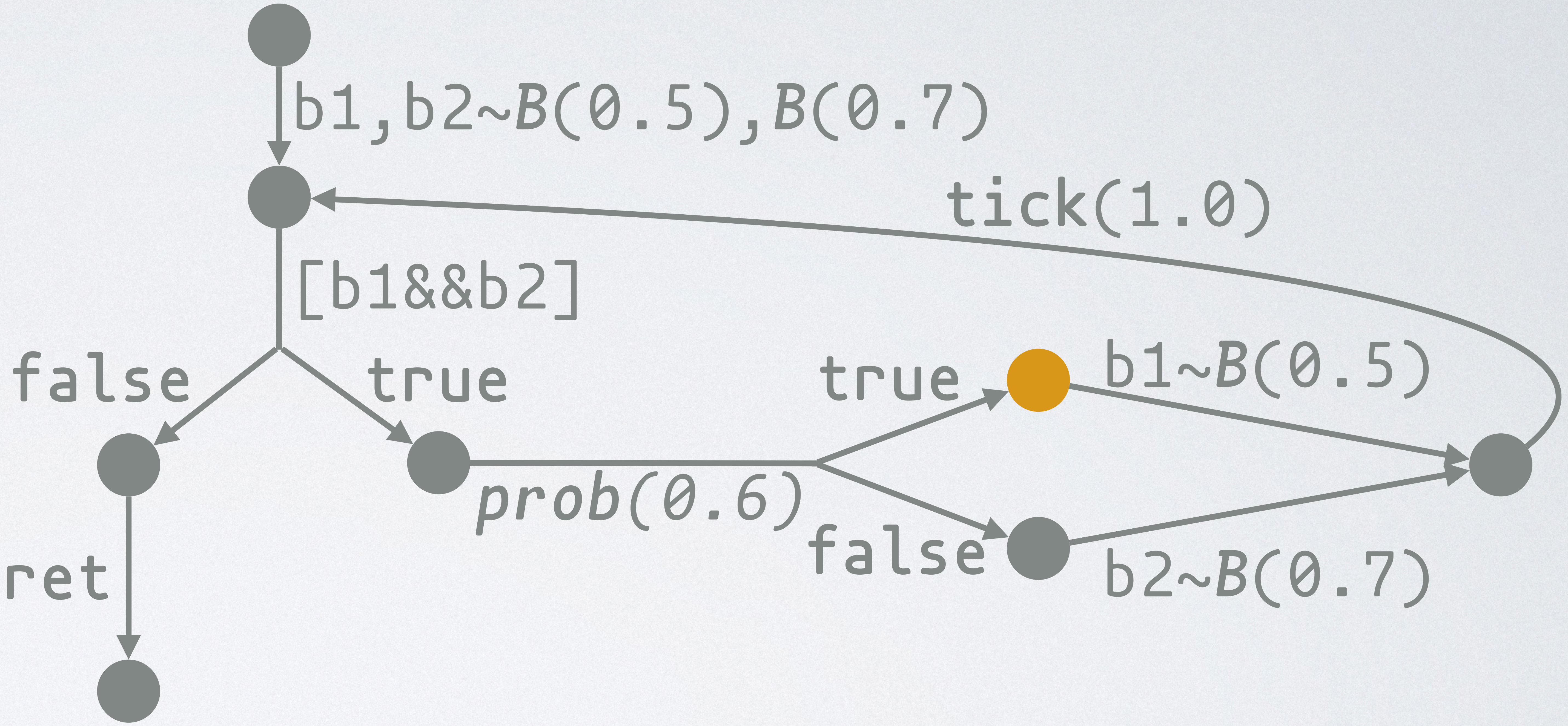

e.g. the property represented by the node is

$$\lambda(\mathsf{b1},\mathsf{b2})\,.\,\textbf{if}\ \mathsf{b2}\ \textbf{then}\ \frac{1}{7}[\mathsf{b1}'=\mathsf{T},\mathsf{b2}'=\mathsf{F}]+\frac{6}{7}[\mathsf{b1}'=\mathsf{F},\mathsf{b2}'=\mathsf{T}]\ \textbf{else}\ \frac{1}{2}[\mathsf{b1}'=\mathsf{T},\mathsf{b2}'=\mathsf{F}]+\frac{1}{2}[\mathsf{b1}'=\mathsf{F},\mathsf{b2}'=\mathsf{F}]$$

Pre-Markov Algebras (PMA)

- A PMA is basically a Markov algebra, but we assume a different set of axioms that are suitable for formulating static analysis

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Program properties form a complete lattice

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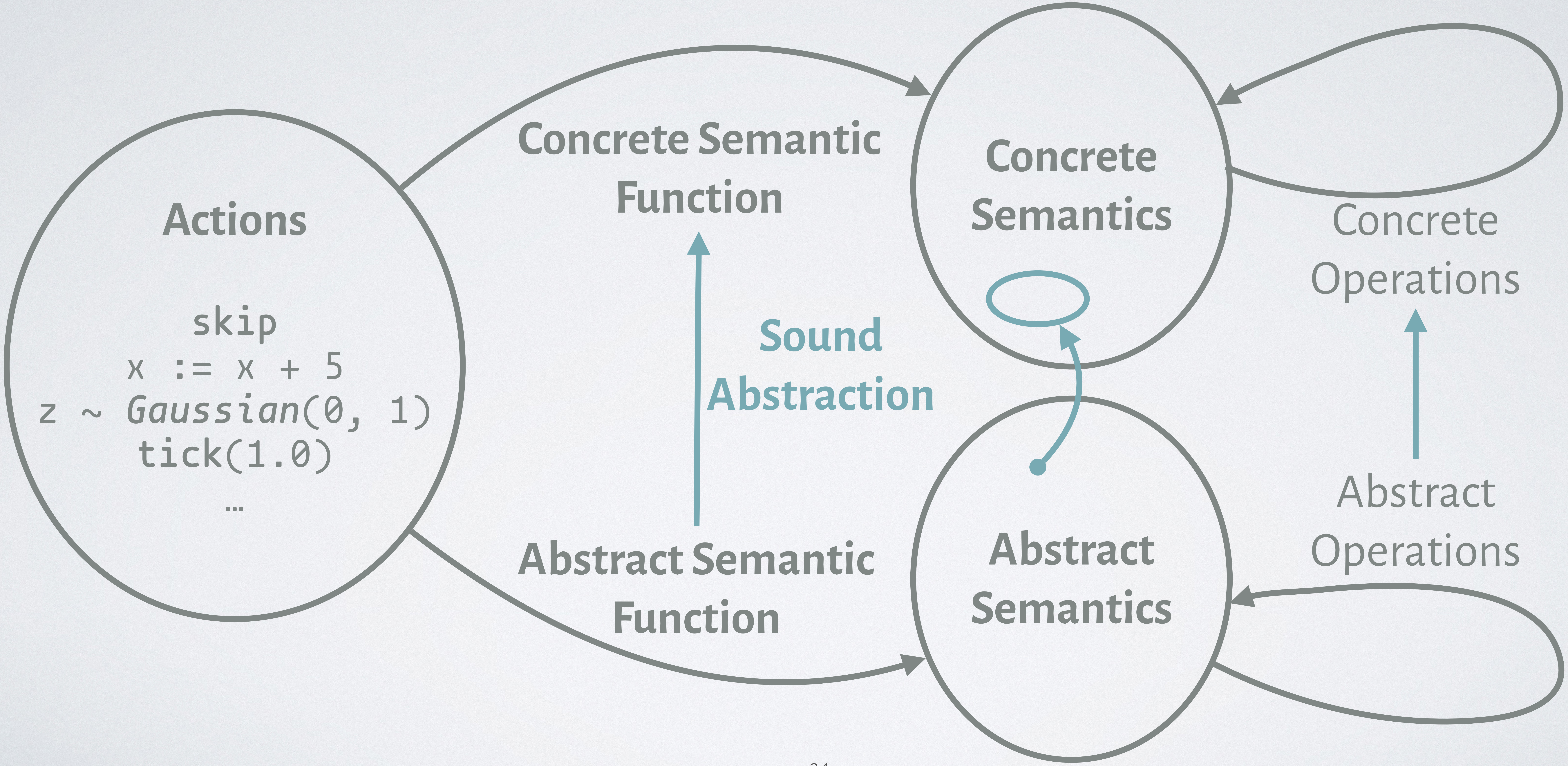
Soundness via Abstract Interpretation



INSTANTIATIONS

PMAF

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New

Interprocedural analyzers!

Instantiations

Existing

PMAF

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Instantiations

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New

- Expectation-Invariant Analysis

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Prove invariants among **initial values** and **expected final values**

Expectation-Invariant Analysis

- Benchmark collected from the literature[7,8] and also handcrafted by us
- Derive expectation invariants as least as precise as them in most case

Expectation-Invariant Analysis

Program	#loc	time (sec)	Expectation Invariants
binom-update	14	0.06	E[4x'-n']=4x-n, E[x']<=x+1/4
eg	8	0.89	E[x'+y']=x+y+4, E[z']=1/4z+3/4
recursive	13	0.37	E[x']=x+9
mot-ex	16	0.06	E[2x'-y']=2x-y, E[4x'-3c']=4x-3c, E[x']<=x+3/4

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