Central Moment Analysis For Cost Accumulators In Probabilistic Programs

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Probabilistic Programs

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Standard programs +

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Standard programs + Probability distributions

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Random control flows

PROBABILISTIC PROGRAMS

Standard programs + Probability distributions / Random control flows

Can be used to implement and analyze

- Randomized algorithms
- Cryptographic protocols
- Machine-learning algorithms

Cost Accumulators

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Quantities that can only be incremented or decremented

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Termination time

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Cash flow

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Termination time	Rewards in MDPs	Cash flow

```
# variables: x, t
# pre-condition: x > 0
func rdwalk() begin
  if x > 0 then
    t ~ uniform(-1, 2); # sample t from a uniform distribution
    x := x - t;
    call rdwalk();
    tick(1)              # add one to the cost accumulator
  fi
end
```

Quantitative Analysis

```
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QUANTITATIVE ANALYSIS

What can we know about the **accumulated cost**?

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The program produces a **distribution** on possible accumulated costs.

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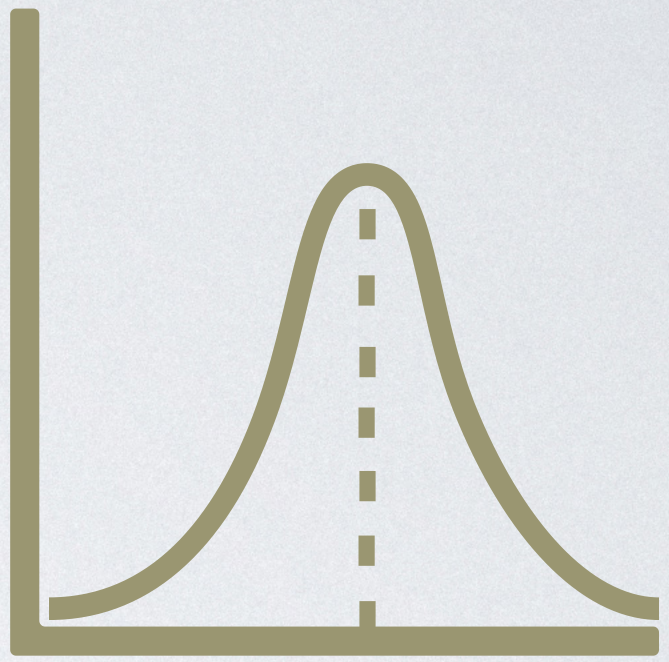
By **simulation**, we can obtain an empirical estimation of this distribution.



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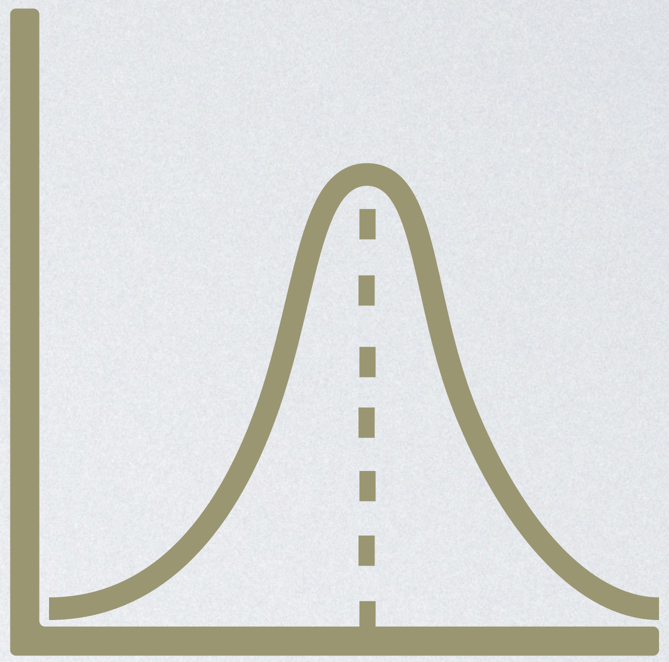

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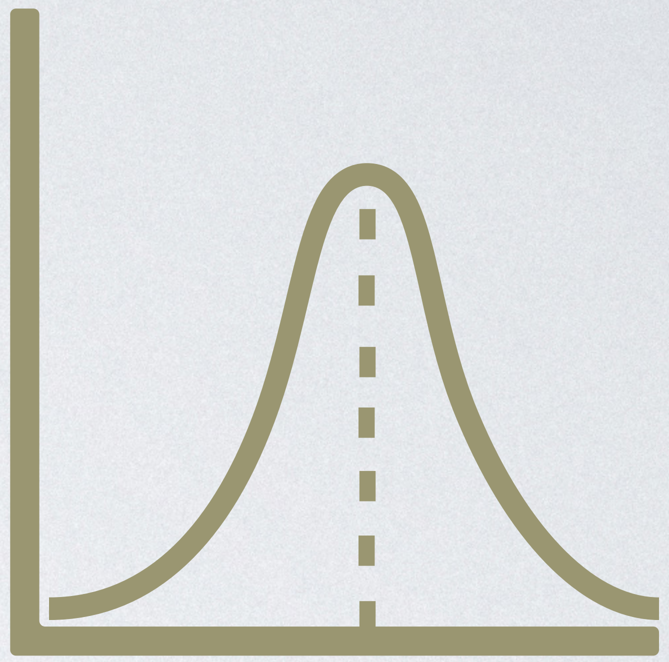

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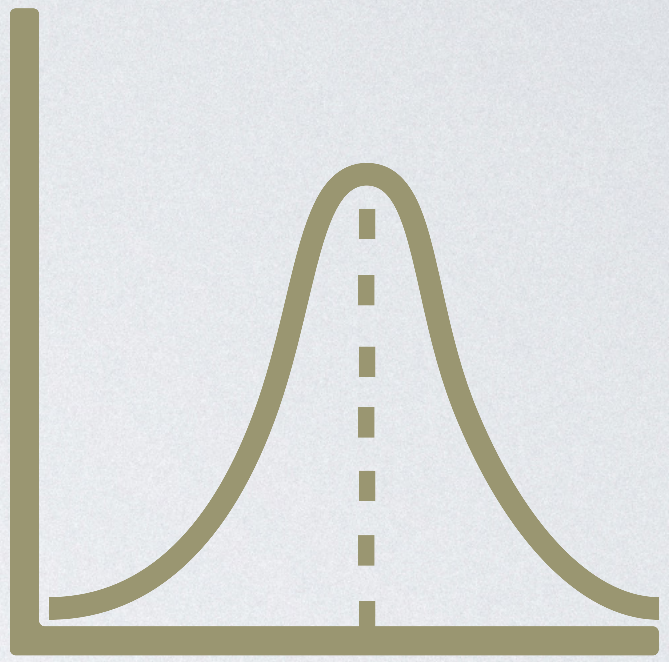

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Such as Moments!

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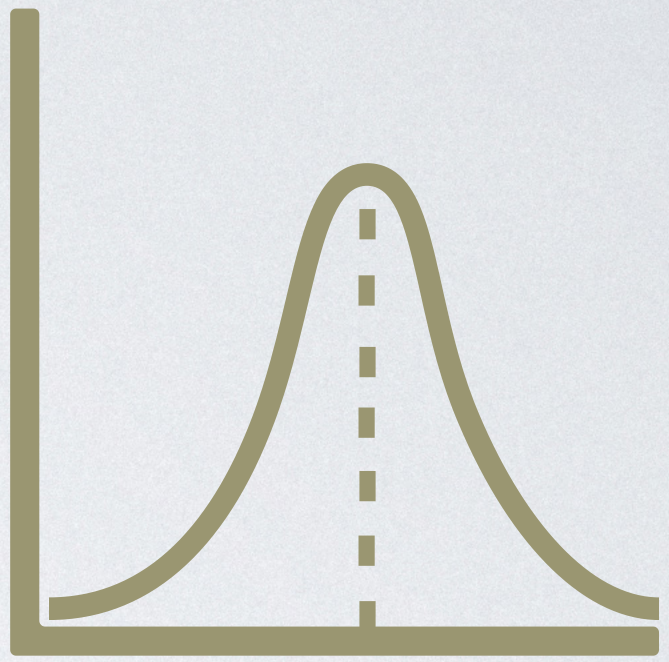

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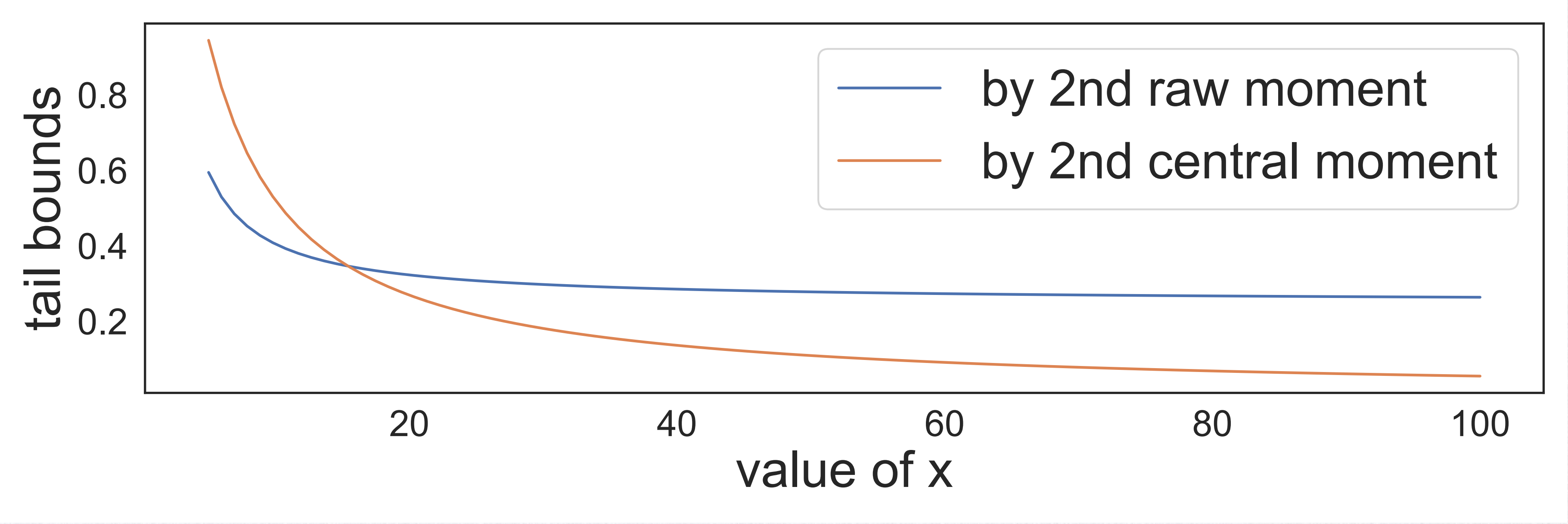

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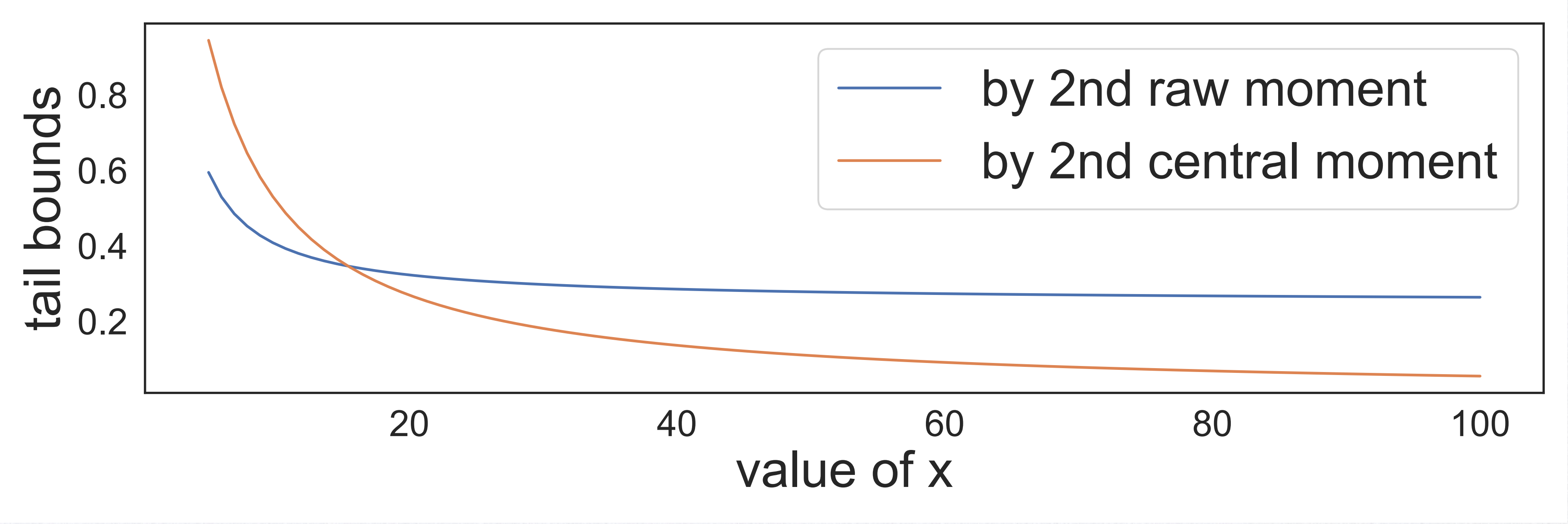

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Central moments can provide more information!

Our Work

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for deriving symbolic interval bounds
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in probabilistic programs with general recursion

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- A program logic for moment inference, and proof of its soundness

Case Study: Timing Attack

password checker

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Our tool still points out that a timing attack is very likely to succeed!

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$$13N - 5K \leq \mathbb{E}[T_0] \leq 13N - 3K \qquad\qquad 13N \leq \mathbb{E}[T_1] \leq 15N$$

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So if the runtime is less than $13N - 1.5K$, the next bit is likely to be zero

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$$\mathbb{E}[(T_0 - \mathbb{E}[T_0])^2] \leq 8N - 36K^2 + 52NK + 24K \qquad \mathbb{E}[(T_1 - \mathbb{E}[T_1])^2] \leq 26N^2 + 42N$$

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$\mathbb{P}[\text{The attack succeeds}] \geq 0.830561$ when $N = 32$

More in the Paper

- Full formalism of the program logic for moment inference
- How our system supports inter-procedural reasoning
- Proof of the soundness for the program logic
- An implementation and experiments on analysis capability and scalability
- Comparison with prior work on raw-moment analysis for termination time and expected-cost bound analysis