Liquid Resource Types

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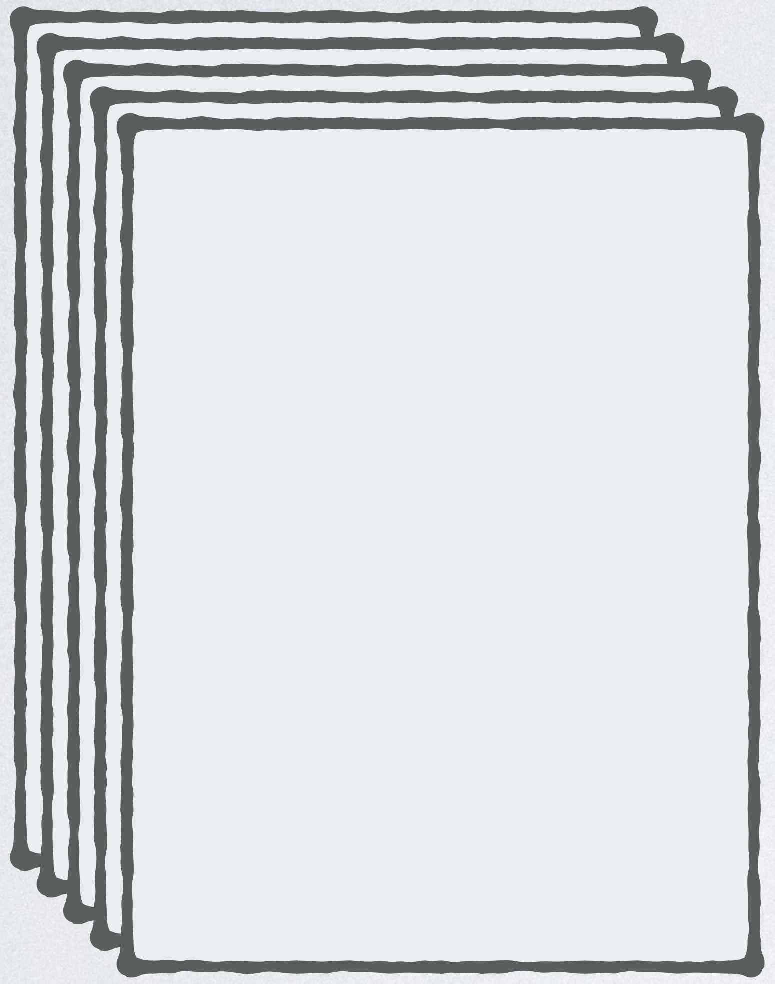
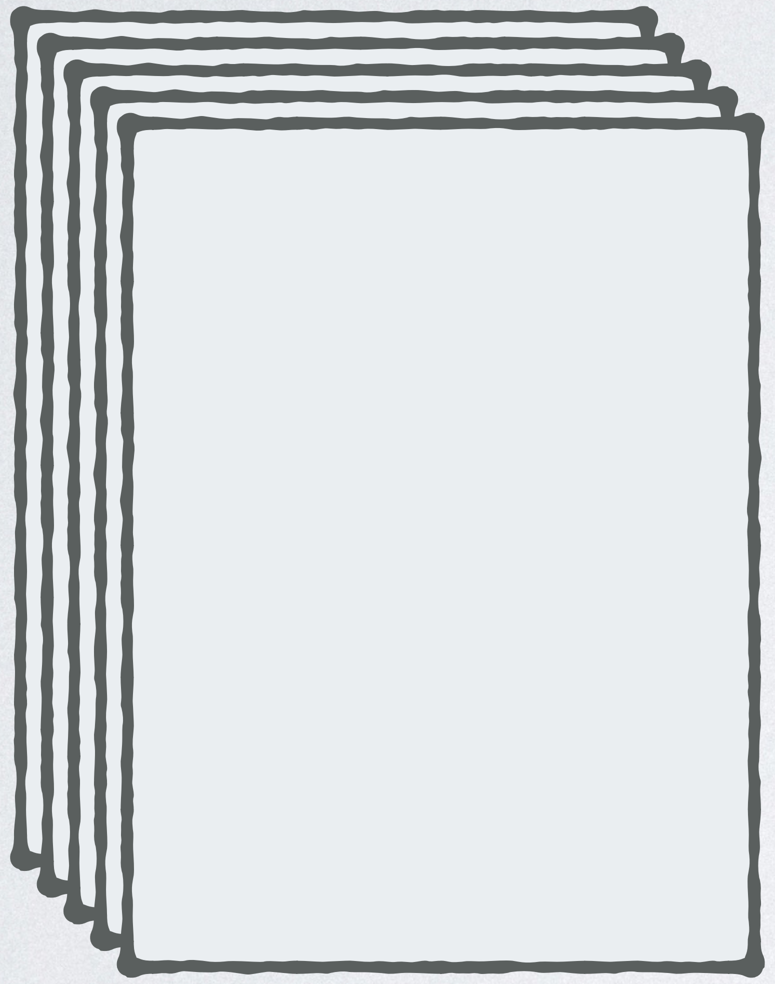
RESOURCE ANALYSIS



Programs

Resource Analysis



Programs

Performance

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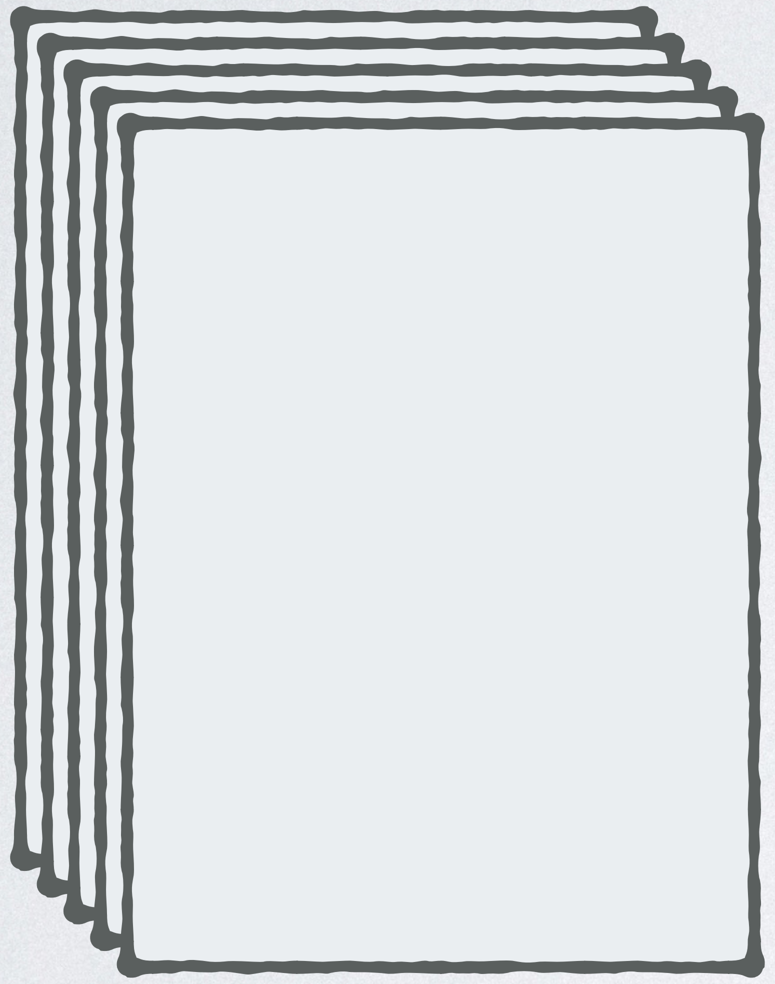

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- Memory
- Power
- …

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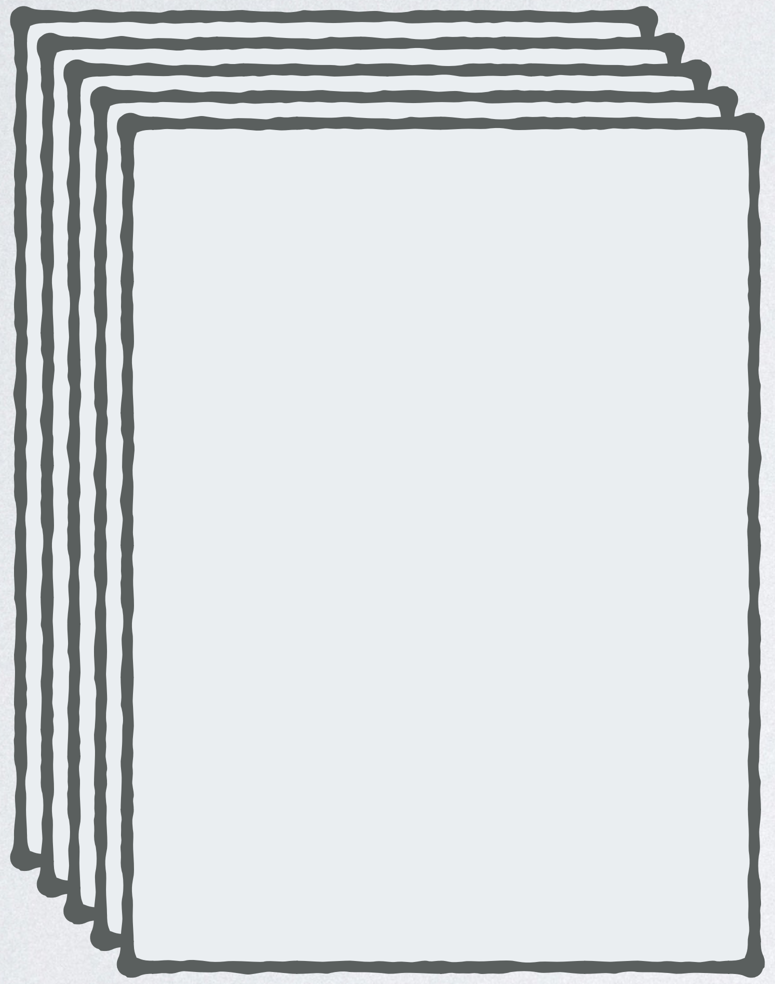

Programs

- Time
- Memory
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- …

Performance

Scenarios

- Algorithmic complexity
- Gas usage in blockchains
- Side-channel detection

Example: Insertion Sort

Quick Sort

Insertion Sort

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Quick Sort	Insertion Sort

- These two are functionally equivalent. Which one performs better?

EXAMPLE: INSERTION SORT

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- Both run in quadratic time in the worst case.

EXAMPLE: INSERTION SORT

Quick Sort	Insertion Sort

- These two are functionally equivalent. Which one performs better?
- Both run in quadratic time in the worst case.
- But insertion sort can be linear-time on nearly-sorted data.

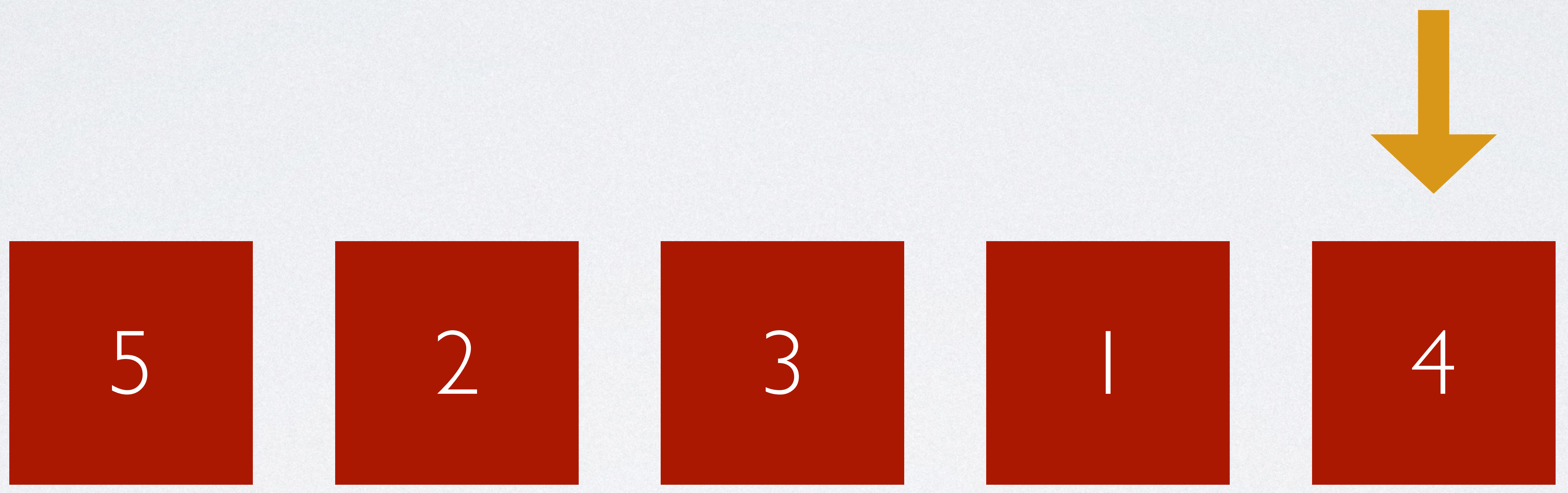
EXAMPLE: INSERTION SORT

Insertion Sort

5 2 3 1 4

Example: Insertion Sort

Insertion Sort



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How many **swaps** does the algorithm need?

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Insertion Sort

How many **swaps** does the algorithm need?

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- #swaps is proportional to #out-of-order-pairs in the input.
- **Challenge:** Express and automatically verify such a complex bound.

Liquid Types and Resources

[1] P. M. Rondon, M. Kawaguchi, and R. Jhala. 2008. Liquid Types. In *PLDI'08*.
[2] T. Knoth, D. Wang, N. Polikarpova, and J. Hoffmann. 2019. Resource-Guided Program Synthesis. In *PLDI'19*.

LIQUID TYPES AND RESOURCES

Liquid Types[1]

"A function returns the absolute value of the input integer"

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Liquid Types[1]	"A function returns the absolute value of the input integer"
RESYN[2]	"A list where each element carries one unit of potential"

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LIQUID TYPES AND RESOURCES

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Liquid Resource Types (This Work) "Potentials can be inductively specified from the formation of data structures themselves"

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Liquid Types and Resources

Liquid Types[1] "A function returns the absolute value of the input integer"

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ReSyn[2] "A list where each element carries one unit of potential"

Liquid Resource Types (This Work) "Potentials can be **inductively** specified from the formation of data structures themselves"

can be used to express the bound of insertion sort

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Contributions

CONTRIBUTIONS

- Liquid resource **types** for verifying super-linear value-dependent bounds

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- Proof of type **soundness** w.r.t. a cost semantics
- Prototype **implementation** and **evaluation**

Outline

- [x] Motivation
- [ ] Background: Liquid Types and the Potential Method
- [ ] Liquid Resource Types
- [ ] Evaluation

Liquid Types

Liquid Types

$\{ B \mid \psi \}$

A value of type B that satisfies ψ

LIQUID TYPES

`{ B	ψ }`	A value of type B that satisfies ψ
`{ Int	v ≥ 0 }`	A non-negative integer

Liquid Types

`{ B | ψ }` A value of type B that satisfies ψ

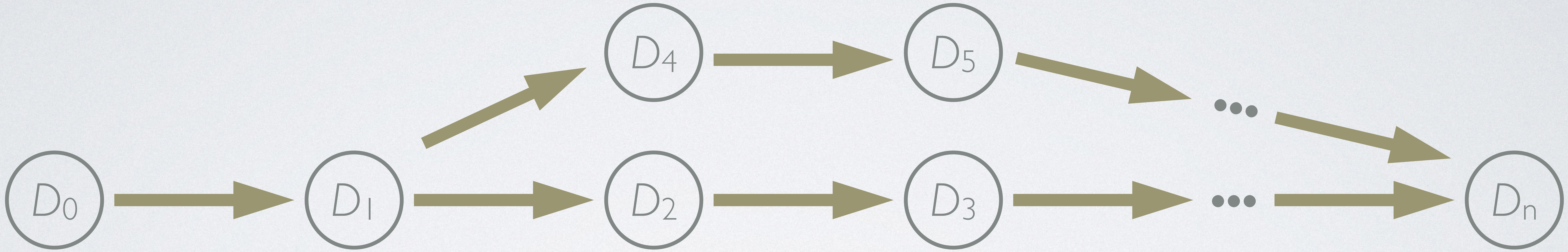
`{ Int | v ≥ 0 }` A non-negative integer

```
(xs : List a) -> { List a | len(v) = len(xs) + 1 }
```

A function that returns a list whose length is one plus the length of its input

The Potential Method

THE POTENTIAL METHOD



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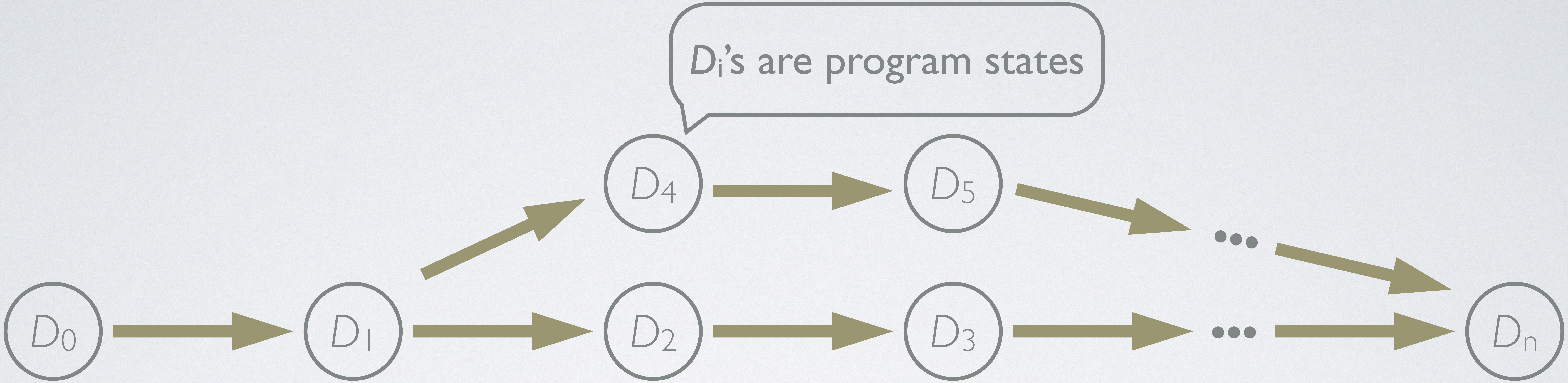

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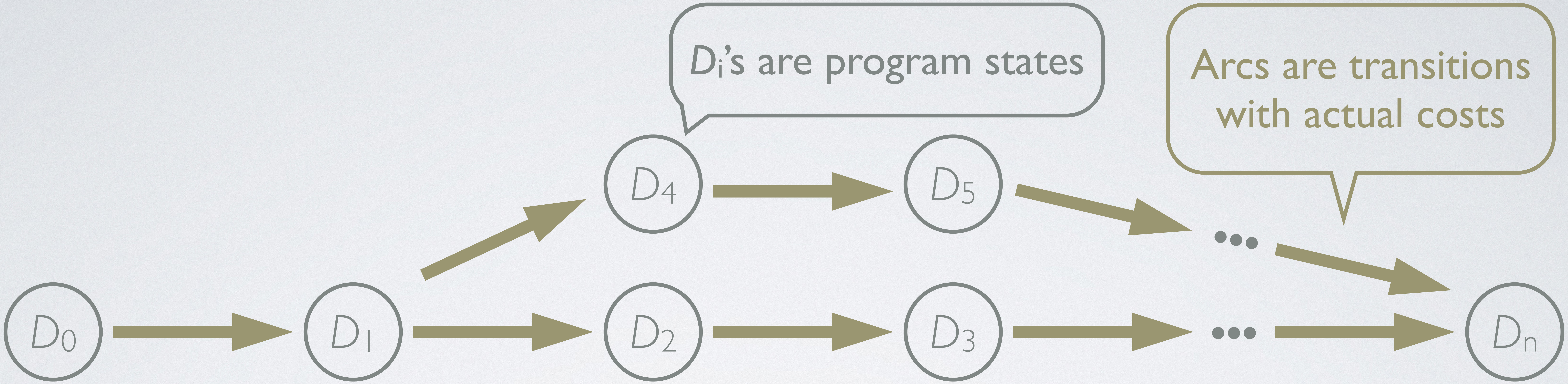

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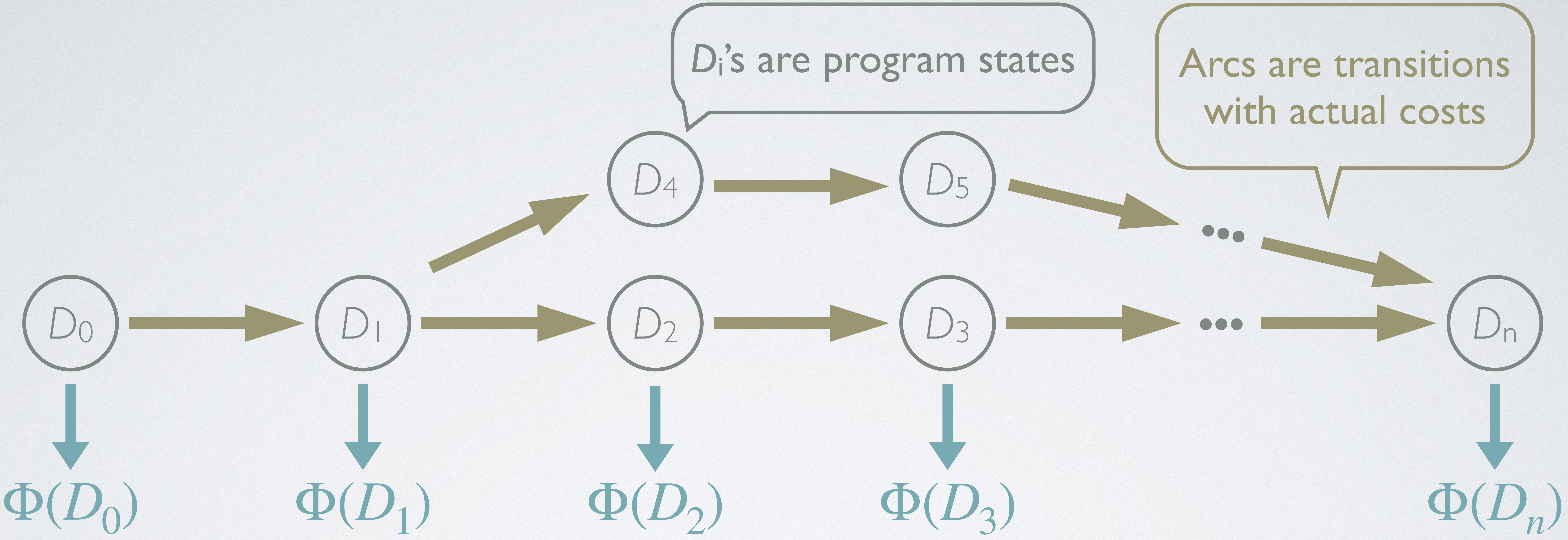

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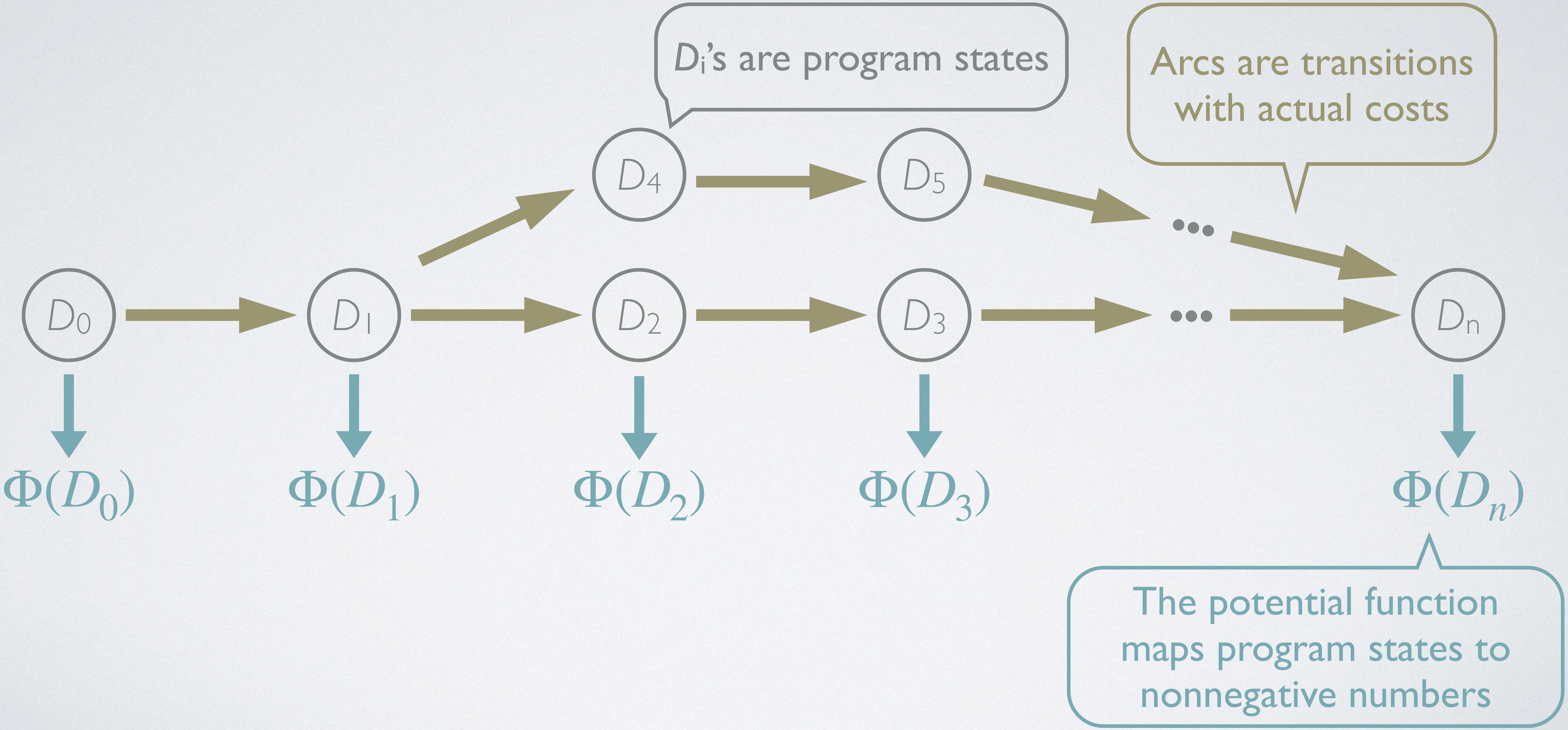

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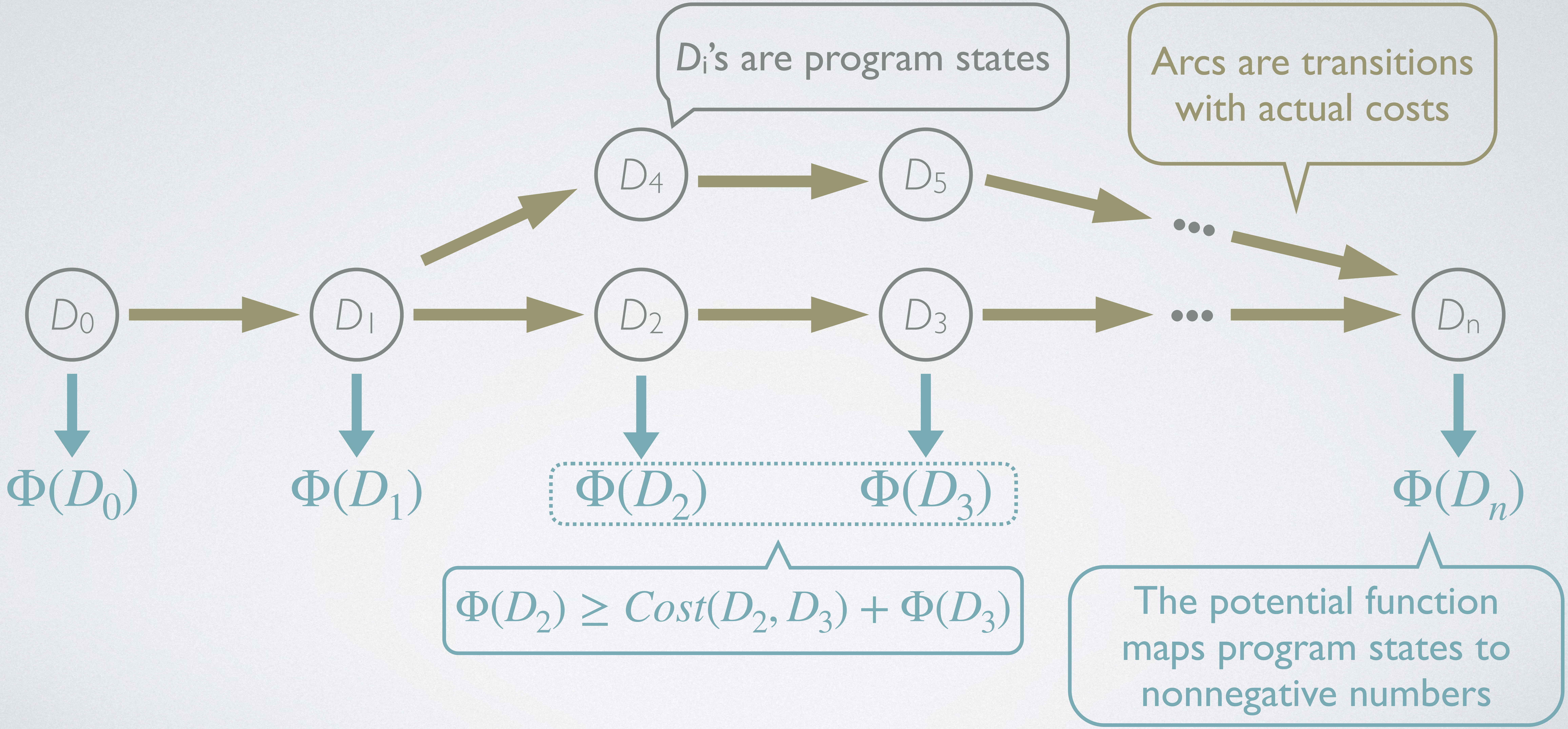

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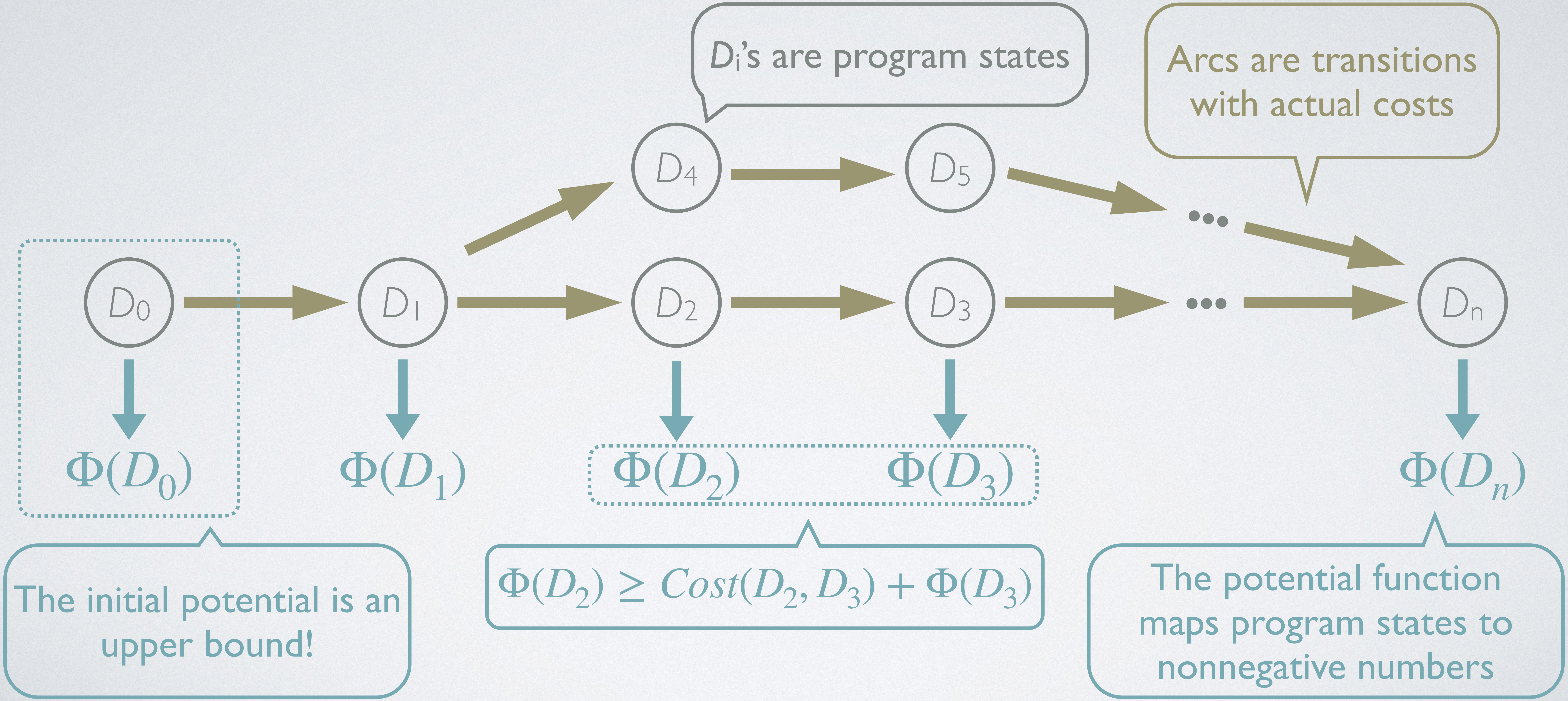

ReSyn: Liquid Types + Linear Potentials

Potential: numeric

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Refinement: boolean

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A non-negative integer carrying potential equal to 5 times of its value

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A non-negative integer carrying potential equal to 5 times of its value

$$\mathtt{List\ a}^{\mathtt{ite}(v \geq 0,1,0)}$$

A list of numbers carrying potential equal to #non-negative elements in it

RESYN: LIQUID TYPES + LINEAR POTENTIALS

```
insert = λx. λxs.
 match xs with
   Nil -> Cons x xs
   Cons hd tl -> if hd < x
     then Cons hd (tick 1 (insert x tl))
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Each element that is less than **x** carries **one unit** of potential

- In ReSyn, you **cannot** express a type for insertion sort.
- Because you can only distribute potential **uniformly** within a list.

OUTLINE

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- [ ] Liquid Resource Types
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Inductive Potentials

```
data QList a where
  QNil :: QList a
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$$\begin{aligned}\Phi(L) &= \sum_i p + \sum_i \sum_{j>i} 1 \\ &= np + \sum_i (n-i) \\ &= \frac{n(n+2p-1)}{2}\end{aligned}$$

p is the potential of type T

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$$= \boxed{\frac{n(n + 2p - 1)}{2}}$$

Quadratic

p is the potential of type T

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☑ Can be used to express the bound of insertion sort

Abstract Potentials

```haskell
data QList a where
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```
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```
QList a = List a <λ_.1>
ISList a = List a <λ(x1,x2).ite(x1>x2,1,0)>
```

Constraint-Based Type Checking

```
insert :: (x : a^0) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^0 <λ_.0>

sort :: (xs : List a^1 <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^0 <λ_.0>
```

```
sort = λxs.
  match xs with
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[hd: a^p1, tl: List a^q1(hd,v) <q1>]
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    Cons hd tl ->
      insert hd
        (tick 1
          (sort tl))
```

```
[xs: List a^1 <λ(x1,x2).ite(x1>x2,1,0)>]
[]
[hd: a^1, tl: List a^{1+ite(hd>v,1,0)} <λ(x1,x2).ite(x1>x2,1,0)>]
[hd: a^{p1}, tl: List a^{q1(hd,v)} <q1>]
[hd: a^{p2}, tl: List a^{q2(hd,v)} <q2>]
```

Constraint-Based Type Checking

```
insert :: (x : a^{0}) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^{0} <λ_.0>

sort :: (xs : List a^{1} <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^{0} <λ_.0>
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sort = λxs.
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[hd: a^{p1}, tl: List a^{q1(hd,v)} <q1>]
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[hd: a^{p2-1}, tl: List a^{q2(hd,v)} <q2>]
```

CONSTRAINT-BASED TYPE CHECKING

```
insert :: (x : a^0) -> (xs : List a^ite(x>v,1,0) <λ_.0>) -> List a^0 <λ_.0>

sort :: (xs : List a^1 <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^0 <λ_.0>
```

```
sort = λxs.
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    Nil -> Nil
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```
[xs: List a^1 <λ(x1,x2).ite(x1>x2,1,0)>]
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[hd: a^1, tl: List a^(1+ite(hd>v,1,0)) <λ(x1,x2).ite(x1>x2,1,0)>]
[hd: a^p1, tl: List a^q1(hd,v) <q1>]
[hd: a^p2, tl: List a^q2(hd,v) <q2>]
[hd: a^(p2-1), tl: List a^q2(hd,v) <q2>]
```

$$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,.$$

CONSTRAINT-BASED TYPE CHECKING

$\texttt{insert :: (x : a}^{0}\texttt{) -> (xs : List a}^{\texttt{ite(x>v,1,0)}}\ \texttt{<}\lambda\texttt{\_.0>) -> List a}^{0}\ \texttt{<}\lambda\texttt{\_.0>}$

$\texttt{sort :: (xs : List a}^{1}\ \texttt{<}\lambda\texttt{(x1,x2).ite(x1>x2,1,0)>) -> List a}^{0}\ \texttt{<}\lambda\texttt{\_.0>}$

```
sort = λxs.
 match xs with
   Nil -> Nil
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```

$\texttt{[xs: List a}^{1}\ \texttt{<}\lambda\texttt{(x1,x2).ite(x1>x2,1,0)>]}$

$\texttt{[]}$

$\texttt{[hd: a}^{1}\texttt{, tl: List a}^{\texttt{1+ite(hd>v,1,0)}}\ \texttt{<}\lambda\texttt{(x1,x2).ite(x1>x2,1,0)>]}$

$\texttt{[hd: a}^{\texttt{p1}}\texttt{, tl: List a}^{\texttt{q1(hd,v)}}\ \texttt{<q1>]}$

$\texttt{[hd: a}^{\texttt{p2}}\texttt{, tl: List a}^{\texttt{q2(hd,v)}}\ \texttt{<q2>]}$

$\texttt{[hd: a}^{\texttt{p2-1}}\texttt{, tl: List a}^{\texttt{q2(hd,v)}}\ \texttt{<q2>]}$

$$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,. \quad p_1 + p_2 = 1 \land q_1(hd, v) + q_2(hd, v) = 1 + \textbf{ite}(hd > v, 1, 0)$$

Constraint-Based Type Checking

```
insert :: (x : a^0) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^0 <λ_.0>

sort :: (xs : List a^1 <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^0 <λ_.0>
```

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sort = λxs.
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[hd: a^{p1}, tl: List a^{q1(hd,v)} <q1>]
[hd: a^{p2}, tl: List a^{q2(hd,v)} <q2>]
[hd: a^{p2-1}, tl: List a^{q2(hd,v)} <q2>]
```

$$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,.$$

CONSTRAINT-BASED TYPE CHECKING

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insert :: (x : a^0) -> (xs : List a^ite(x>v,1,0) <λ_.0>) -> List a^0 <λ_.0>

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```

$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,.$ $\qquad p_2 - 1 \geq 0$

Constraint-Based Type Checking

```
insert :: (x : a^{0}) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^{0} <λ_.0>

sort :: (xs : List a^{1} <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^{0} <λ_.0>
```

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sort = λxs.
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Constraint-Based Type Checking

```
insert :: (x : a^0) -> (xs : List a^ite(x>v,1,0) <λ_.0>) -> List a^0 <λ_.0>
sort :: (xs : List a^1+s <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^0+s <λ_.0>
```

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sort = λxs.
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```
[xs: List a^1 <λ(x1,x2).ite(x1>x2,1,0)>]
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[hd: a^p2, tl: List a^q2(hd,v) <q2>]
[hd: a^p2-1, tl: List a^q2(hd,v) <q2>]
```

$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,.$ $\quad q_2(hd, v) \geq 1 + s(hd, v)$

Constraint-Based Type Checking

```
insert :: (x : a^{0}) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^{0} <λ_.0>
sort :: (xs : List a^{1+s} <λ(x1,x2).ite(x1>x2,1,0)>) -> List a^{0+s} <λ_.0>
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    Nil -> Nil
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insert :: (x : a^0) -> (xs : List a^ite(x>v,1,0) <λ_.0>) -> List a^0 <λ_.0>
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```

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[hd: a^p2, tl: List a^q2(hd,v) <q2>]
[hd: a^p2-1, tl: List a^q2(hd,v) <q2>]
```

$\exists p_1, p_2, q_1, q_2, s \,.\, \forall hd, v \,.$ $\quad 0 + s(hd, v) \geq \mathbf{ite}(hd > v, 1, 0)$

Constraint-Based Type Checking

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insert :: (x : a^0) -> (xs : List a^{ite(x>v,1,0)} <λ_.0>) -> List a^0 <λ_.0>
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[hd: a^{p2-1}, tl: List a^{q2(hd,v)} <q2>]
```

Second-Order Conditional Linear Arithmetic Constraints

Type Soundness

Type Soundness

If a closed program ***E*** of type ***T*** is well-typed with ***Q*** units of initial potential, then the evaluation of ***E*** with ***Q*** units of initial resource will not get stuck, and when ***E*** evaluates to a value ***V***, ***V*** will satisfy the constraints specified by ***T***.

OUTLINE

- ☑ Motivation
- ☑ Background: Liquid Types and the Potential Method
- ☑ Liquid Resource Types
- ☐ Evaluation

Reusable Datatypes

Datatype	Type	Potential

Reusable Datatypes

Datatype	Type	Potential

List	data List t <q::t->t->Nat> where
	  Nil :: List t <q>
	  Cons :: (x : t) -> List t$^{q(x,v)}$ <q> -> List t <q>
		Quadratic
		$\sum_{i<j} q(v_i, v_j)$

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		Quadratic
		$\sum_{i<j} q(v_i, v_j)$

List	`data EList t <q::Nat> where`
	`  Nil :: EList t <q>`
	`  Cons :: (x : t^q) -> EList t <2*q> -> EList t <q>`
		Exponential
		$q \cdot (2^n - 1)$

REUSABLE DATATYPES

Datatype	Type	Potential

List	data List t <q::t->t->Nat> where
	  Nil :: List t <q>
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List	data EList t <q::Nat> where
	  Nil :: EList t <q>
	  Cons :: (x : t^{q}) -> EList t <2*q> -> EList t <q>
		Exponential
		$q \cdot (2^n - 1)$

Binary tree	data LTree t <q::Nat> where
	  Leaf :: (x : t) -> LTree t <q>
	  Node :: LTree t^{q} <q> -> LTree t^{q} <q> -> Ltree t <q>
		Size * Height
		$\approx q \cdot n \log_2 n$

Reusable Datatypes

Datatype	Type	Potential

List	`data List t <q::t->t->Nat> where`
	`  Nil :: List t <q>`
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		$\sum_{i<j} q(v_i, v_j)$

List	`data EList t <q::Nat> where`
	`  Nil :: EList t <q>`
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		Exponential
		$q \cdot (2^n - 1)$

Binary tree	`data LTree t <q::Nat> where`
	`  Leaf :: (x : t) -> LTree t <q>`
	`  Node :: LTree t^q <q> -> LTree t^q <q> -> Ltree t <q>`
		Size * Height
		$\approx q \cdot n \log_2 n$

Pathed potential tree	`data PTree t <p::t->Bool, q::Nat> where`
	`  Leaf :: PTree t <p, q>`
	`  Node :: (x : t^q) -> PTree t <p, ite(p(x),q,0)>`
	`    -> PTree t <p, ite(p(x),0,q)> -> PTree t <p, q>`
		Parameterized by a specific path
		$q \cdot \lvert \ell \rvert$

Benchmark Programs

Kind	Description	Bound on #recursive-calls	Time (in sec.)
Polynomial Quadratic Potential	All Ordered Pairs	n^2+n	0.5
	List Reverse (Slow)	$0.5n^2+1.5n$	0.4
	List Remove Duplicates	$0.5n^2+1.5n$	0.4
	Insertion Sort (Coarse)	$0.5n^2+1.5n$	0.6
	Selection Sort	$1.5n^2+2.5n$	0.5
	Quick Sort	$1.5n^2+1.5n$	1.0
	Merge Sort	n^2+n	0.9
Non-Polynomial Potential	Subset Sum	$2(2^n-1)$	0.3
	Merge Sort Flatten	$(n+1)*h$	0.9
Value-Dependent Potential	Insertion Sort (Fine)	#out-of-order-pairs	5.4
	BST Insert	|insertion path|	2.4
	BST Member	|search path|	6.0

LIQUID RESOURCE TYPES

```
data List t <q::t->t->Nat> where
  Nil :: List t <q>
  Cons :: (x : t) -> List tq(x,v) <q> -> List t <q>
```

LIQUID RESOURCE TYPES

```
data List t <q::t->t->Nat> where
  Nil :: List t <q>
  Cons :: (x : t) -> List t^q(x,v) <q> -> List t <q>
```

Contributions:

- Verification of value-dependent super-linear resource bounds
- Soundness proof of the type system
- Effective prototype implementation

Liquid Resource Types

```
data List t <q::t->t->Nat> where
  Nil :: List t <q>
  Cons :: (x : t) -> List t^q(x,v) <q> -> List t <q>
```

Contributions:

- Verification of value-dependent super-linear resource bounds
- Soundness proof of the type system
- Effective prototype implementation

Limitations:

- Only inductively defined potentials
- Only univariate bounds
- No bound inference