Algebraic Program Analysis of Probabilistic Programs

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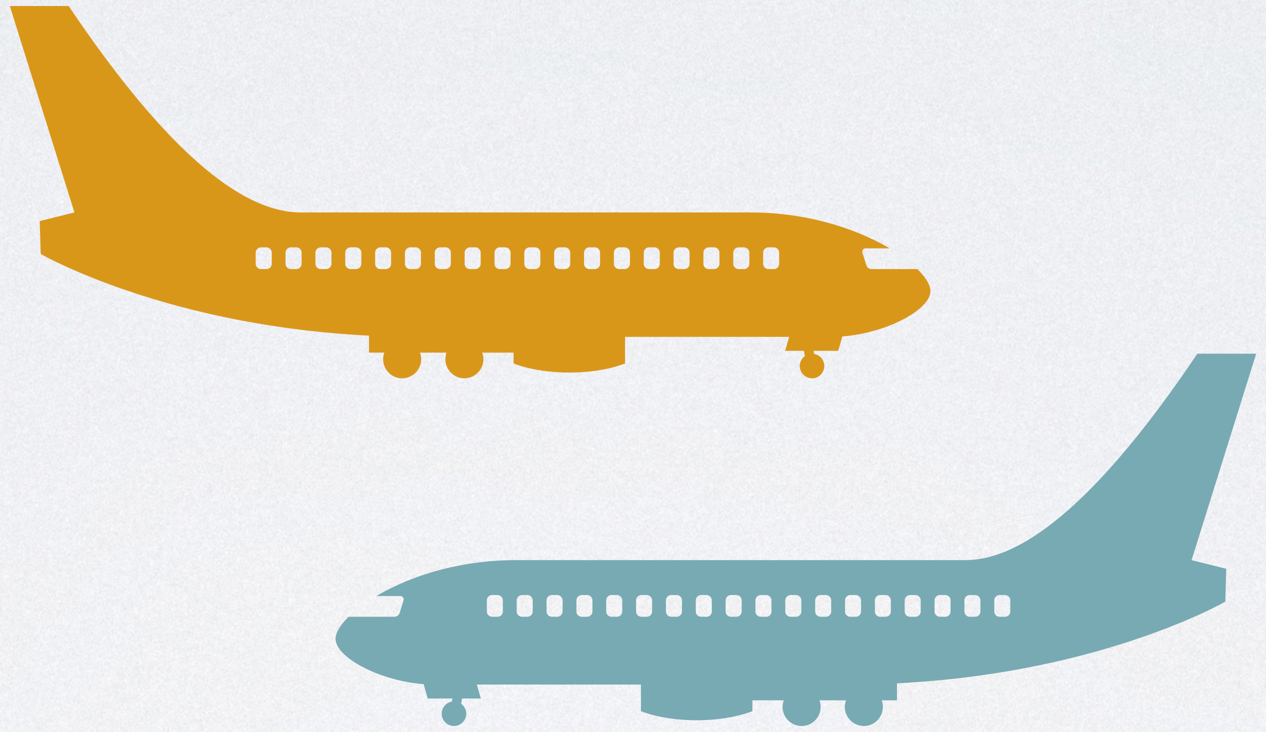
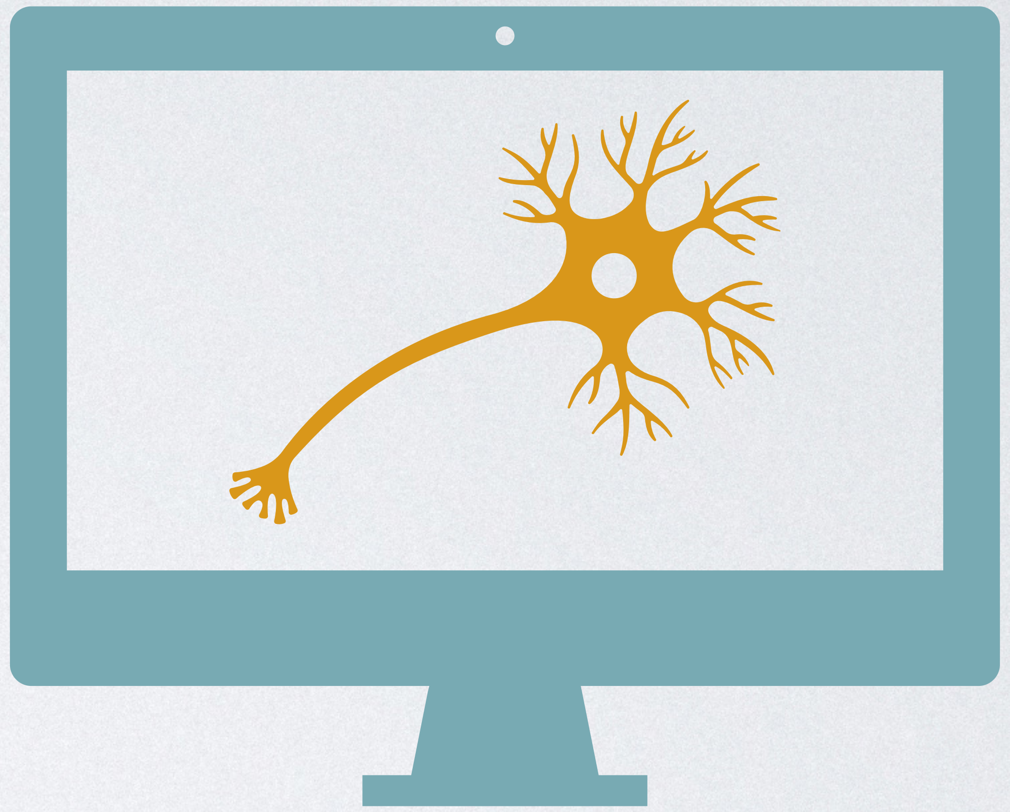
Joint Work with **Jan Hoffmann** and **Thomas Reps**

Probabilistic Systems are Becoming Pervasive

Randomized Algorithms
(improve efficiency)



Cyber-Physical Systems
(model uncertainty)



Artificial Intelligence
(describe statistical models)

Application: Randomized Quicksort

- Improve efficiency
- From $\Theta(n^2)$ to $\Theta(n \log n)$ (expected)
- Samplesort
  - Use >1 random samples as pivots

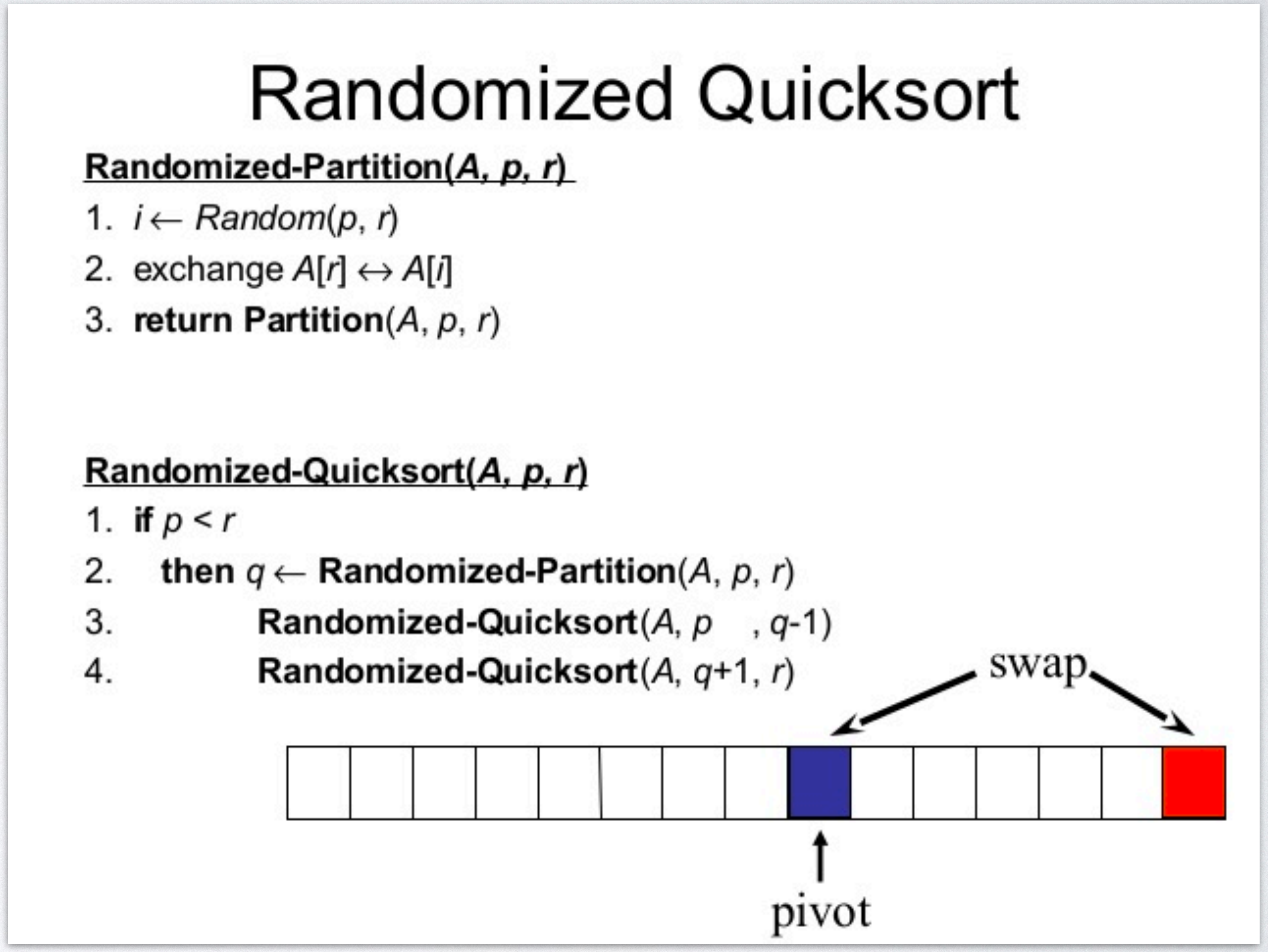


Image source: https://geekfactorial.blogspot.com/2016/08/randomized-quick-sort-algorithm.html.

Application: Airborne Collision Avoidance

- Model uncertainty
- Probabilistic dynamics
- Probabilistic sensors
- High-confidence system

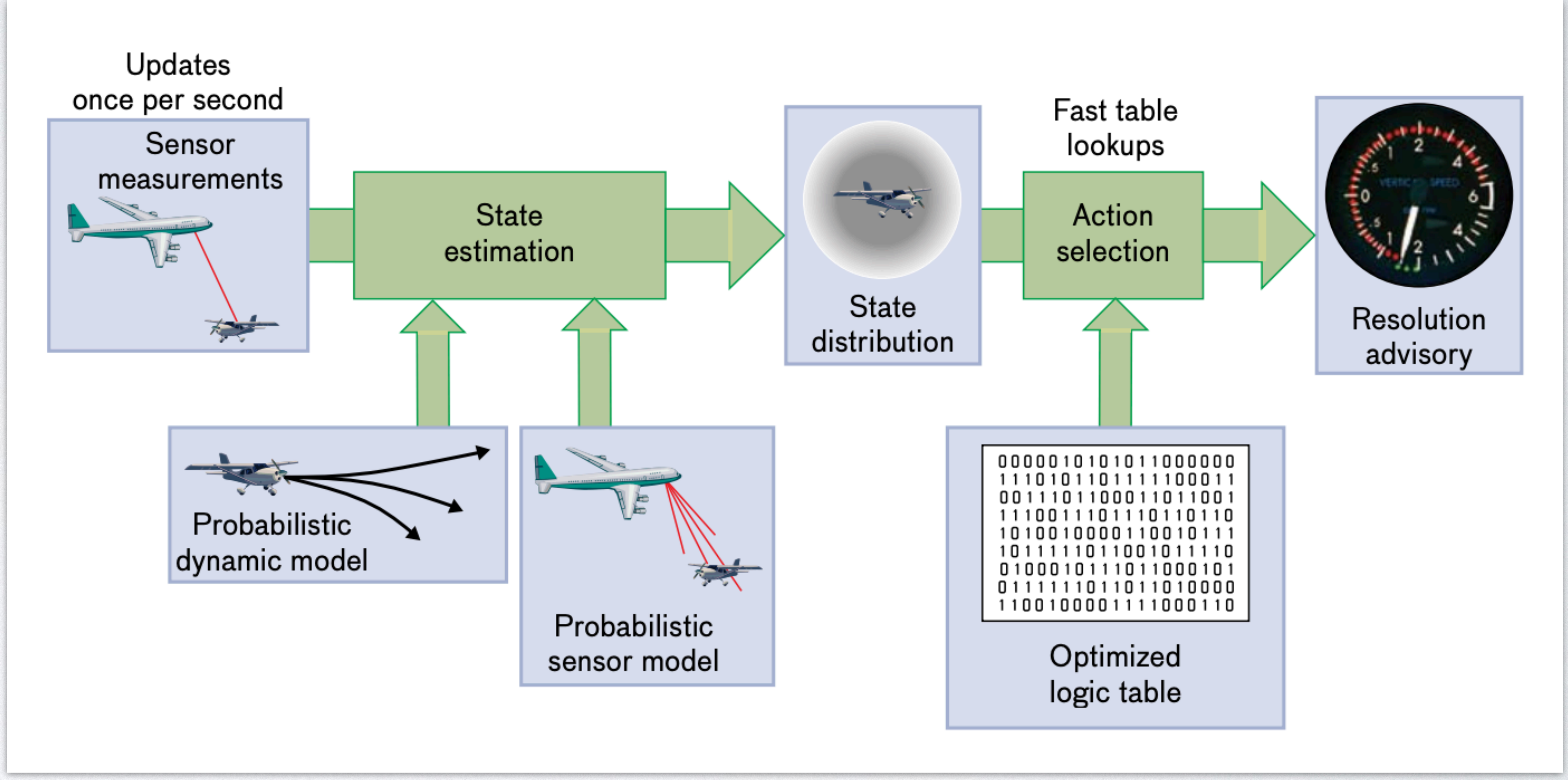
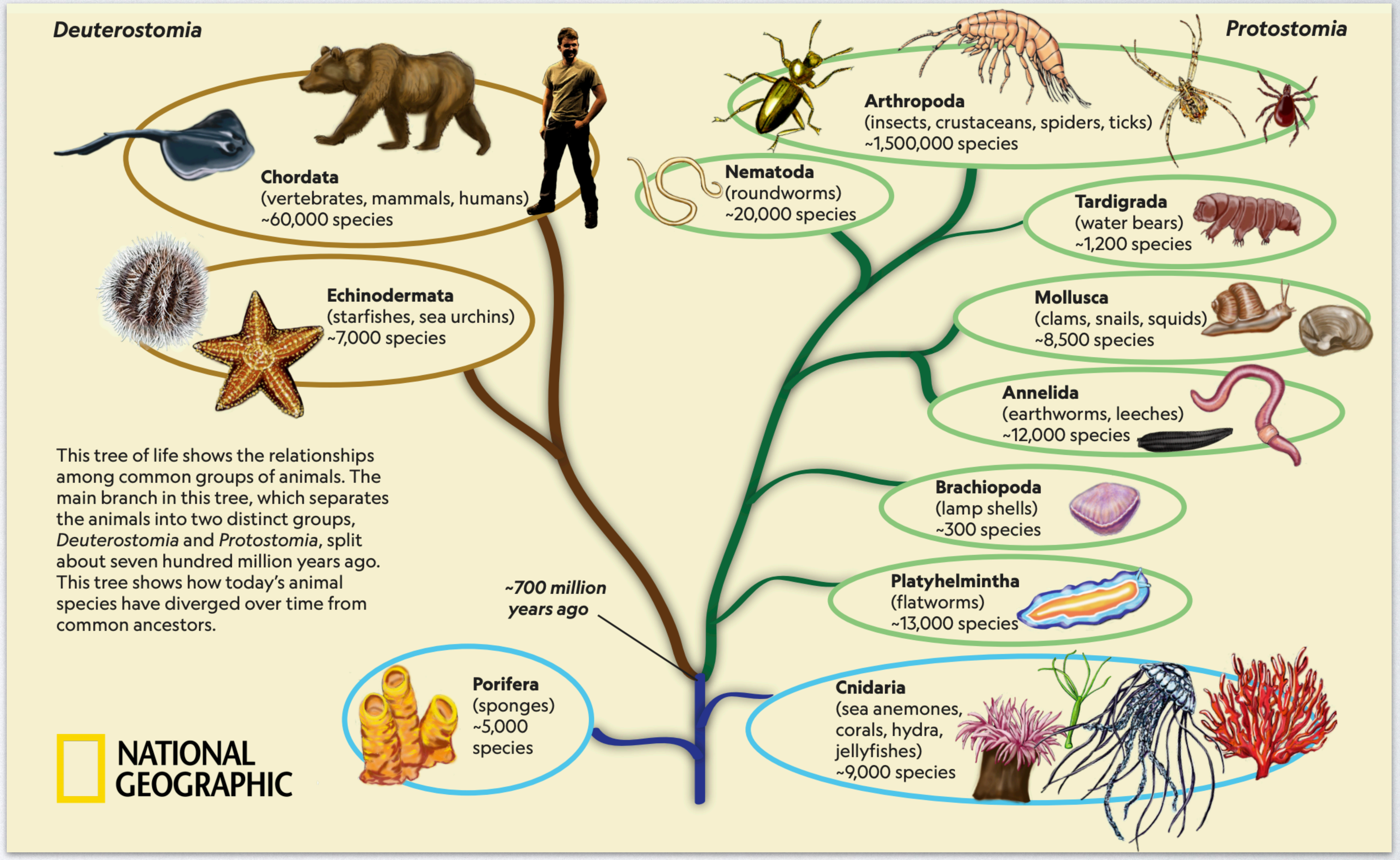


Image source: Kochenderfer, et al. "Next-Generation Airborne Collision Avoidance System." Lincoln Laboratory Journal (2012).

Application: Statistical Phylogenetics

- Describe statistical models
- Automated reasoning
- Apply Bayesian inference to infer evolutionary history
- Solve previously intractable problems



Ronquist, et al. "Universal probabilistic programming offers a powerful approach to statistical phylogenetics." Communications Biology (2021).
Image source: https://www.nationalgeographic.org/media/tree-life/.

Probabilistic Programs

Draw random **data** from distributions

Change **control-flow** at random

Image sources: https://www.libertystorch.info/2022/02/21/the-grab-bag/; https://www.stockvault.net/photo/179872/at-a-crossroads-decisions-and-choices-concept-with-large-arrow-signs.

Probabilistic Programs

- True randomness
- A distribution on execution paths
- Probabilistic nondeterminism

```
if
| prob(1/3) → choice ≔ 1
| prob(1/3) → choice ≔ 2
| prob(1/3) → choice ≔ 3
fi
```

Probabilistic Programs

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```
choice :∈p (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3)
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Demonic Programs

- Dijkstra's **Guarded Command Language** (GCL)
- A set of execution paths
- Demonic nondeterminism

```
if
| true → prize ≔ 1
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Demonic Programs

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$\boxed{\texttt{prize} :\in_d \{1,2,3\}}$

Example: Monty Hall

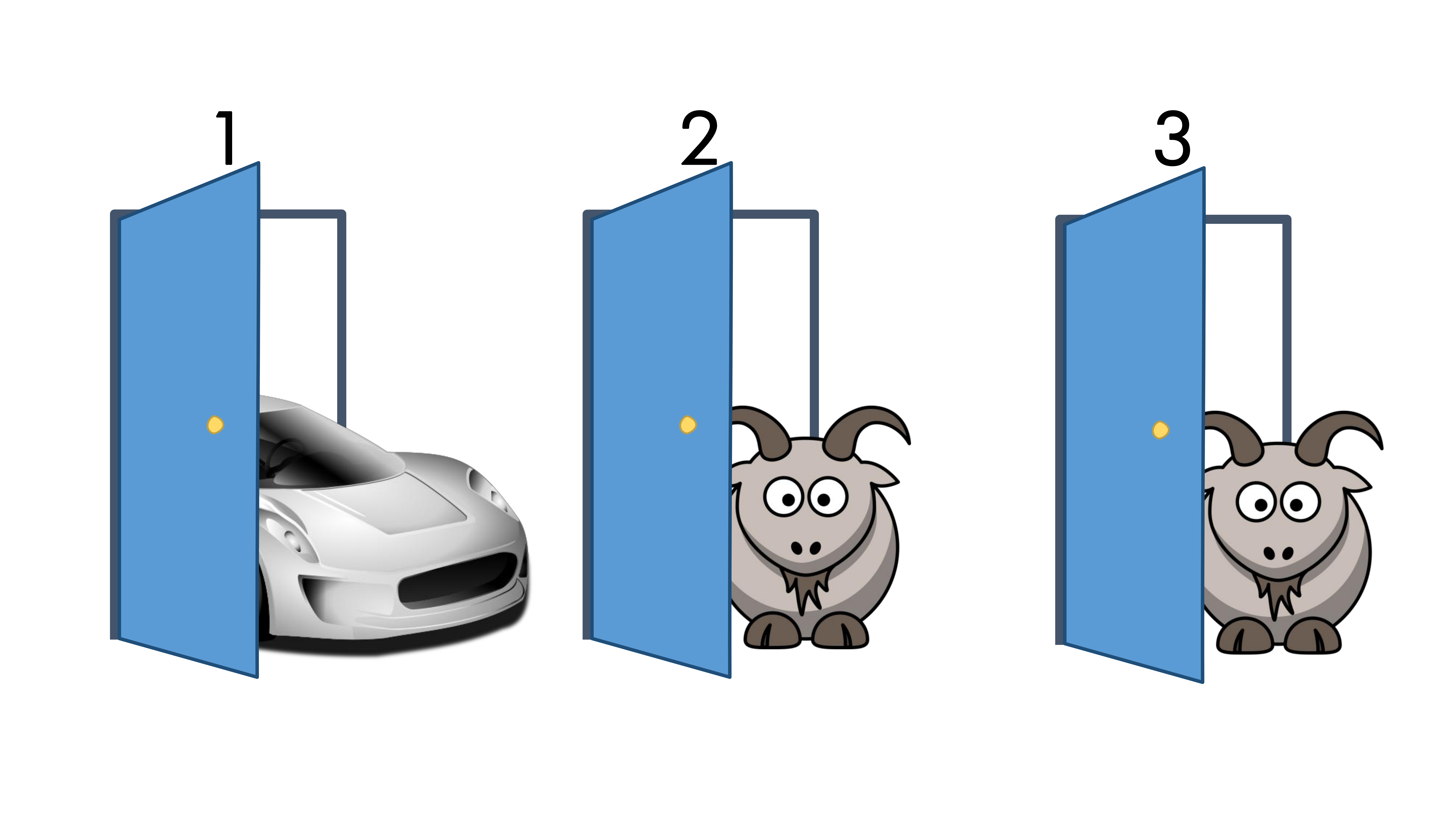

Image source: Maria Gorinova's Advances in Programming Languages (Guest Lecture) slides on Probabilistic Programming.

Example: Monty Hall

Example: Monty Hall

- McIver and Morgan's **probabilistic Guarded Command Language** (pGCL)
- Combine two forms of nondeterminism:
  - Probabilistic
  - Demonic

```
prize :∈d {1,2,3};
choice :∈p (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3);
host :∈d {1,2,3} \ {prize,choice};
if switch then
  choice :∈d {1,2,3} \ {choice,host}
fi
```

Example: Monty Hall

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$\mathbb{P}(choice = prize) = ?$

Example: Monty Hall

- McIver and Morgan's **probabilistic Guarded Command Language** (pGCL)
- Combine two forms of nondeterminism:
    - Probabilistic
    - Demonic
- "Demons" minimize the probability

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Example: Failure Modeling

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- An example of **probabilistic modeling checking**
- Send c messages, each with a failure probability `0.1`

```
fail := FALSE;
c :∈d {0,1,2};
while not(fail) and c > 0 do
  fail :∈p (TRUE @ 0.1 | FALSE @ 0.9 );
  c := c - 1
od
```

Example: Failure Modeling

- An example of **probabilistic modeling checking**
- Send c messages, each with a failure probability `0.1`
- What is the probability of success?

```
fail := FALSE;
c :∈d {0,1,2};
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$\mathbb{P}(\mathit{fail} = \mathit{FALSE}) = ?$

Example: Abstraction

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- Program analysis introduces **abstraction**
- **Predicate Abstraction**
  - [c=0] is a Boolean variable

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Example: Abstraction

- Program analysis introduces **abstraction**
- **Predicate Abstraction**
  - [c=0] is a Boolean variable
- Abstraction nondeterminism
  - Maximize —> Upper bound
  - Minimize —> Lower bound

```
fail := FALSE;
[c=0] :∈d {TRUE,FALSE};
while not(fail) and not([c=0]) do
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od;
```

$\mathbb{P}(fail = FALSE) = ?$

How to automate such $\boxed{\text{quantitative}}$ reasoning about probabilistic programs?

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Examples

What is the probability that an assertion holds?

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What is the expected value of an expression?

How to automate such $\boxed{\text{quantitative}}$ reasoning about probabilistic programs?

Examples

What is the probability that an assertion holds?

What is the expected value of an expression?

What is the expected time complexity of a program?

Challenge I:
How to support multiple confluence operations?

… $:\in_p$ …

… $:\in_d$ …

… $:\in_a$ …

Semantic Algebras

- **Kleene Algebras**: A **compositional** and **flexible** framework for program semantics

Program Construct	Algebraic Representation
A program S	An interpretation $\mathbb{S}$ of S into the algebra
Branching between A and B	$\mathbb{A} \oplus \mathbb{B}$
Sequencing of A and B	$\mathbb{A} \otimes \mathbb{B}$
Iteration (i.e., loop) of A	$\mathbb{A}^*$
"`abort`", "`skip`"	$\underline{0}, \underline{1}$

Do Kleene Algebras Suffice?

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```
if
| true → x ≔ 1
| true → x ≔ 2
| true → x ≔ 3
fi
```

Do Kleene Algebras Suffice?

```
if
| true → x := 1
| true → x := 2
| true → x := 3
fi
```

$$([\mathbf{true}] \otimes x \coloneqq 1)$$
$$\oplus\ ([\mathbf{true}] \otimes x \coloneqq 2)$$
$$\oplus\ ([\mathbf{true}] \otimes x \coloneqq 3)$$

Do Kleene Algebras Suffice?

```
if
| true → x ≔ 1
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fi
```

$$
\begin{aligned}
&([\mathbf{true}] \otimes x \coloneqq 1) \\
\oplus\ &([\mathbf{true}] \otimes x \coloneqq 2) \\
\oplus\ &([\mathbf{true}] \otimes x \coloneqq 3)
\end{aligned}
$$

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$$(([\mathbf{prob}(1/2)] \otimes \mathrm{x} \coloneqq 1) \oplus ([\mathbf{prob}(1/2)] \otimes \mathrm{x} \coloneqq 2))$$
$$\oplus\ (([\mathbf{prob}(1/2)] \otimes \mathrm{x} \coloneqq 3) \oplus ([\mathbf{prob}(1/2)] \otimes \mathrm{x} \coloneqq 4))$$

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$$((\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 1) \oplus (\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 2))$$
$$\oplus\ ((\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 3) \oplus (\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 4))$$

$$= (\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 1)$$
$$\oplus\ (\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 2)$$
$$\oplus\ (\mathbf{[prob(1/2)]} \otimes \mathrm{x} \coloneqq 3)$$
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$$((\mathbf{[prob(1/2)]} \otimes x := 1) \oplus (\mathbf{[prob(1/2)]} \otimes x := 2))$$
$$\oplus\ ((\mathbf{[prob(1/2)]} \otimes x := 3) \oplus (\mathbf{[prob(1/2)]} \otimes x := 4))$$

$$\begin{aligned} = {} & (\mathbf{[prob(1/2)]} \otimes x := 1) \\ & \oplus (\mathbf{[prob(1/2)]} \otimes x := 2) \\ & \oplus (\mathbf{[prob(1/2)]} \otimes x := 3) \\ & \oplus (\mathbf{[prob(1/2)]} \otimes x := 4) \end{aligned}$$

Probabilities sum up to 2!

Our Approach: Markov Algebras

- Key observation: Probabilistic programs have **multiple confluence operations**

$$\left\langle M, \sqsubseteq, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \right\rangle$$

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Program denotations
form a CPO

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Sequencing, branching, and nondeterministic-choice

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Easy to extend with more confluence operations!

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Program denotations form a CPO

Sequencing, branching, and nondeterministic-choice

$\underline{0}$ interprets **`abort`**
$\underline{1}$ interprets **`skip`**

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Program denotations form a CPO

Sequencing, branching, and nondeterministic-choice

Easy to extend with more confluence operations!

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

$$a \otimes \underline{1} = \underline{1} \otimes a = a$$

$$a \,{}_{\phi}\!\oplus\, b = b \,{}_{\overline{\phi}}\!\oplus\, a$$

$$a \sqcap a = a$$

...

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```

$$(x := 1 \,{}_{1/2}\!\oplus\, x := 2) \sqcap (x := 3 \,{}_{1/2}\!\oplus\, x := 4)$$

```
while x>0 do
  x :∈p (x+1 @ 1/2 | x-1 @ 1/2)
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$$\mu S\,.\,((x := x+1\,{}_{1/2}\!\oplus x := x\text{-}1) \otimes S)\,{}_{[x>0]}\!\oplus \mathbf{skip}$$

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Recursive Program Scheme

Flexibility: Different Semantics

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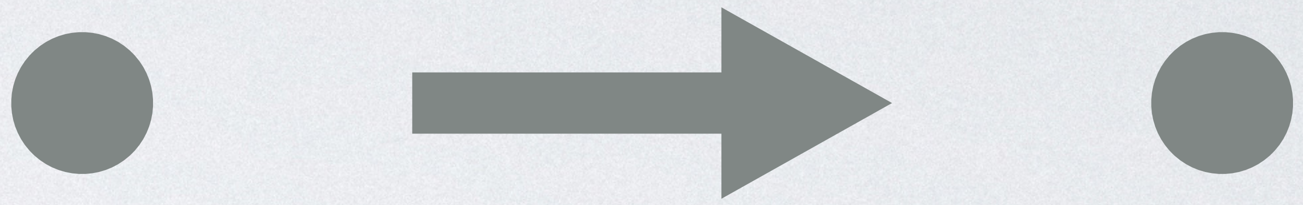
Standard: $State \rightarrow State$



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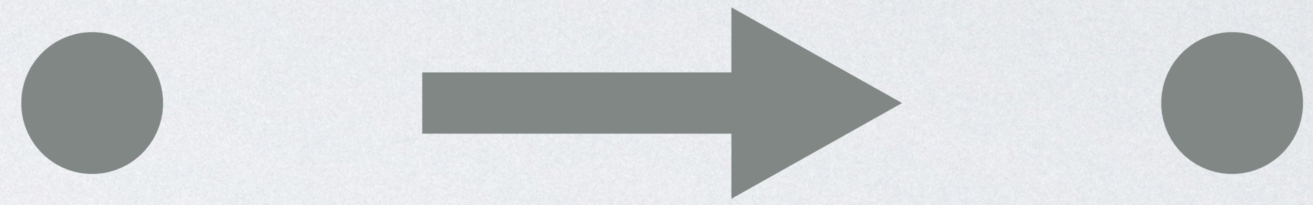


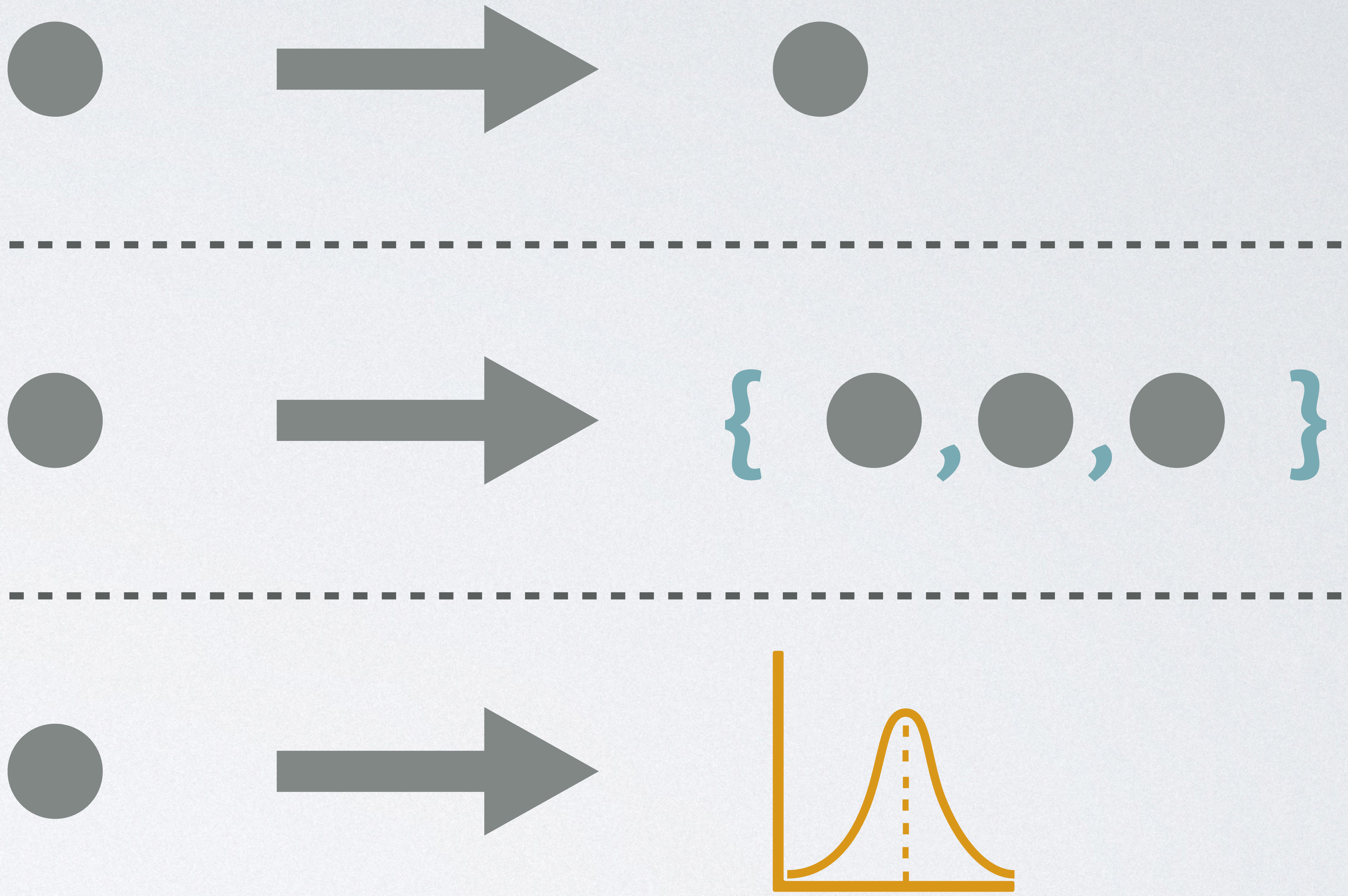
GCL: $State \rightarrow \wp(State)$

Flexibility: Different Semantics

Standard: $State \to State$

GCL: $State \to \wp(State)$

Probabilistic: $State \to \mathbb{D}(State)$



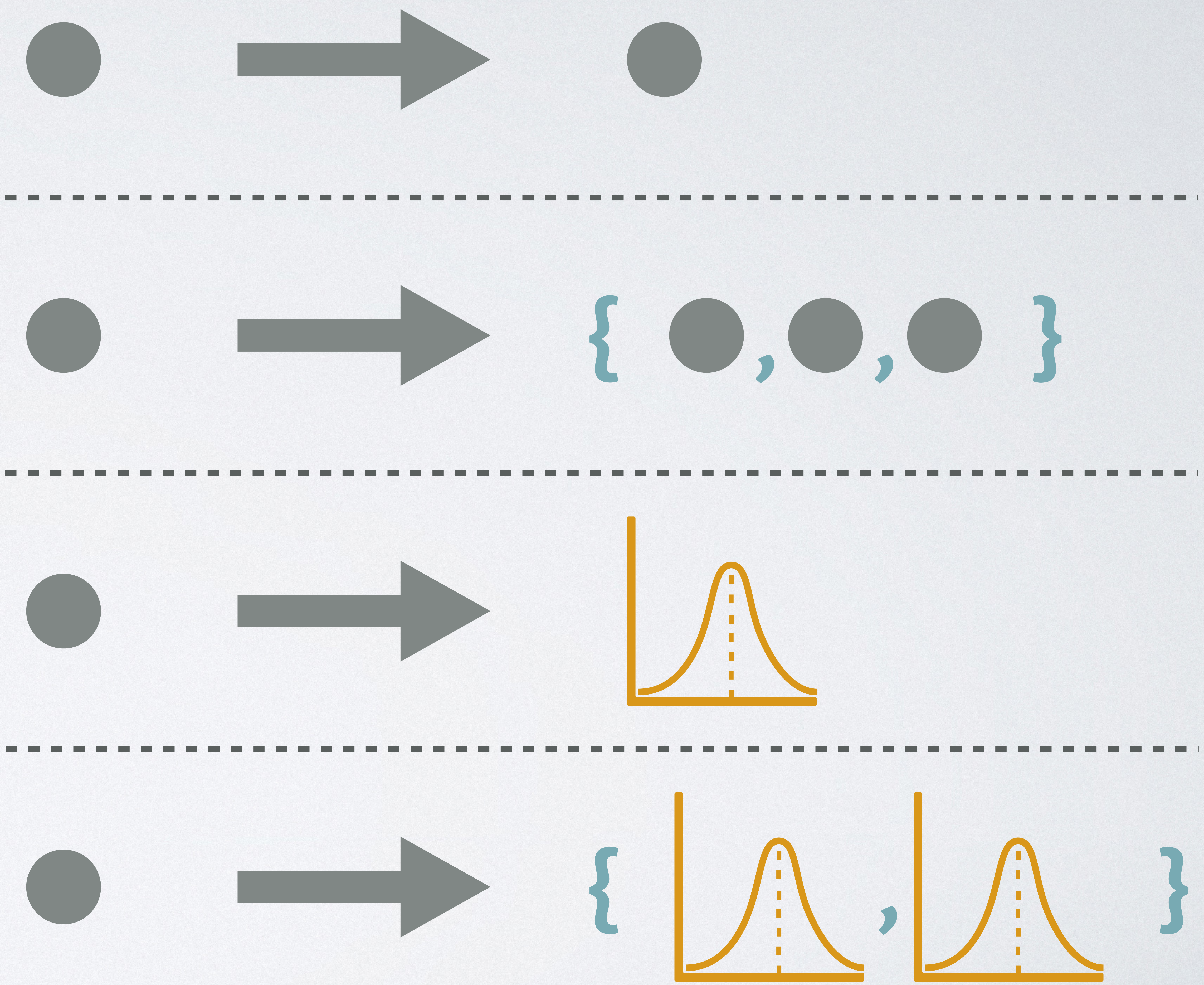
Flexibility: Different Semantics

Standard: $\mathit{State} \to \mathit{State}$

GCL: $\mathit{State} \to \wp(\mathit{State})$

Probabilistic: $\mathit{State} \to \mathbb{D}(\mathit{State})$

pGCL: $\mathit{State} \to \wp(\mathbb{D}(\mathit{State}))$



Flexibility: Different Models for Nondeterminism

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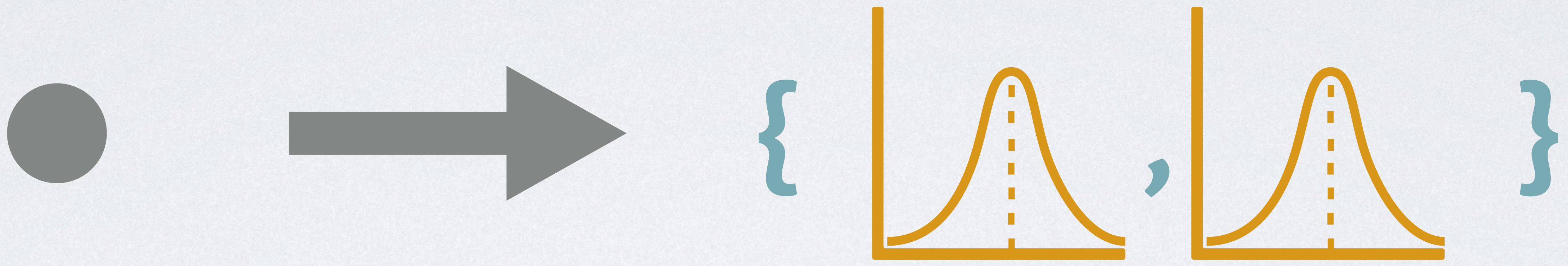
- pGCL: $\mathit{State} \rightarrow \wp(\mathbb{D}(\mathit{State}))$



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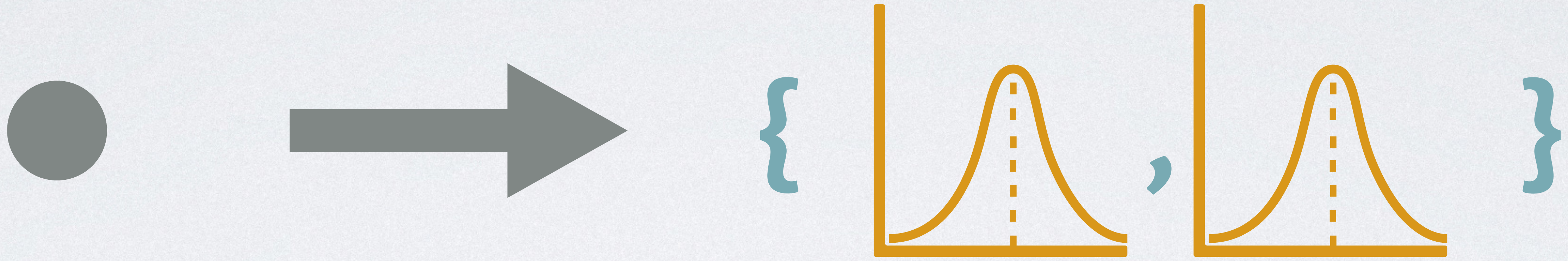
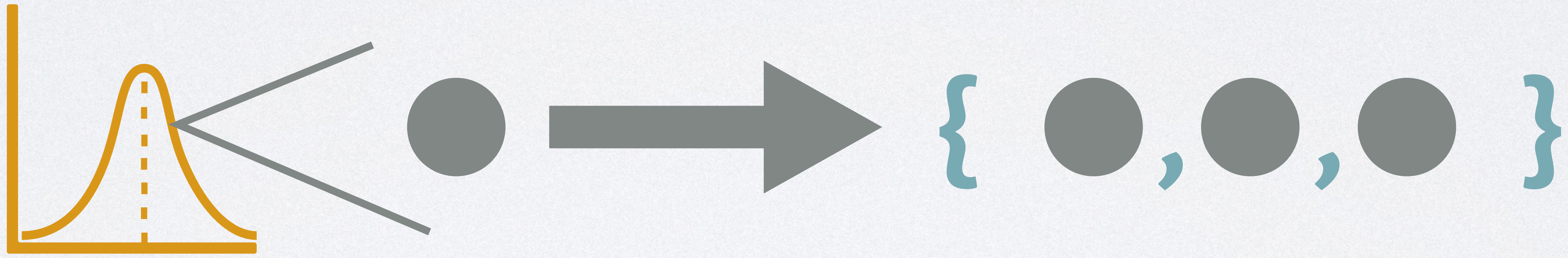


- Cousot's **Probabilistic Abstract Interpretation** (PAI): $\mathbb{D}(\mathit{State} \to \wp(\mathit{State}))$



Flexibility: Different Models for Nondeterminism

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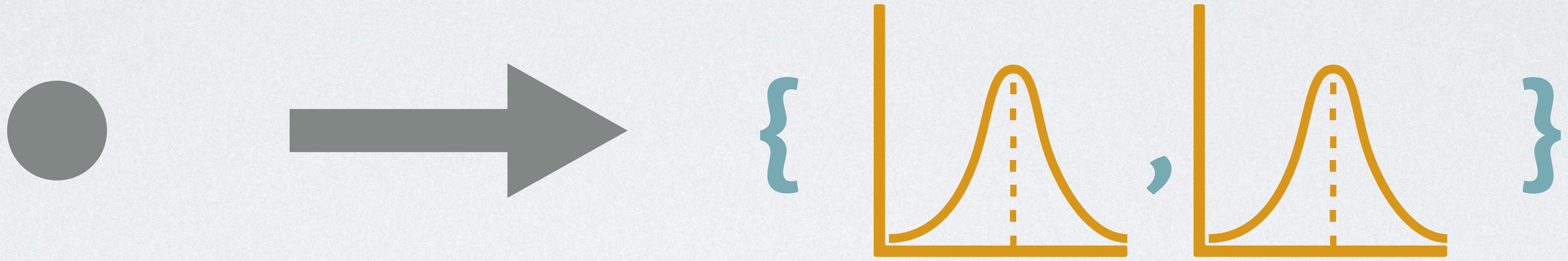

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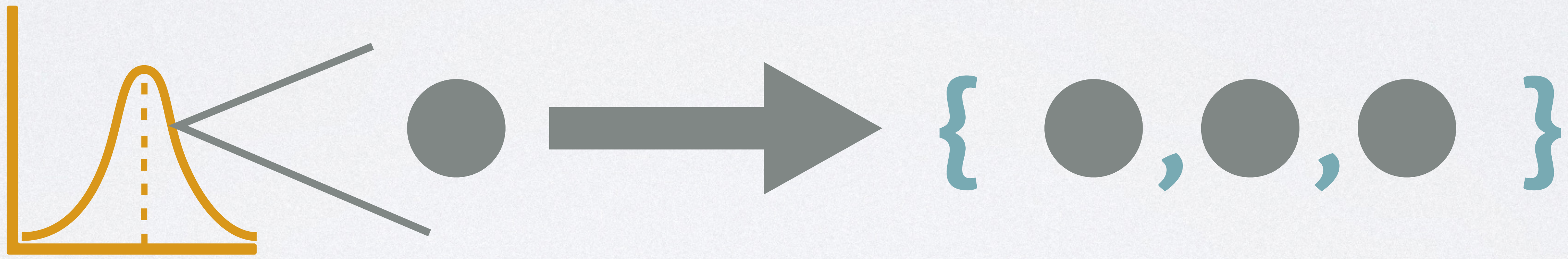
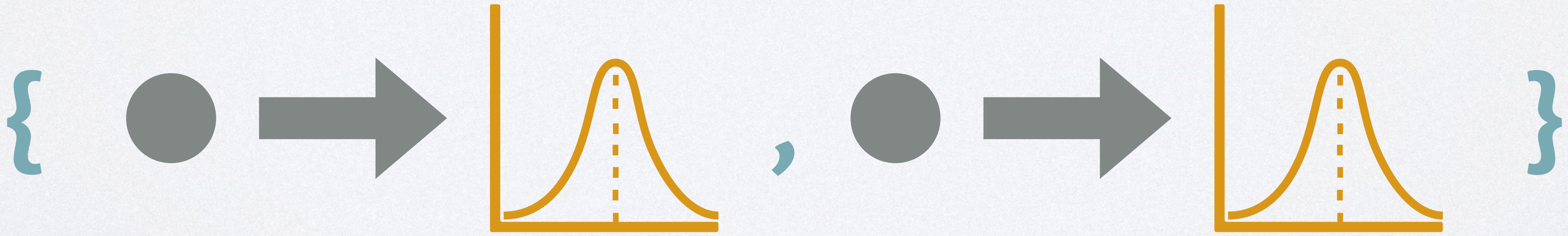


- **Compile-Time/Relational** Nondeterminism: $\wp(\mathit{State} \to \mathbb{D}(\mathit{State}))$



Challenge II:
How to construct recursive program schemes?

```
while x>0 do
  x :∈p (x+1 @ 1/2 | x-1 @ 1/2)
od
```

$$\mu S \, . \, ((x \coloneqq x{+}1 \;{}_{1/2}{\oplus}\; x \coloneqq x{-}1) \otimes S)_{[x>0]} \oplus \textbf{skip}$$

A Control-Flow-Graph's Perspective

- **Kleene Algebras** are compatible with **control-flow graphs** via **regular expressions**

Program Construct	Algebraic Representation
A program S	An interpretation $\mathbb{S}$ of S into the algebra
The **control-flow graph** of S	A **regular expression** over $\underline{0}$, $\underline{1}$, $\oplus$, $\otimes$, and $*$

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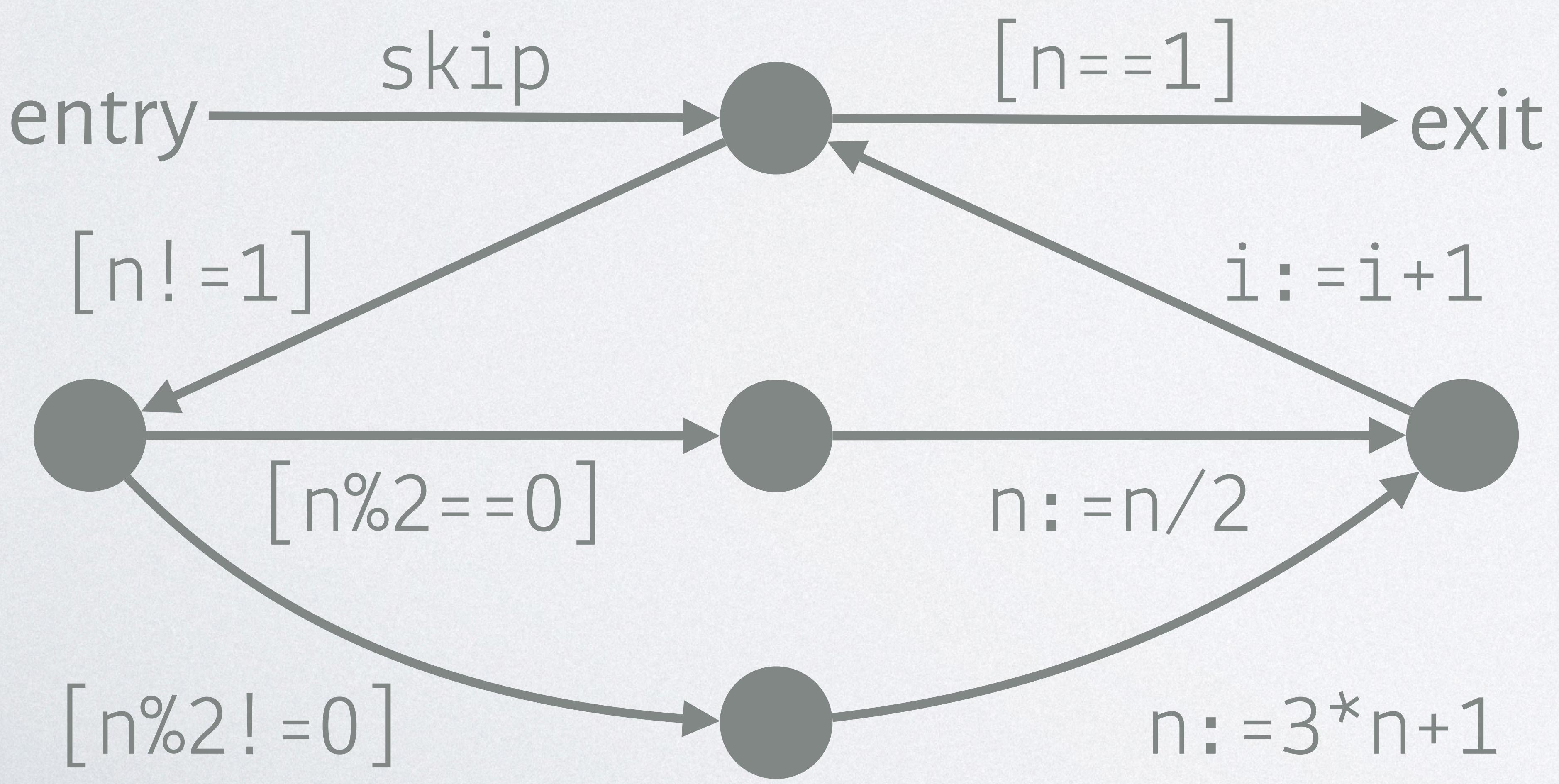

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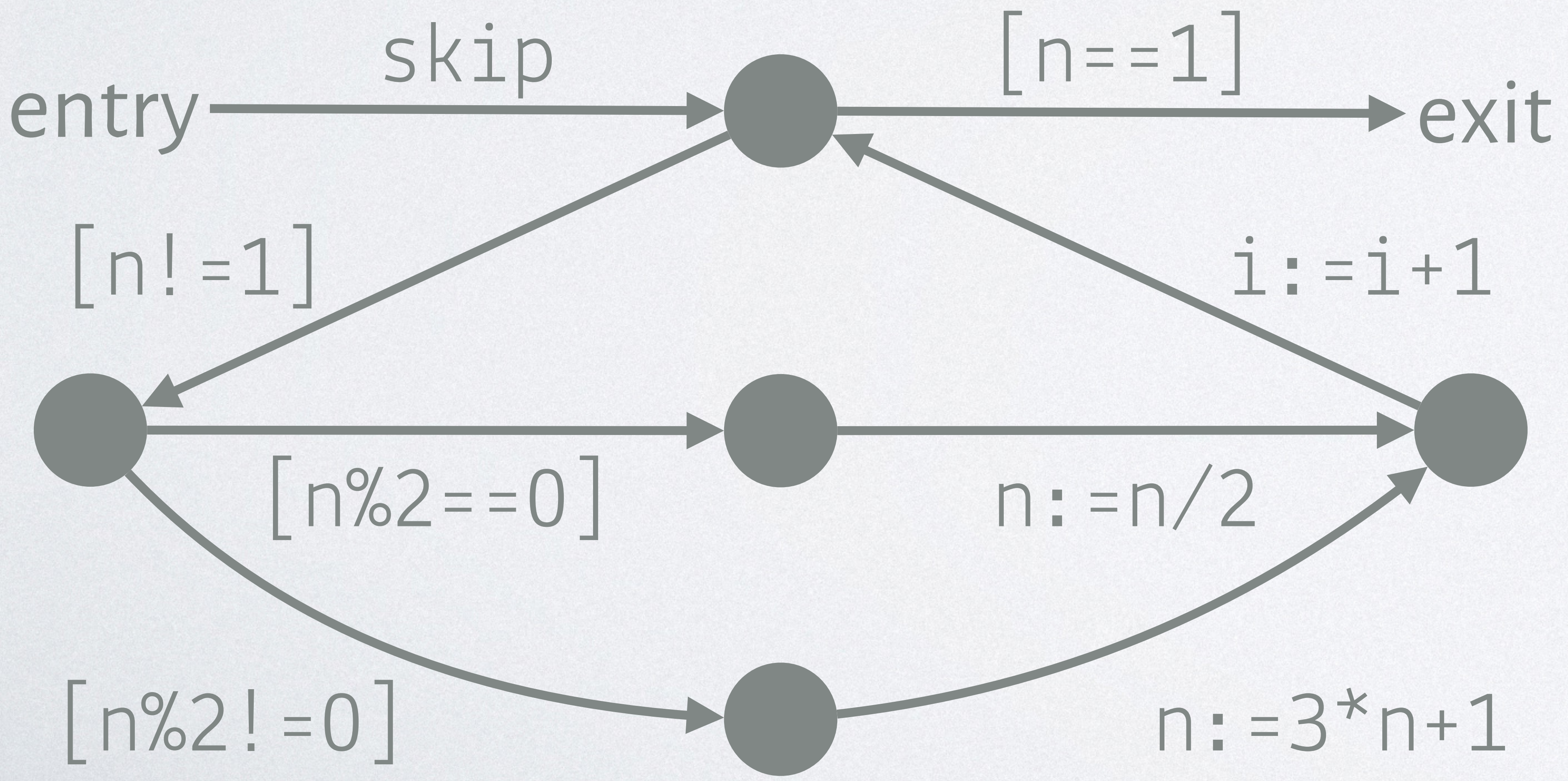

$$\left(\texttt{[n!=1]} \otimes \left(\begin{array}{l}\texttt{[n\%2==0]} \otimes \texttt{n:=n/2} \quad \oplus \\ \texttt{[n\%2!=0]} \otimes \texttt{n:=3*n+1}\end{array}\right) \otimes \texttt{i:=i+1}\right)^{*} \otimes \texttt{[n==1]}$$

Do Control-Flow Graphs Suffice?

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Do Control-Flow Graphs Suffice?

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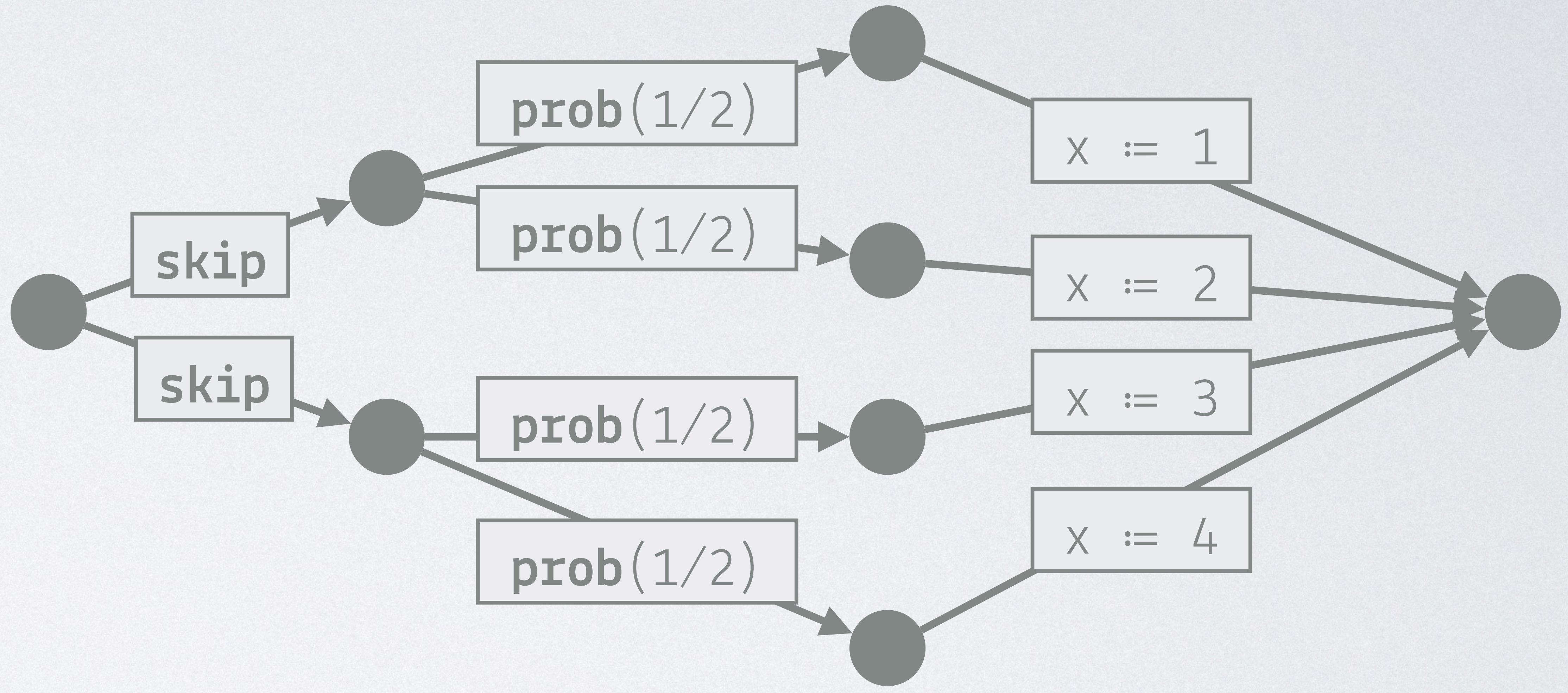

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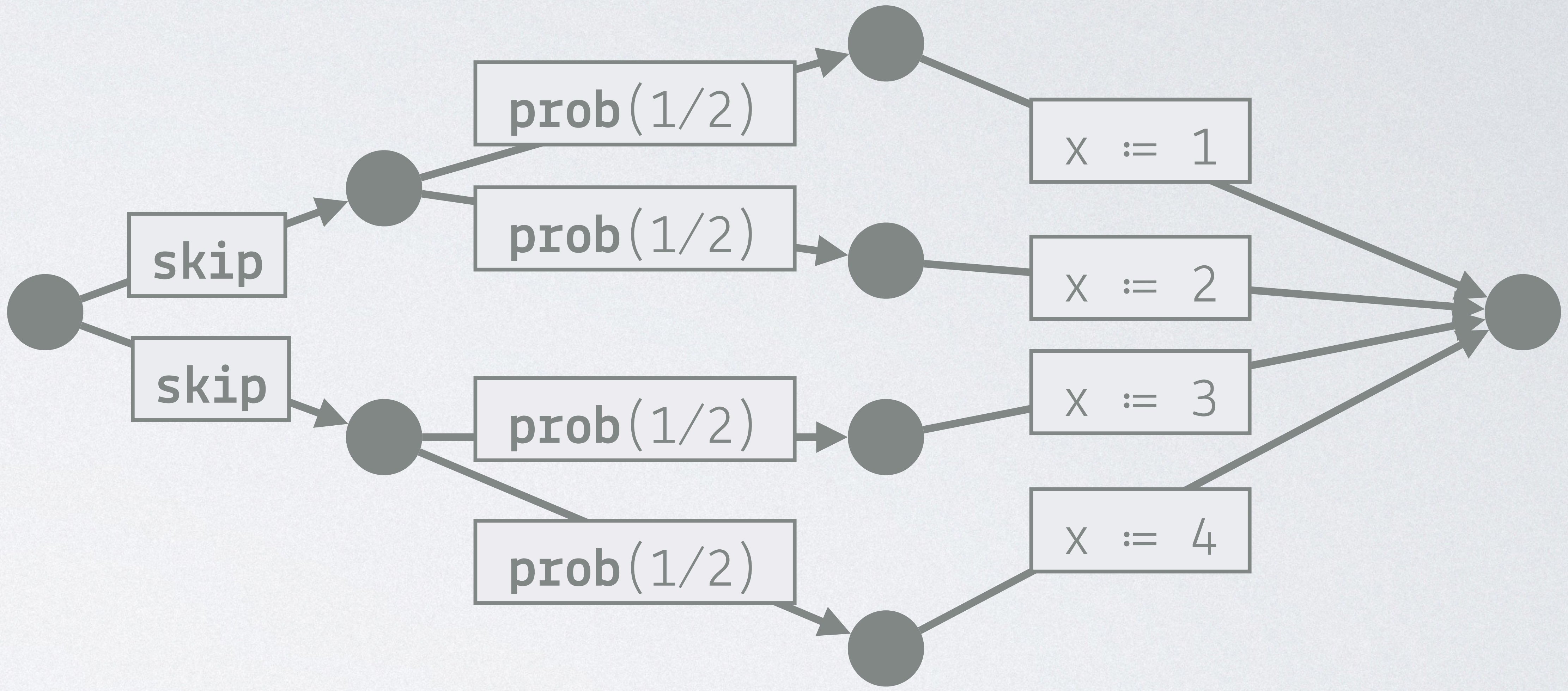

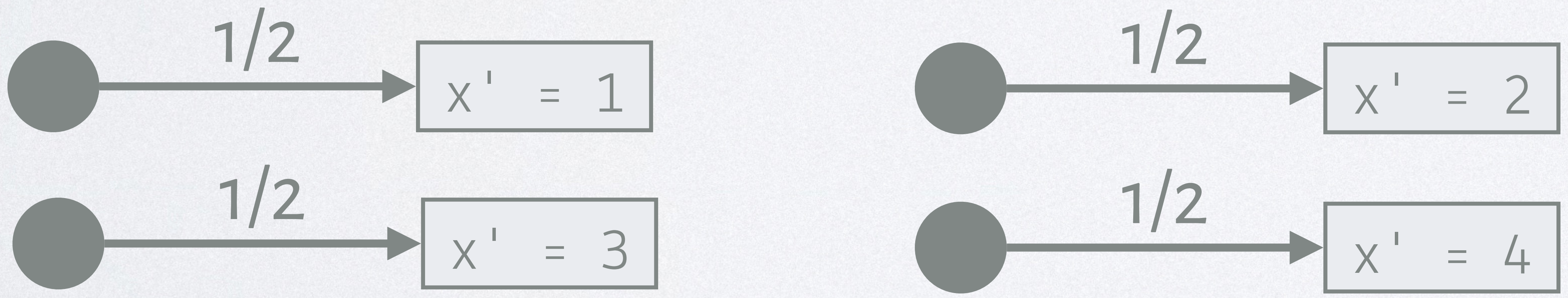

Probabilities sum up to 2!

Our Approach: Control-Flow Hyper-Graphs

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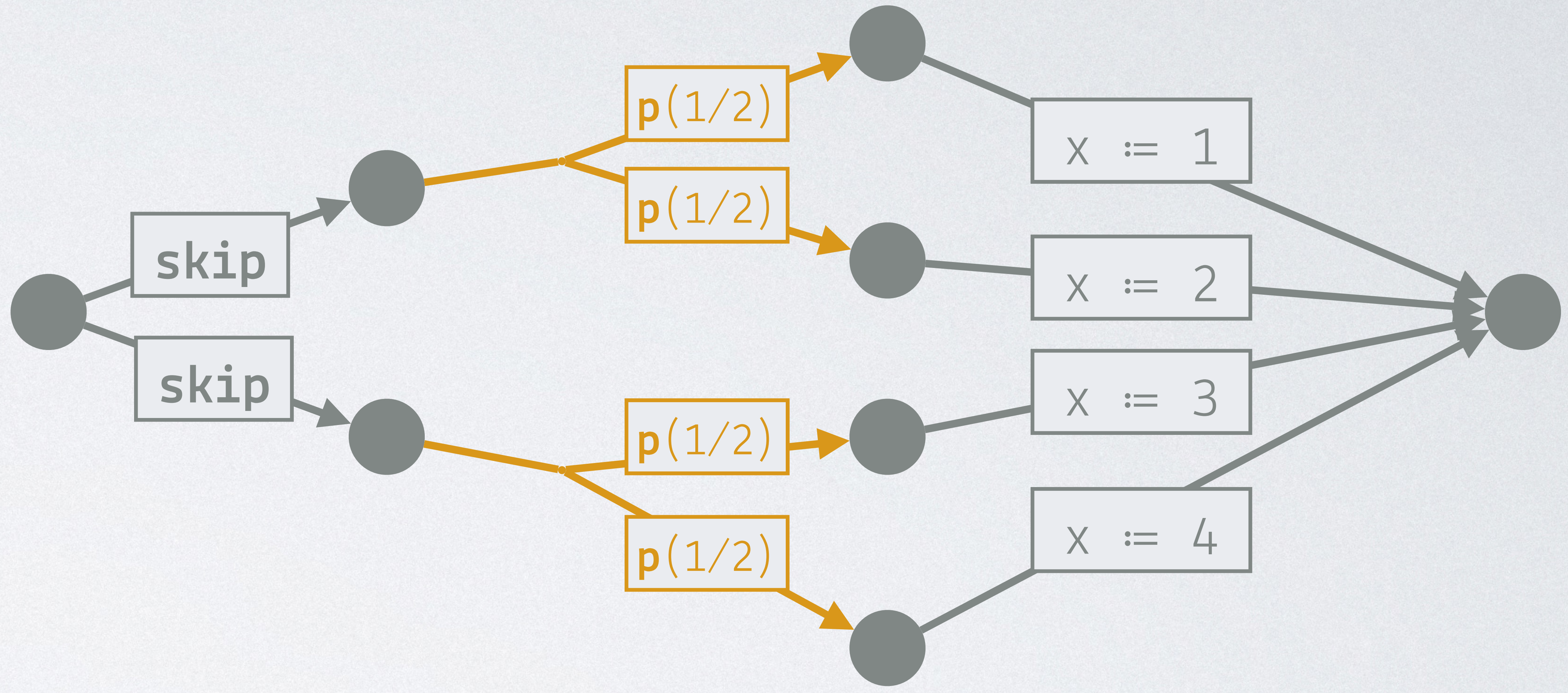

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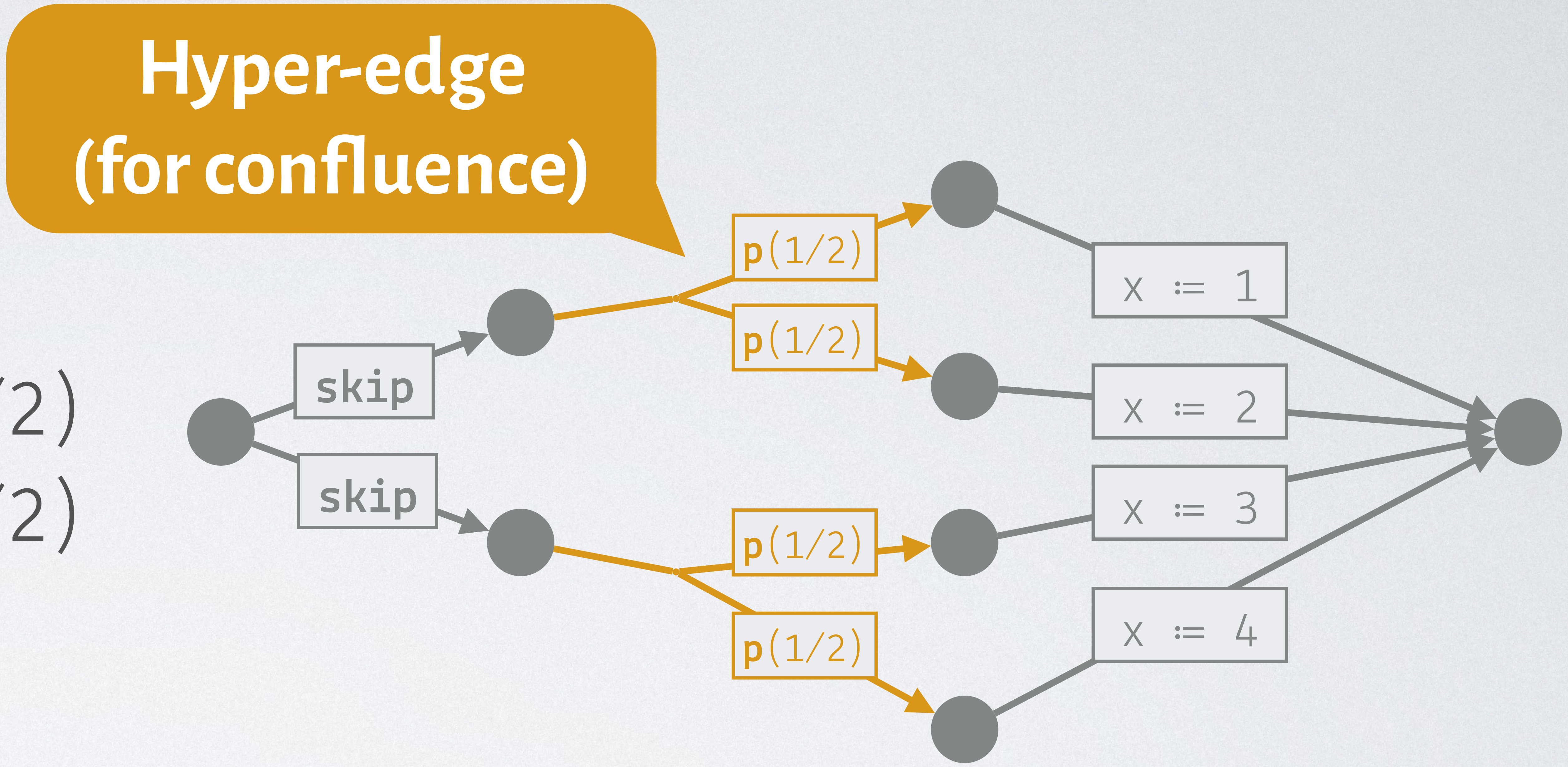

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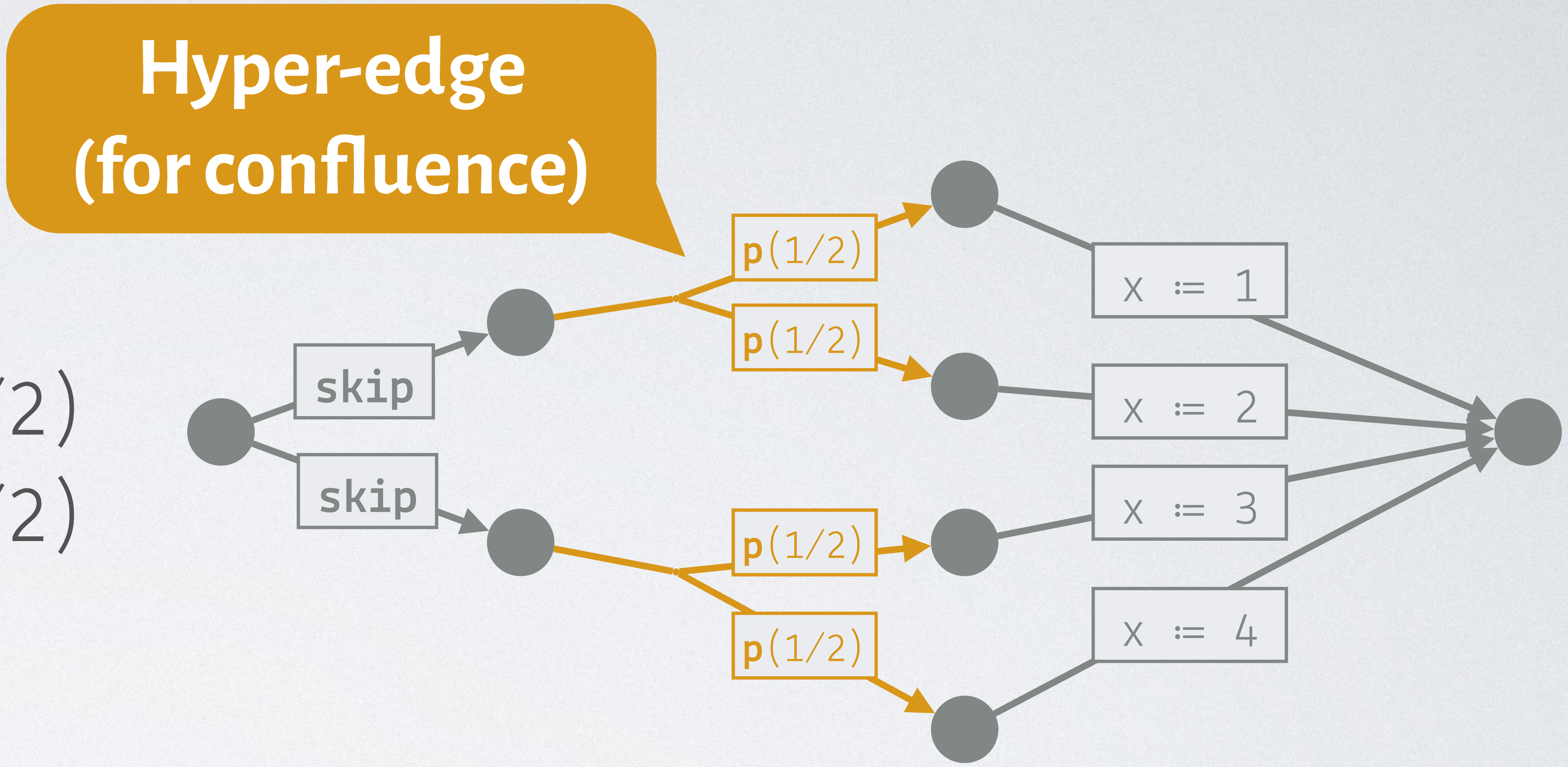

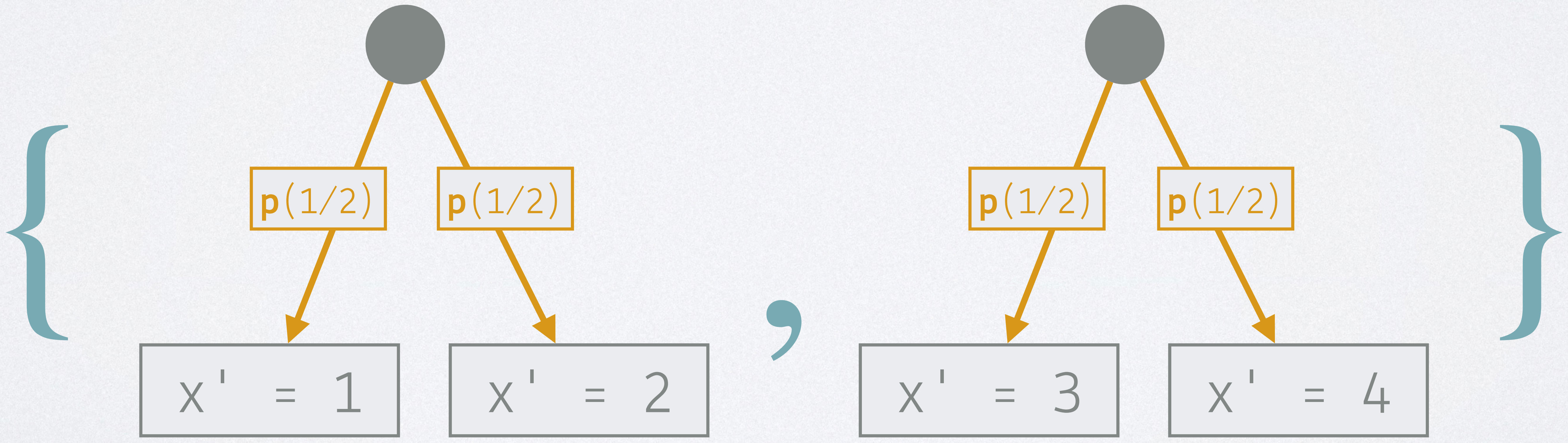

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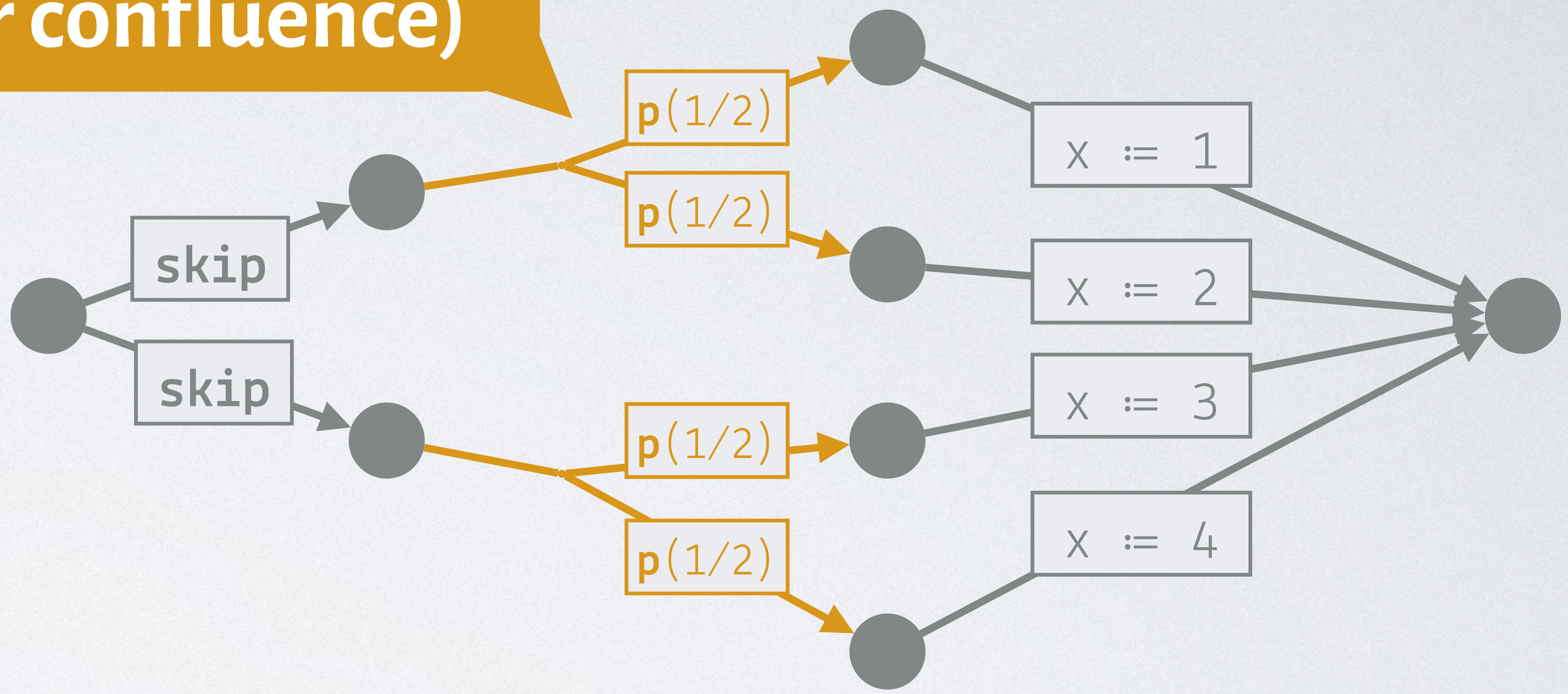
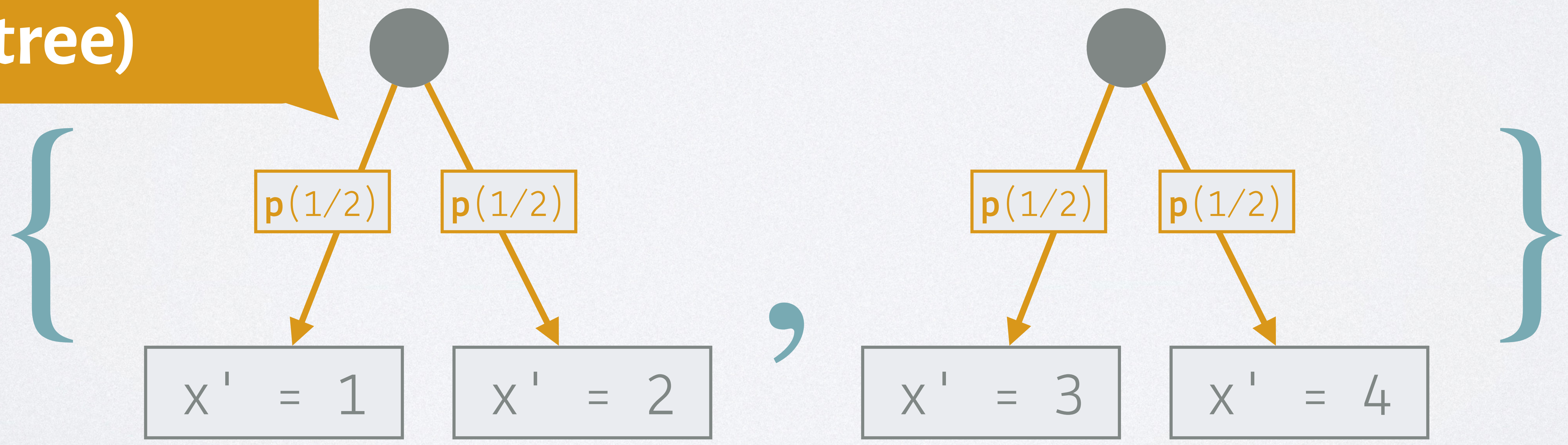
Hyper-edge (for confluence)



Hyper-path (like a tree)



Hyper-Paths are Infinite Trees!

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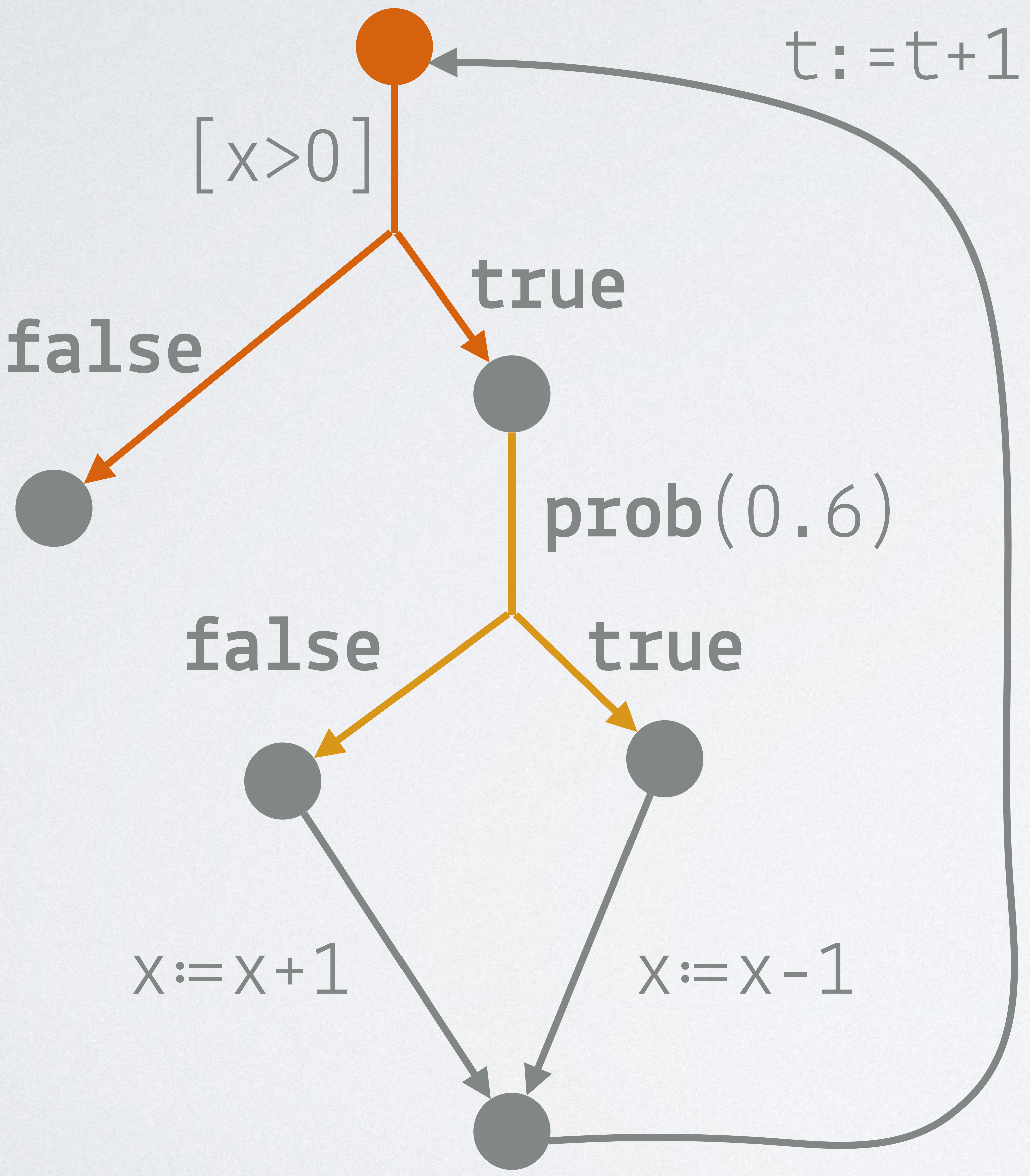

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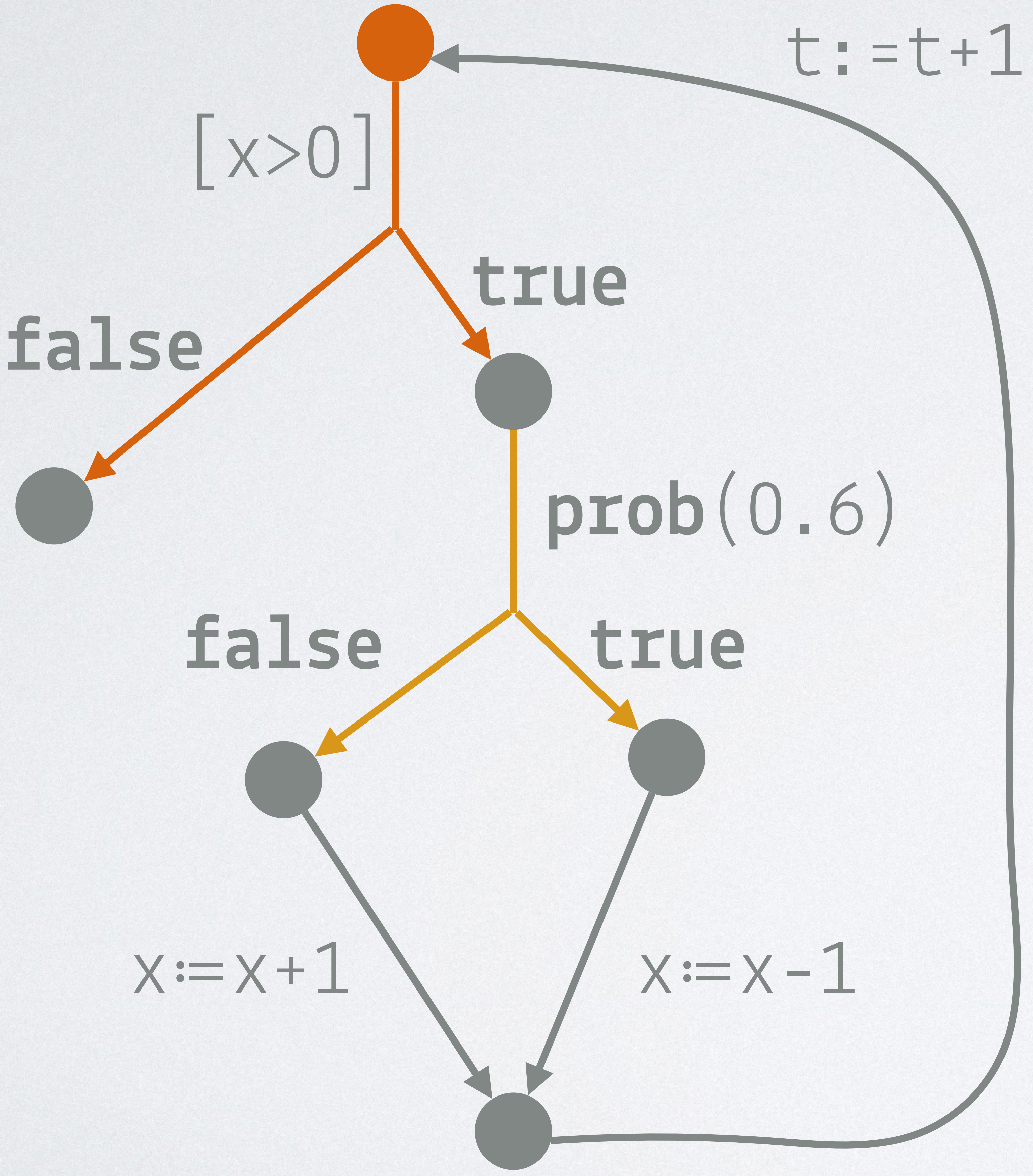
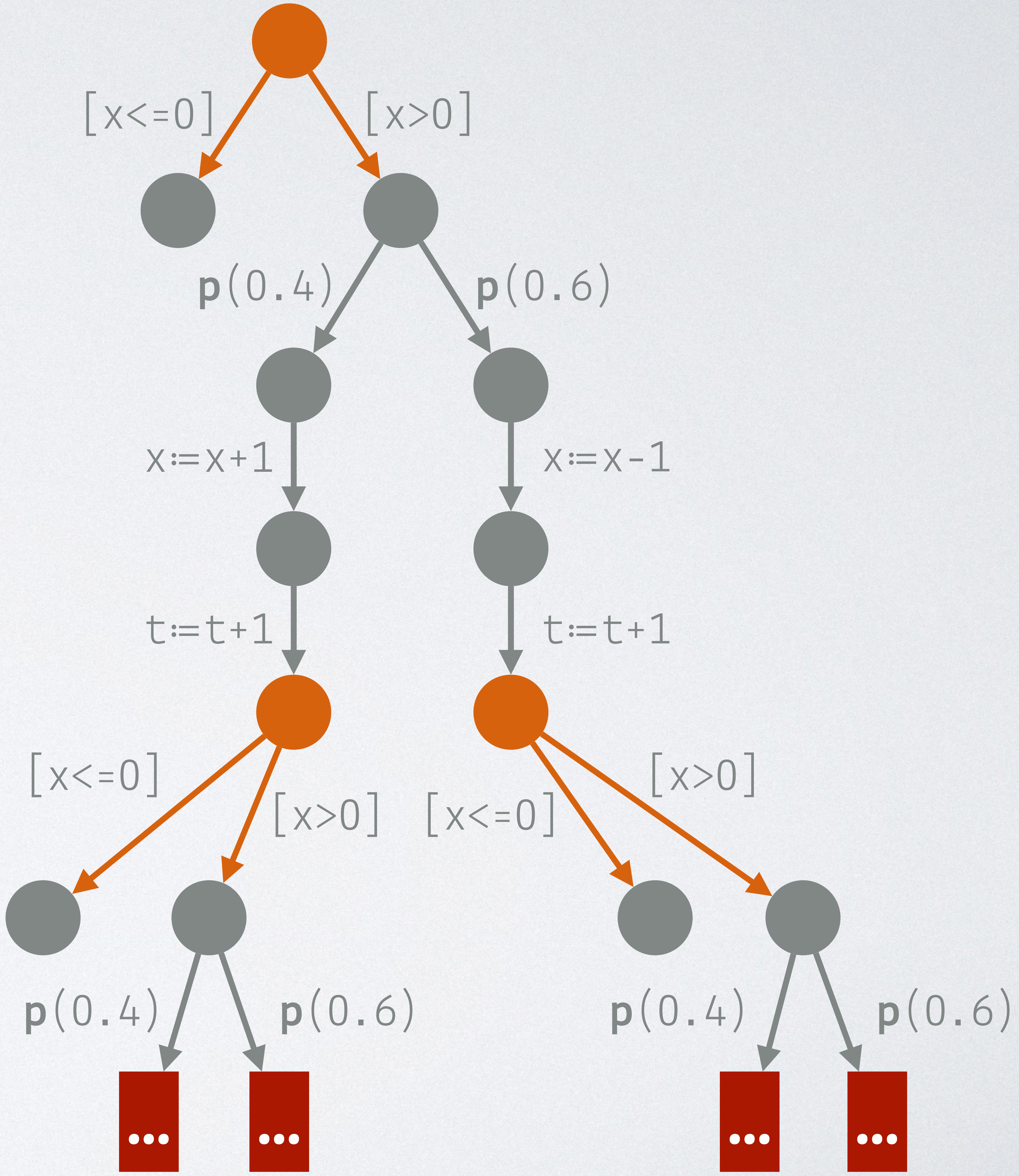
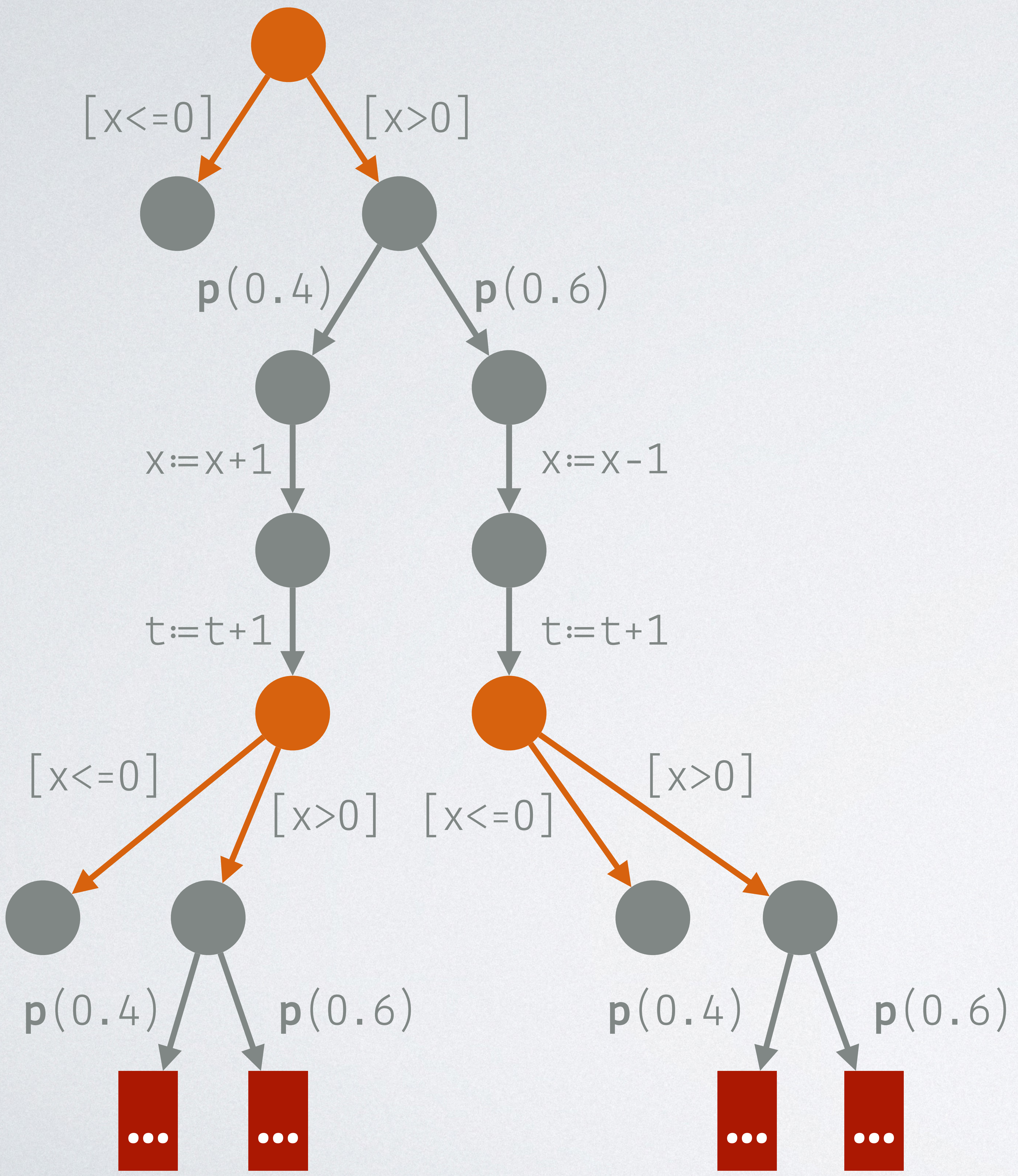




Recursive Program Schemes



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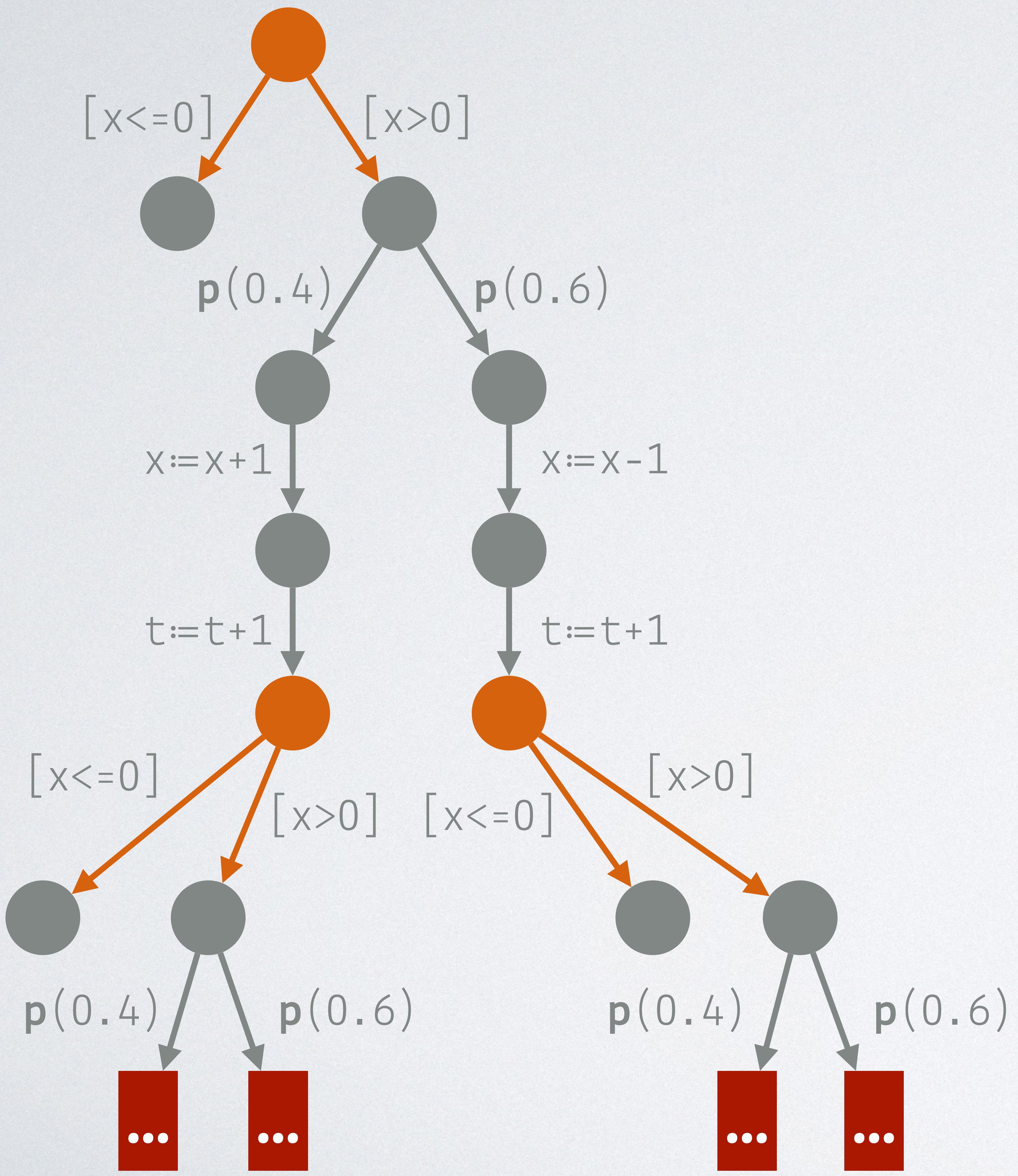

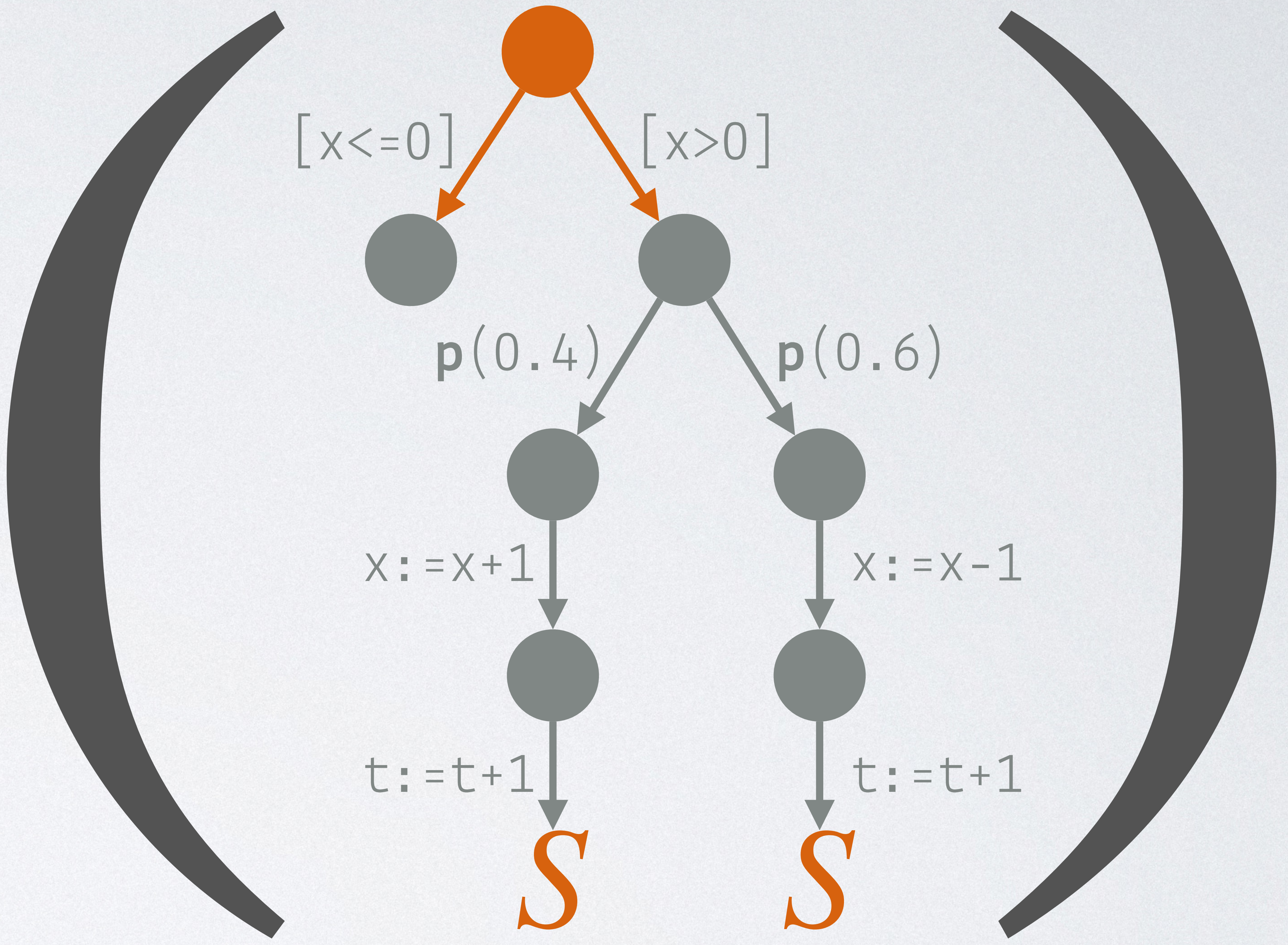

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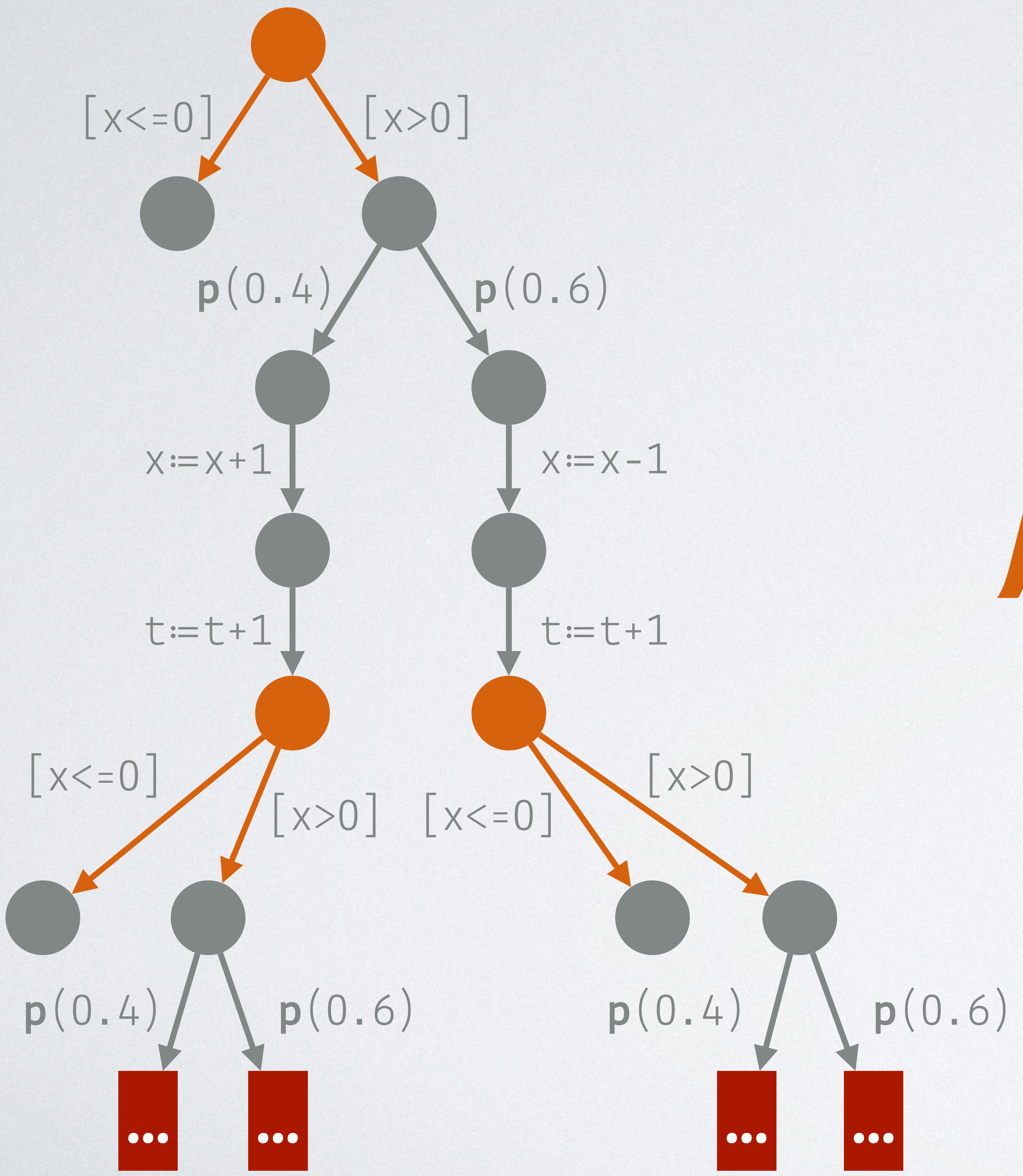

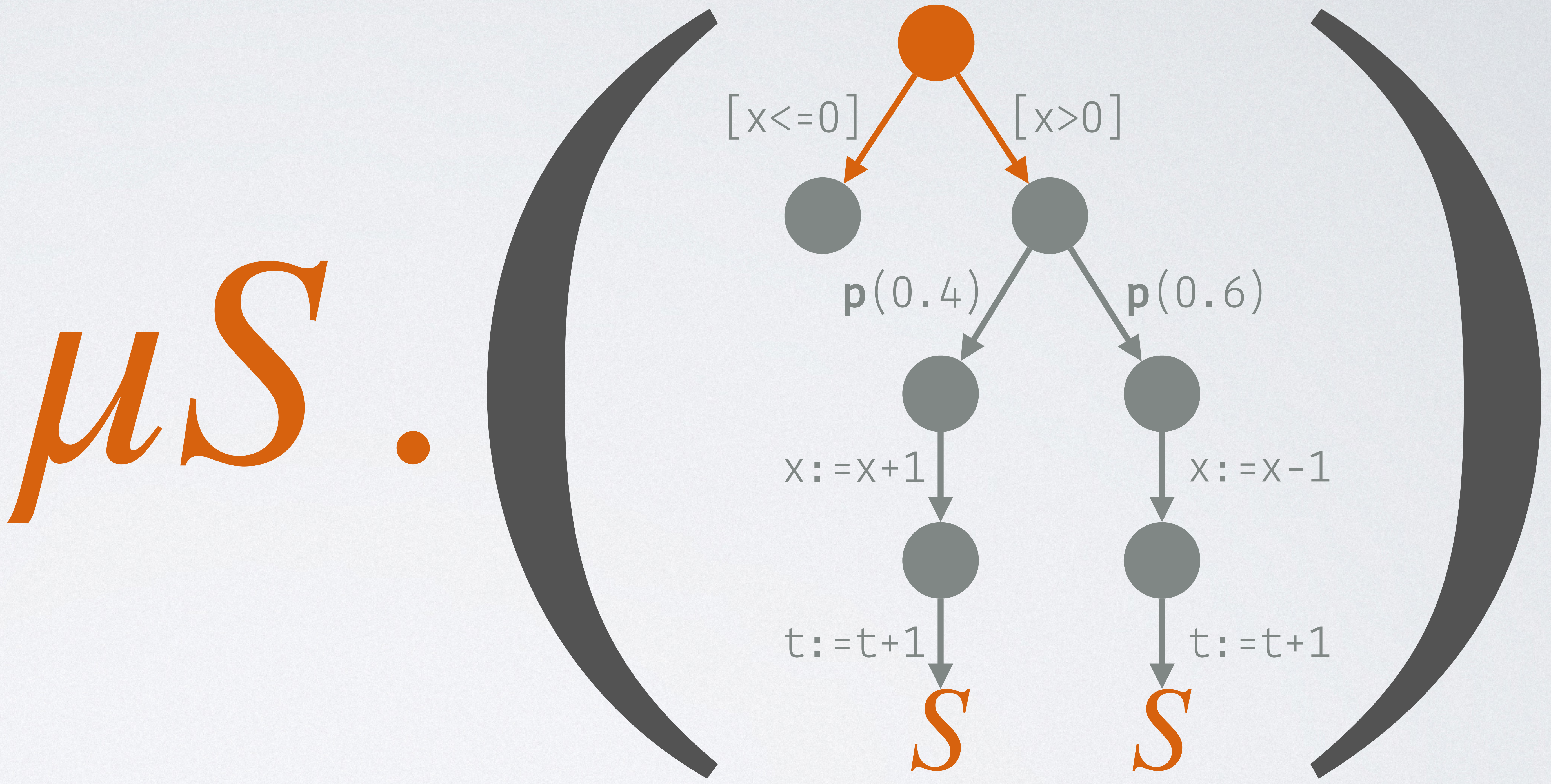

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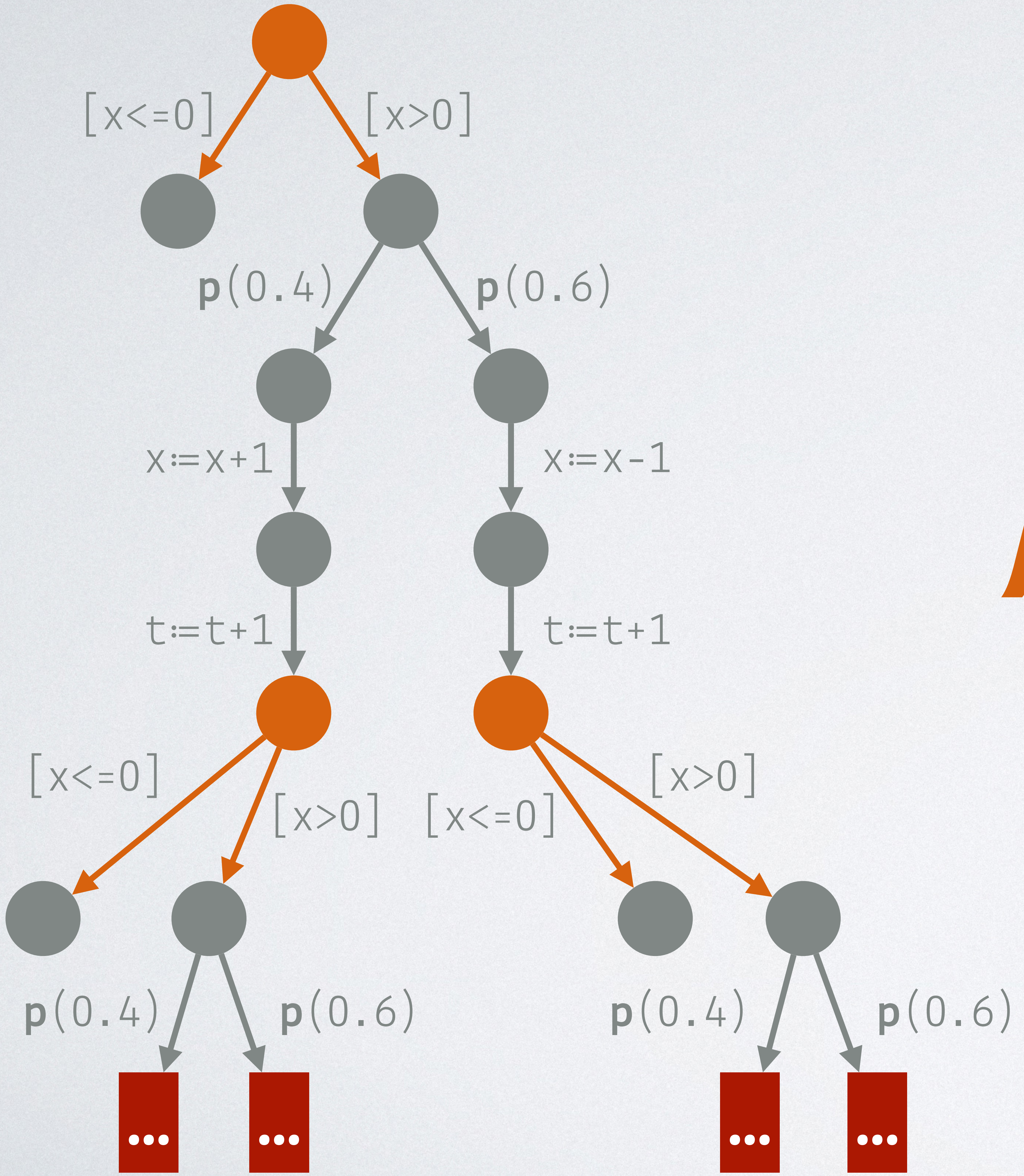

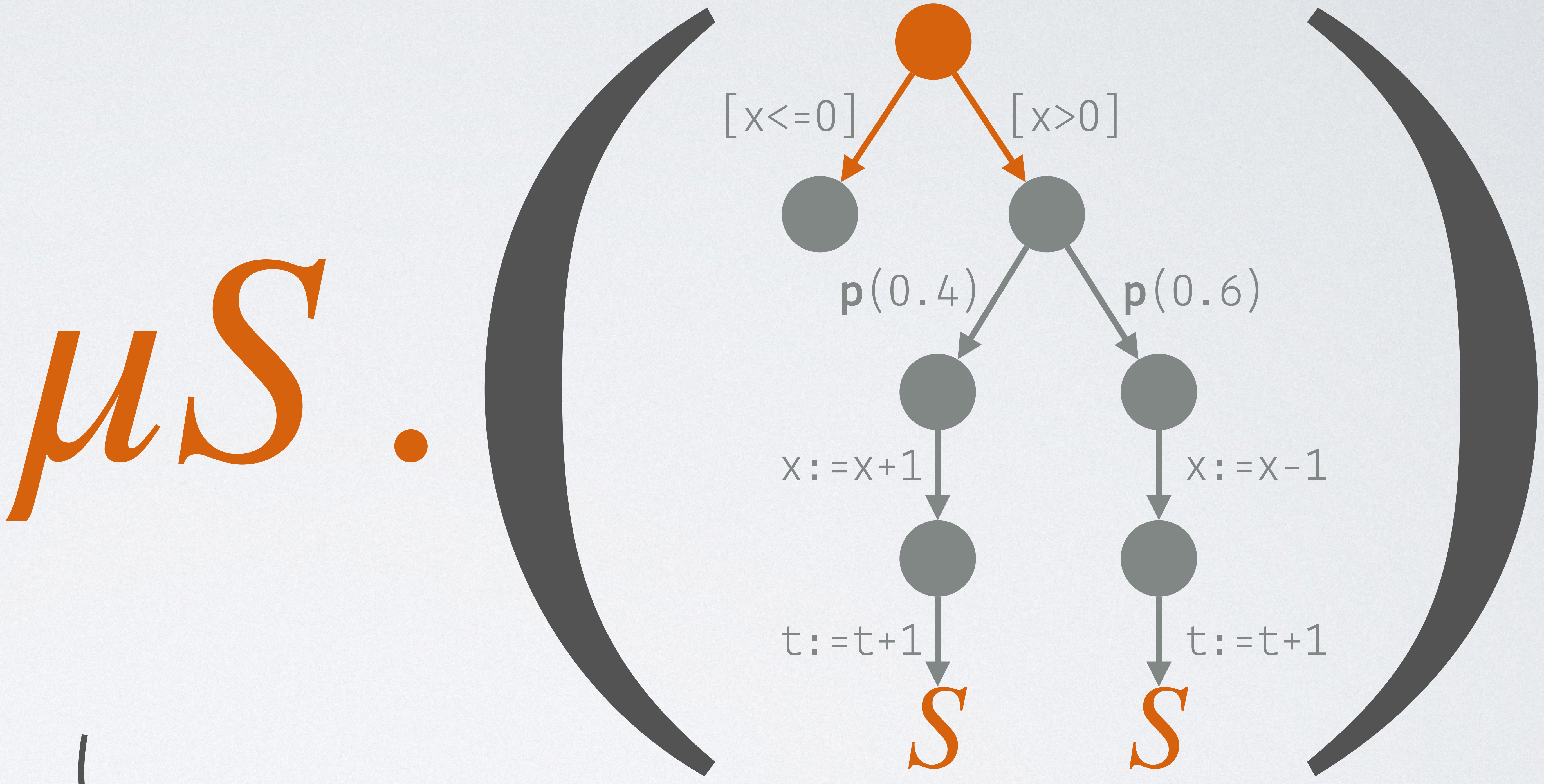

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while x>0 do
  if prob(0.6) then x:=x+1
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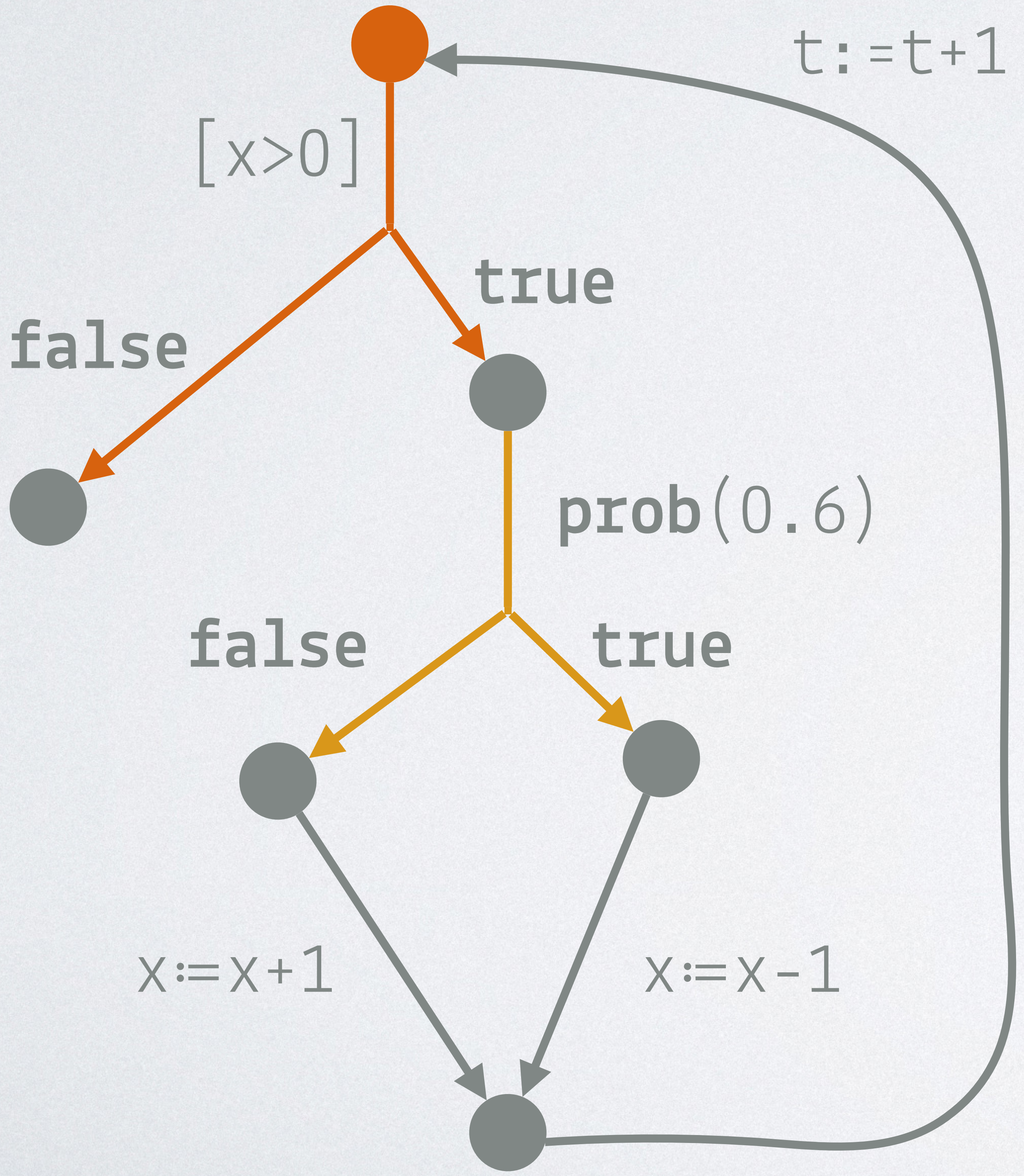


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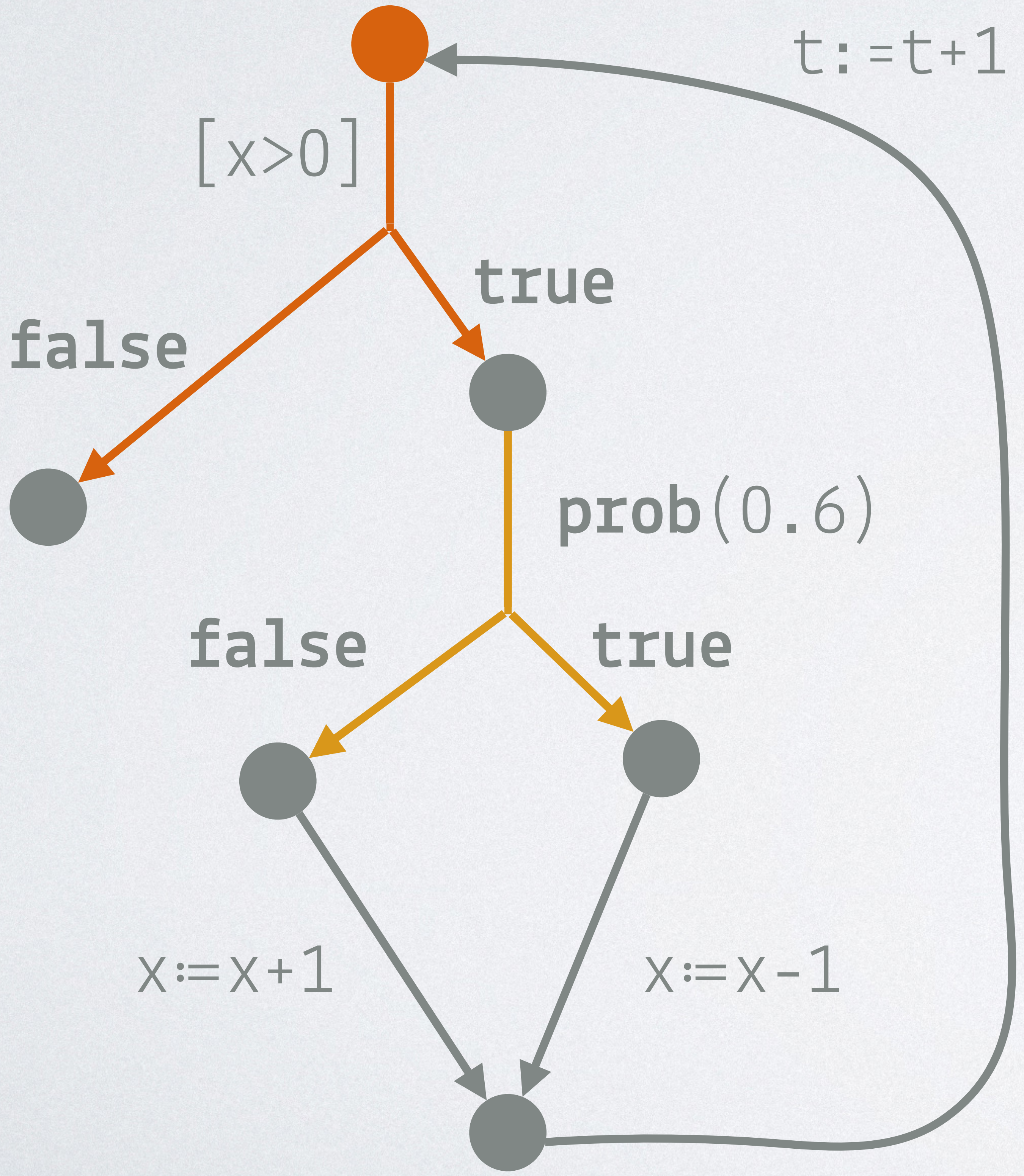


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Challenge III:
How to carry out quantitative analyses efficiently?

```
while prob(2/3) do
  x := x + 1
od
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Iterative Program Analysis

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while prob(2/3) do
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$$\mu S\,.\,((\mathsf{x} \coloneqq \mathsf{x{+}1}) \otimes S)_{[2/3]} \oplus \mathbf{skip}$$

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- Markov algebra for computing $\mathbb{E}[\Delta x]$
  - Sequencing: $r \otimes t \triangleq r + t$
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$$\kappa^{(0)} = 0$$
$$\kappa^{(1)} = 2/3 * (1 + \kappa^{(0)}) + 1/3 * 0 = 2/3$$
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$$\kappa^{(\infty)} = 2$$

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Need ∞ iterations to converge!

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Equivalent to solve:

$$s = 2/3 * (1 + s) + 1/3 * 0,$$

Analytical solution:

$$s = 2$$

No need for iteration!

Non-iterative Intra-procedural Analysis

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- Observation: Loops are (right-)**linear recursions**, thus we can always extract a system of **linear equations**
  - For each $\mu S\,.\,E$, we extract an equation $S = E$
- Techniques to solve linear equation systems extracted from probabilistic programs:
  - **Linear Programming:** Compute probabilities, expectations, or matrices
  - **Loop-Invariant Generation:** Derive probabilistic or expectation invariants

Beyond Loops

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$$p = 1/3 * 1 + 2/3 * (p * p)$$

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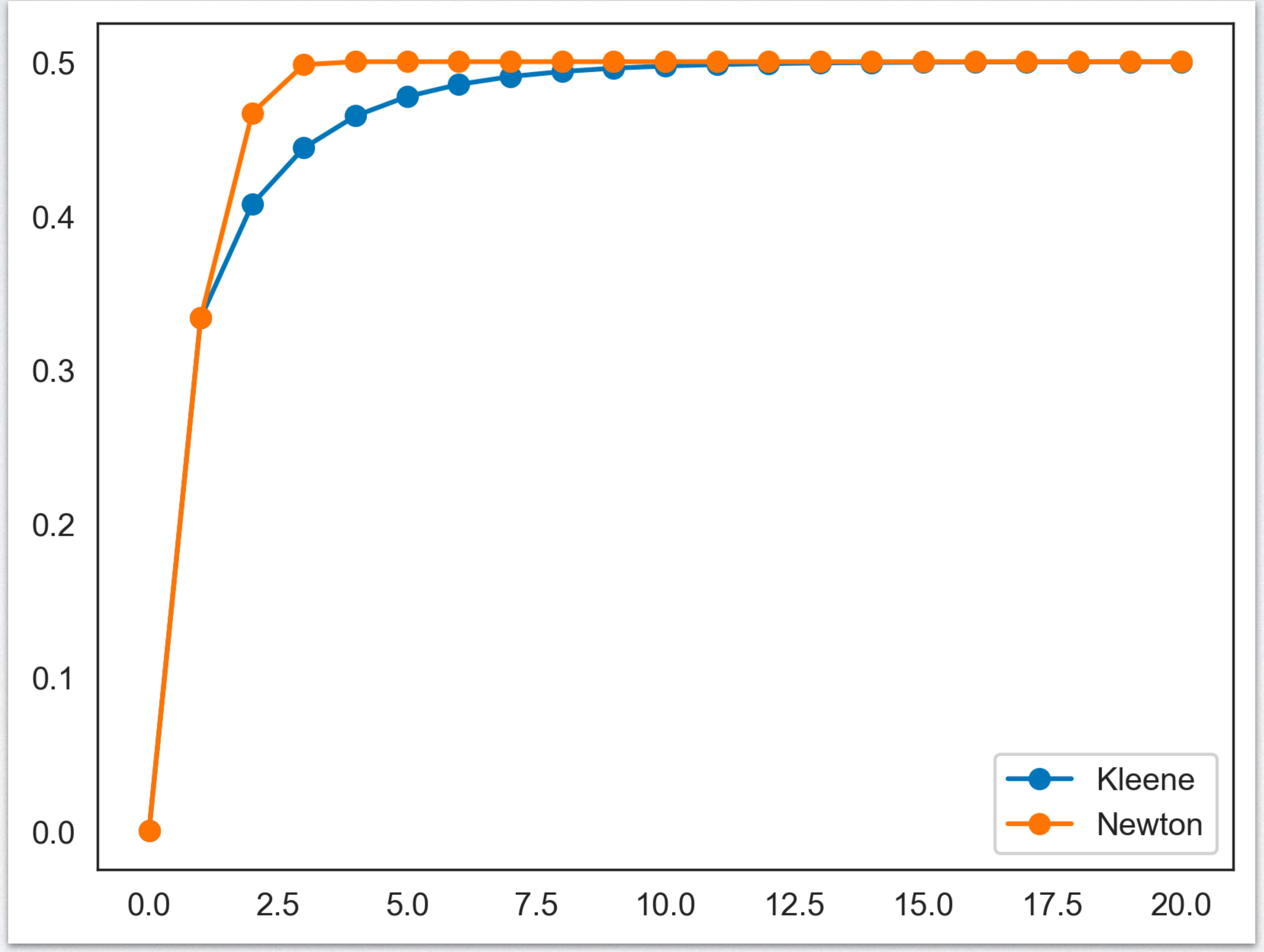
$$\begin{aligned} \nu^{(1)} &= 0 - f(0)/f'(0) = 1/3 \\ \nu^{(2)} &= 1/3 - f(1/3)/f'(1/3) = 7/15 \\ \nu^{(3)} &= 7/15 - f(7/15)/f'(7/15) = 127/255 \\ &\cdots \\ \nu^{(\infty)} &= 1/2 \end{aligned}$$

Newton's Method Converges Faster



Newtonian Program Analysis (NPA)

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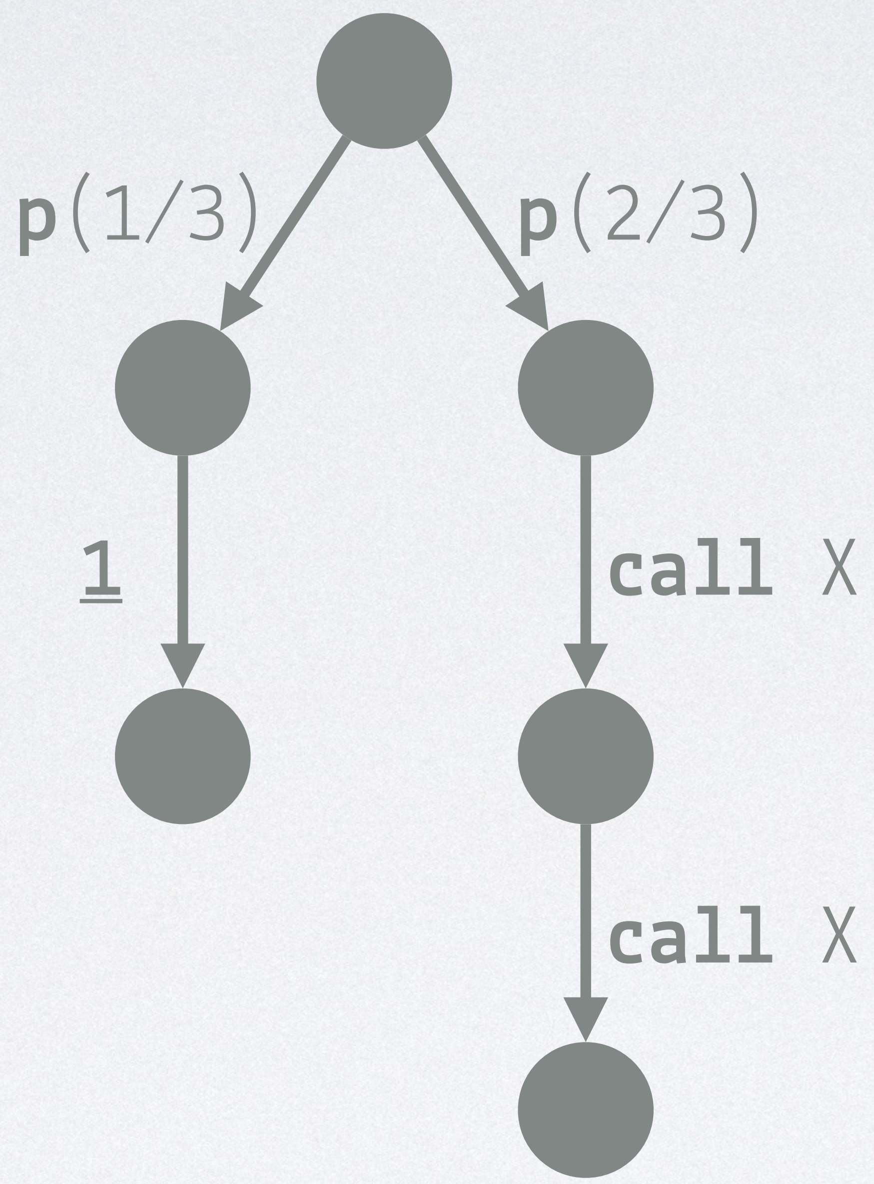

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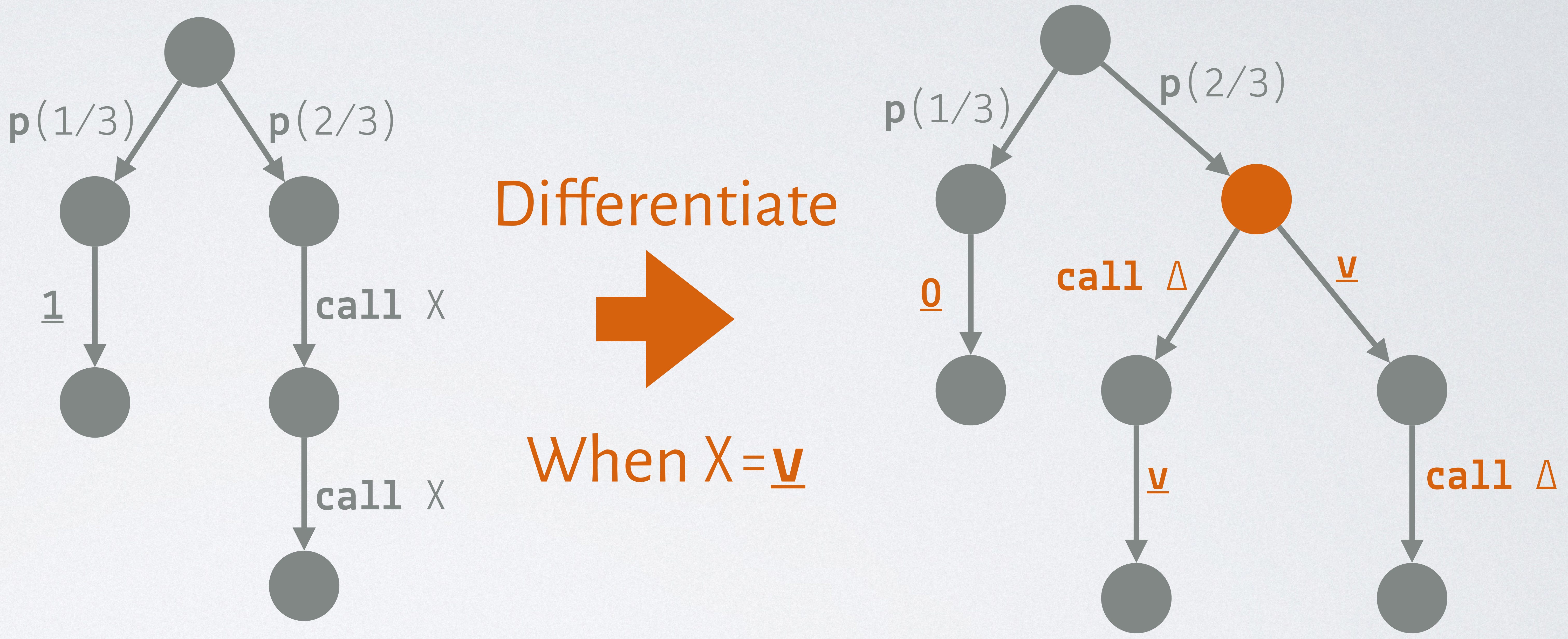

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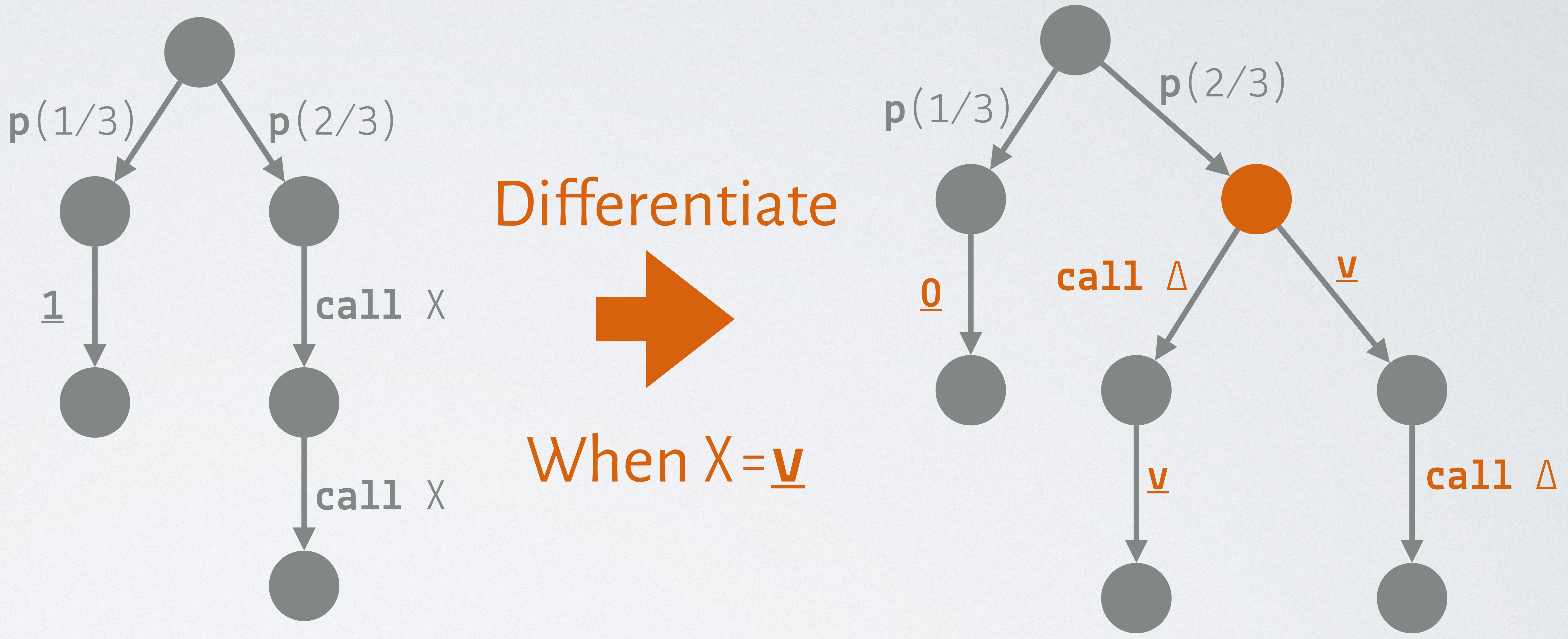

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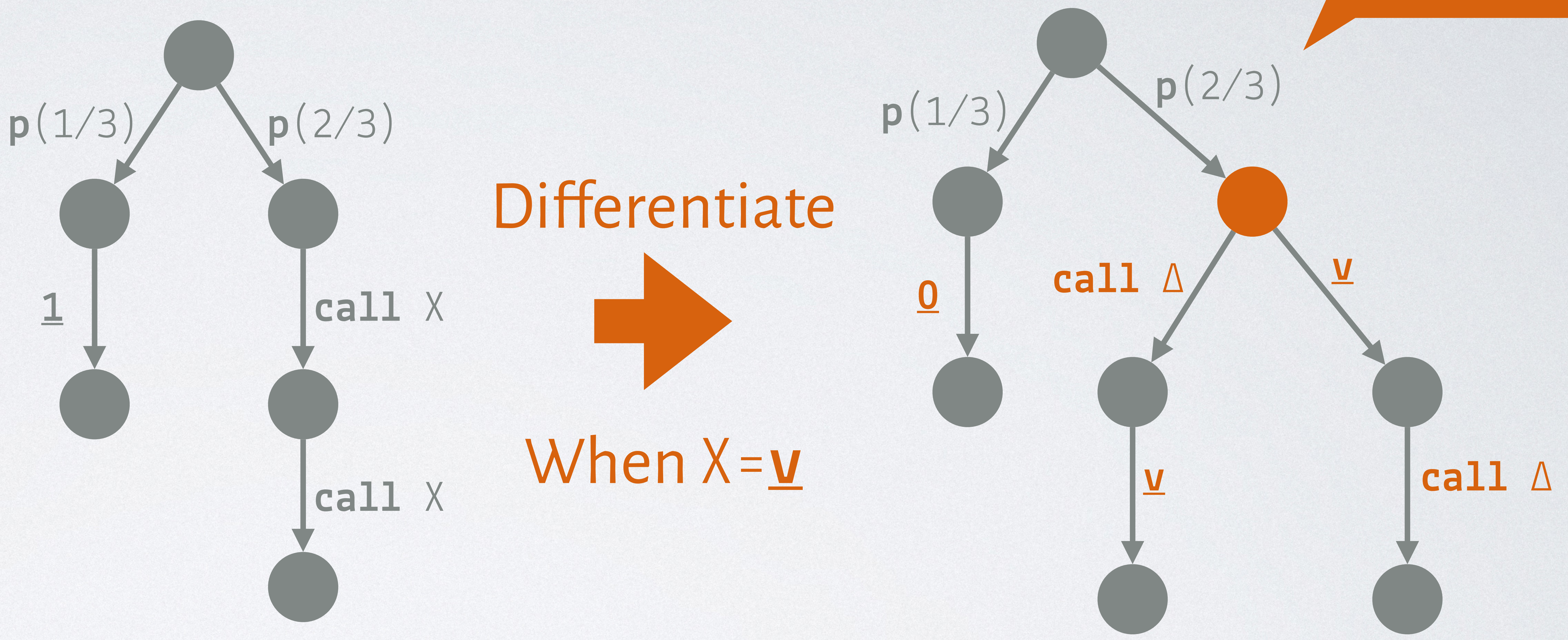

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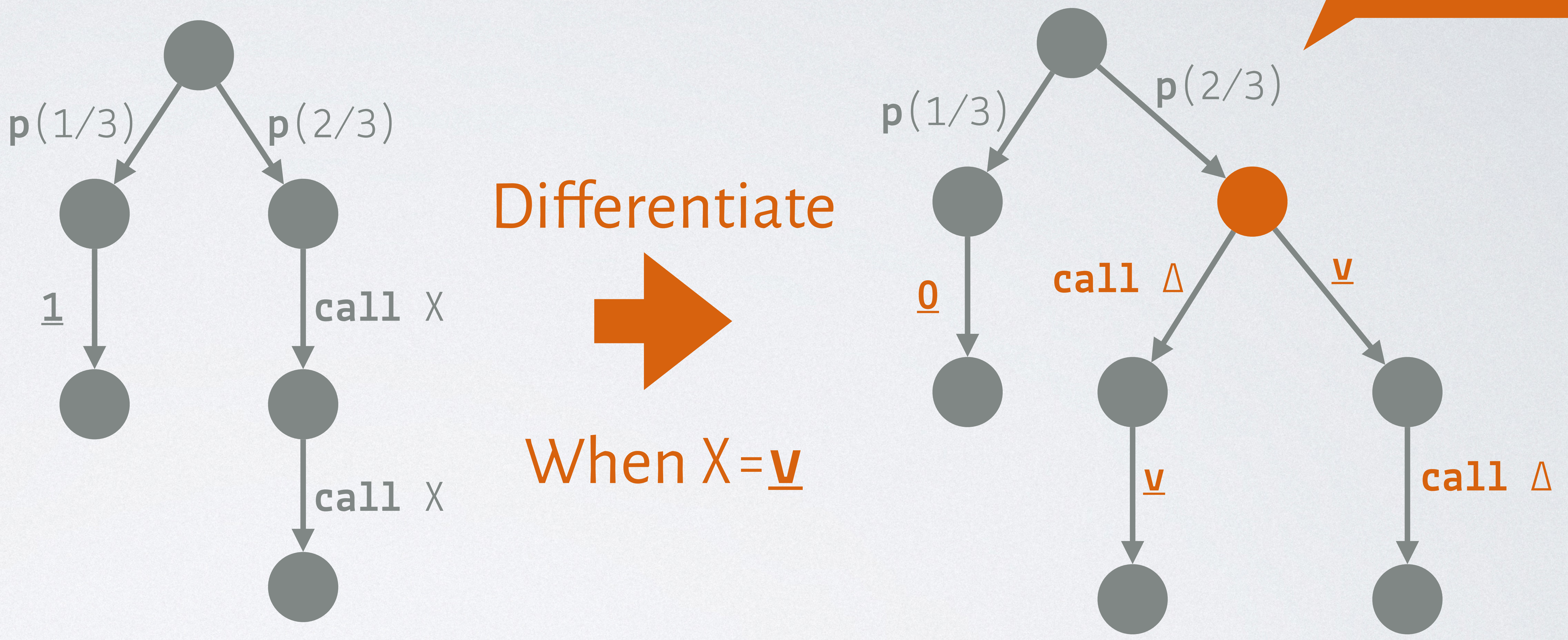

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Procedure call to X

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Branching

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$$\mathscr{D}(f \oplus g)\,|_{\nu}(Y) \triangleq \mathscr{D}f\,|_{\nu}(Y) \oplus \mathscr{D}g_{\nu}(Y)$$

$$\mathscr{D}(f \otimes g)\,|_{\nu}(Y) \triangleq (\mathscr{D}f\,|_{\nu}(Y) \otimes g(\nu)) \oplus (f(\nu) \otimes \mathscr{D}g\,|_{\nu}(Y))$$

Newtonian Program Analysis (NPA)

- Idea: Apply Newton's method to **algebraic structures**, e.g., Kleene algebras
- Derive the **syntactic** differential of algebraic expressions

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Sequencing

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Interpreting calls to X in g by ν

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Our Approach: NPA for pre-Markov Algebras

- Key idea: Apply Newton's method to **pre-Markov algebras**
- We develop a differentiation routine for **recursive program schemes**

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Support multiple confluences, loops, and recursion

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$$\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \rangle$$

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$\oplus$ defines a partial order and gives an additive structure

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$$\text{Equation System} \xrightarrow{\text{Differentiate at } \nu^{(i)}} \text{Linearized Equation System}$$

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Analysis-specified loop-solving strategy

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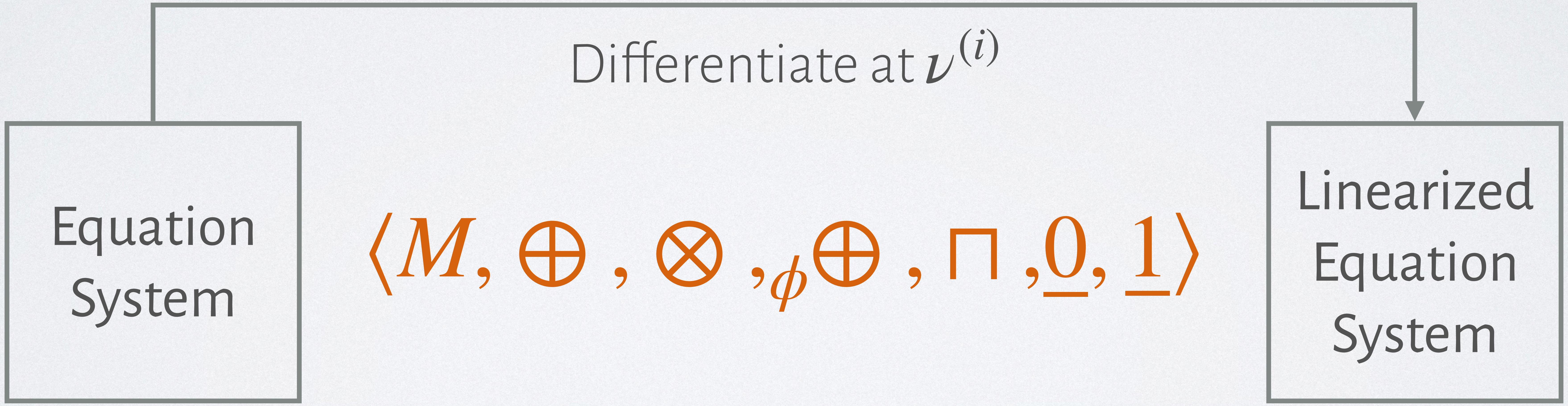

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Differentiate at $\nu^{(i)}$

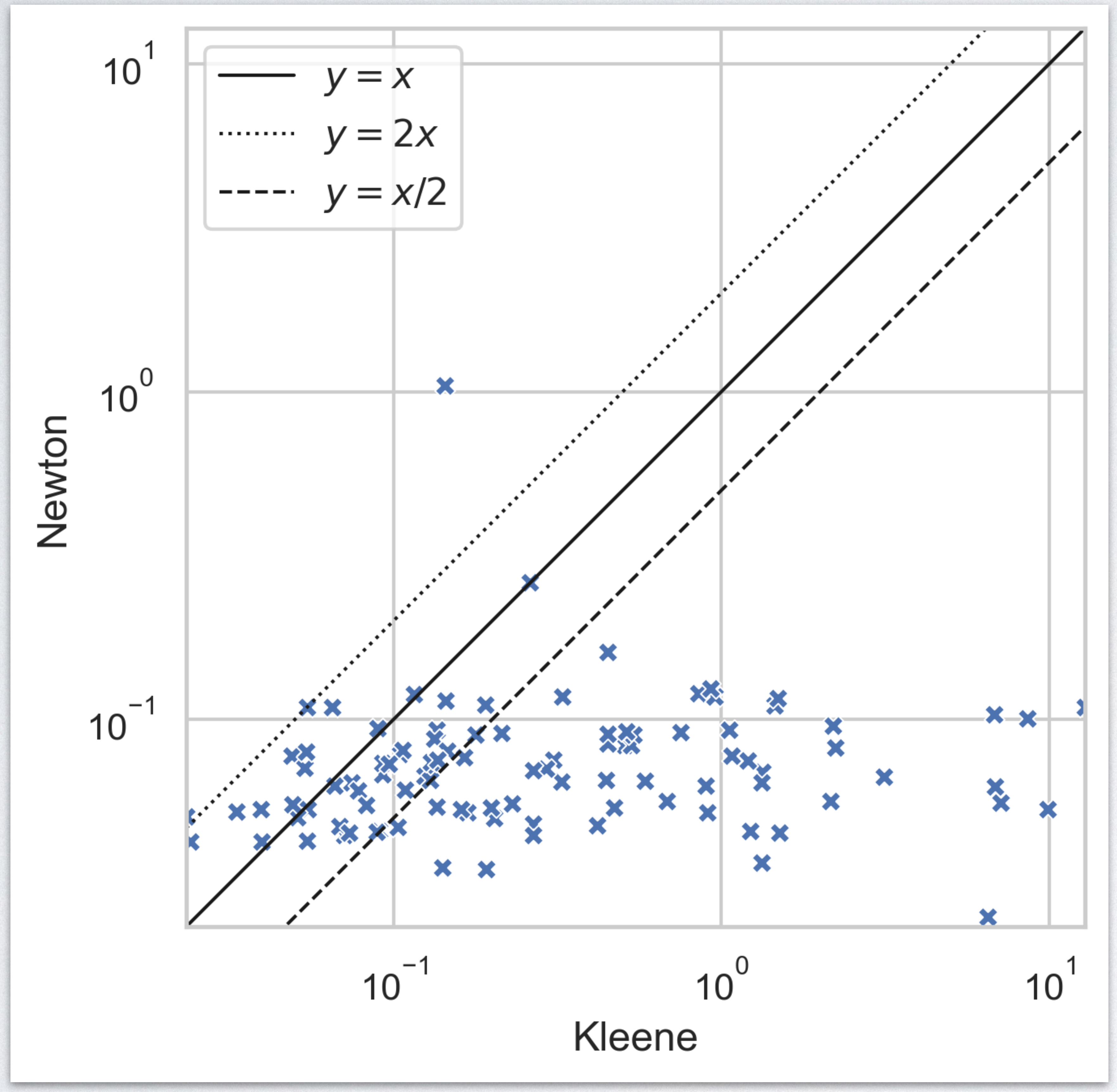
Equation System

$$\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \rangle$$

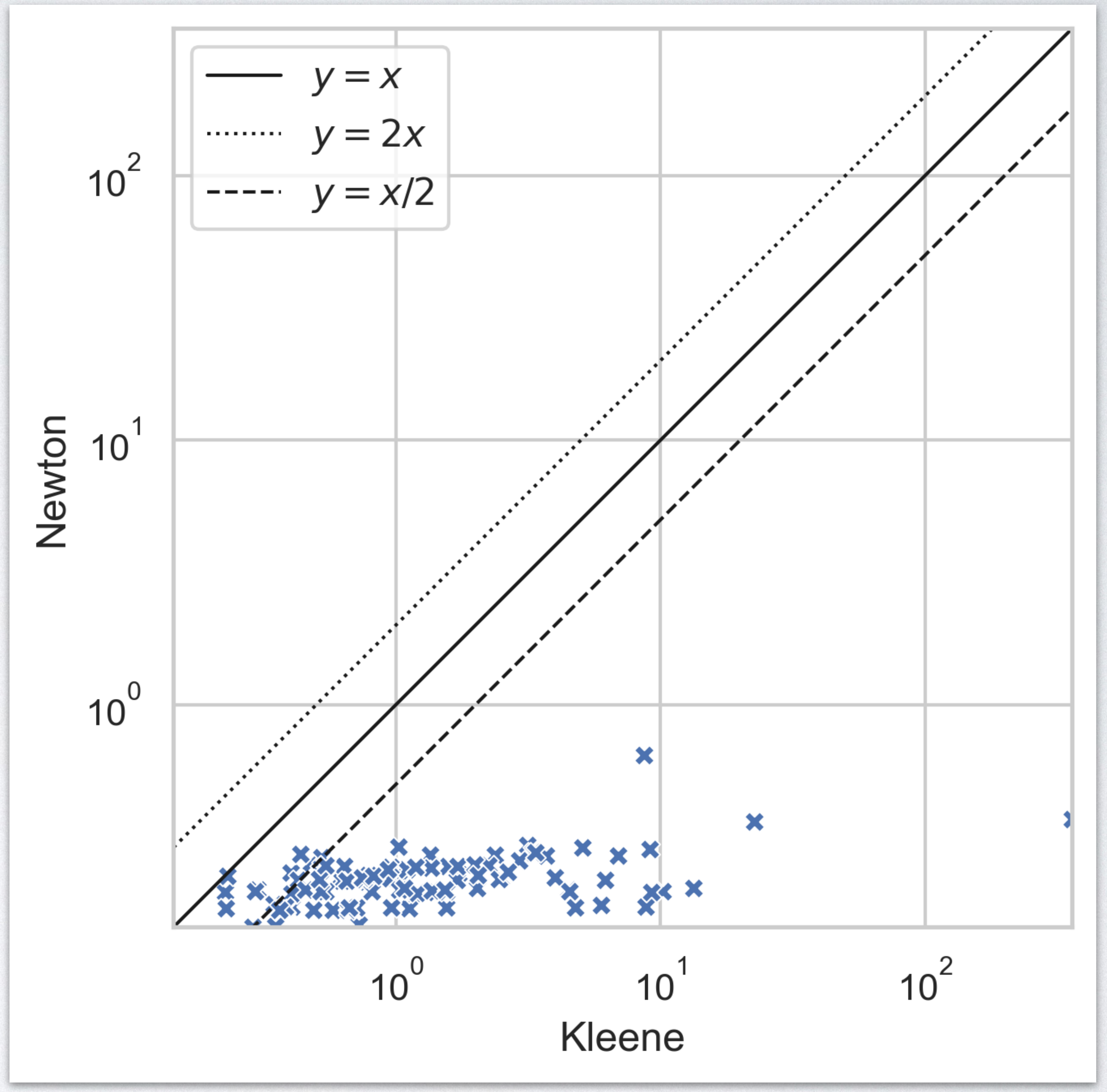
Linearized Equation System

Solve the linear system to get $\nu^{(i+1)}$

Preliminary Evaluation



Reaching Probability



Expected Reward

Our Papers

- Di Wang, Jan Hoffmann, Thomas Reps (2018). **PMAF: An Algebraic Framework for Static Analysis of Probabilistic Programs**. In *PLDI'18*.
- Di Wang, Jan Hoffmann, Thomas Reps (2019). **A Denotational Semantics for Low-Level Probabilistic Programs with Nondeterminism**. In *MFPS'19*.
- Di Wang, Thomas Reps (2024). **Newtonian Program Analysis of Probabilistic Programs**. In *OOPSLA'24*.

Towards a flexible and efficient framework for program analysis of probabilistic programs

- ☑ **Semantics:** Markov Algebras for Multiple Kinds of Confluences
- ☑ **Representation:** Construction of Recursive Program Schemes
- ☑ **Algorithm:** Newton's Method for pre-Markov Algebras