概率程序的代数程序分析

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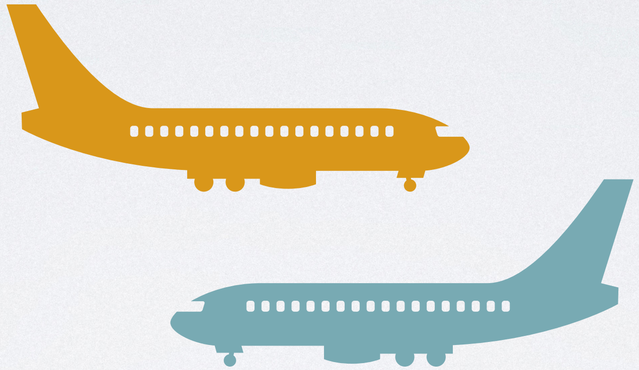
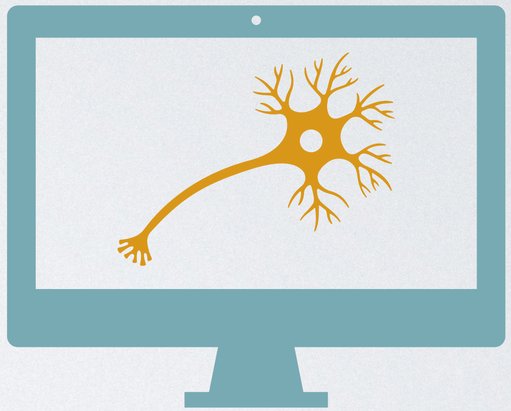
与 Jan Hoffmann 和 Thomas Reps 的合作工作

Probabilistic Systems are Becoming Pervasive

Randomized Algorithms
(improve efficiency)



Cyber-Physical Systems
(model uncertainty)



Artificial Intelligence
(describe statistical models)

Probabilistic Programs

Draw random **data** from distributions

Change **control-flow** at random

Image sources: https://www.libertystorch.info/2022/02/21/the-grab-bag/; https://www.stockvault.net/photo/179872/at-a-crossroads-decisions-and-choices-concept-with-large-arrow-signs.

Application: Randomized Quicksort

- Improve efficiency
- From $\Theta(n^2)$ to $\Theta(n \log n)$ (expected)
- Samplesort
  - Use >1 random samples as pivots

Randomized Quicksort

Randomized-Partition(*A, p, r*)

1. $i \leftarrow$ *Random*(*p*, *r*)
2. exchange $A[r] \leftrightarrow A[i]$
3. **return Partition**(*A*, *p*, *r*)

Randomized-Quicksort(*A, p, r*)

1. **if** $p < r$
2. **then** $q \leftarrow$ **Randomized-Partition**(*A*, *p*, *r*)
3. **Randomized-Quicksort**(*A*, *p* , *q*-1)
4. **Randomized-Quicksort**(*A*, *q*+1, *r*)

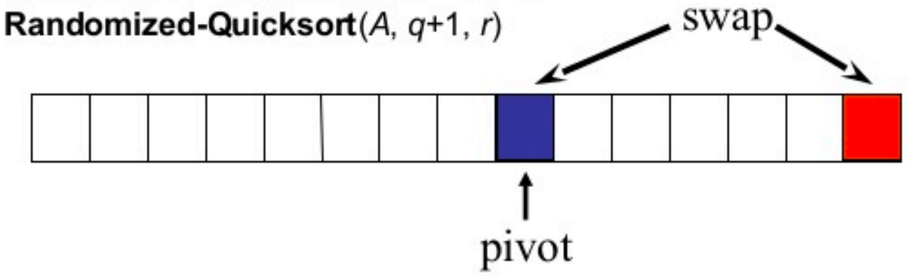


Image source: https://geekfactorial.blogspot.com/2016/08/randomized-quick-sort-algorithm.html.

Application: Airborne Collision Avoidance

- Model uncertainty
- Probabilistic dynamics
- Probabilistic sensors
- High-confidence system

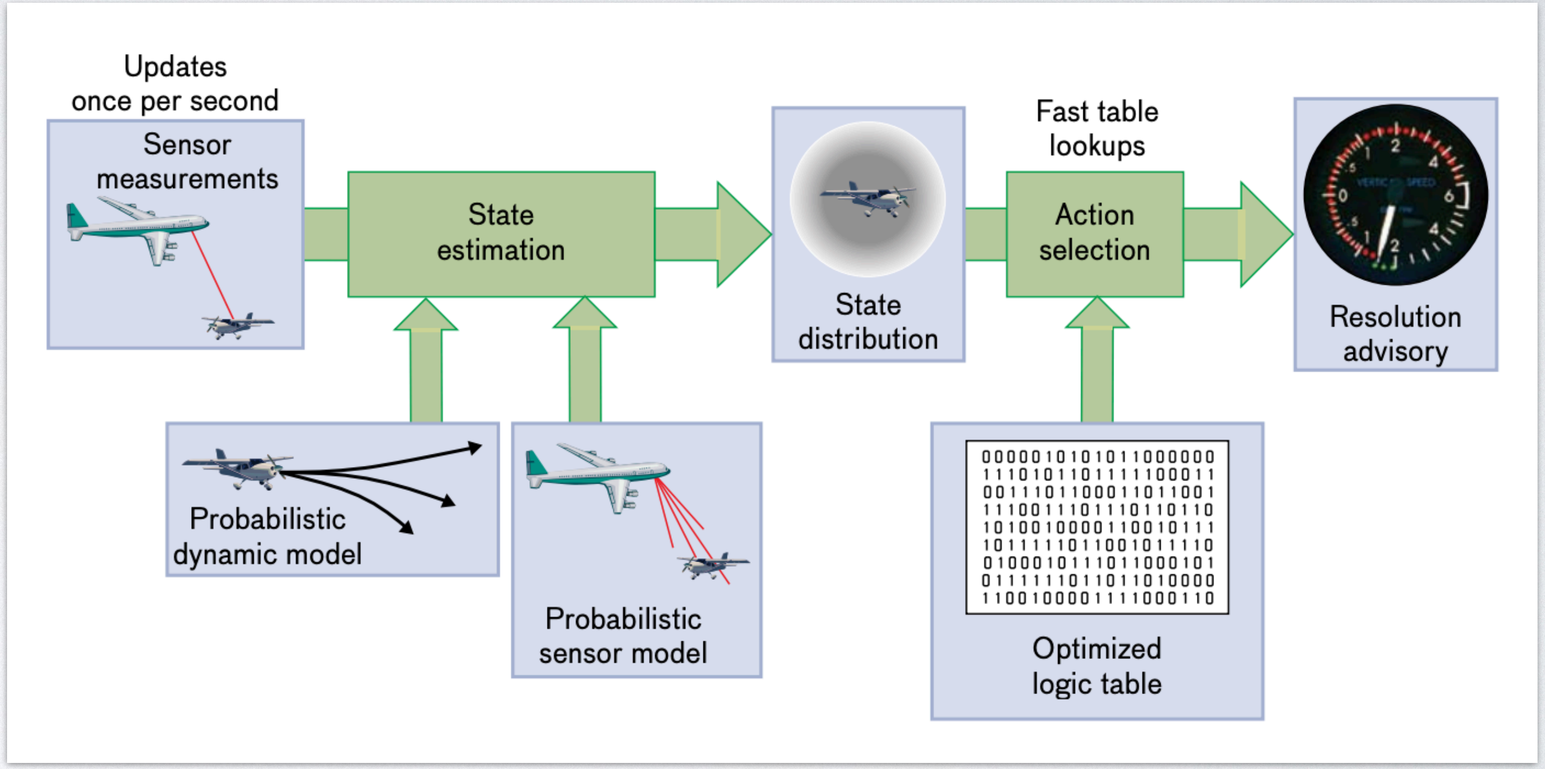
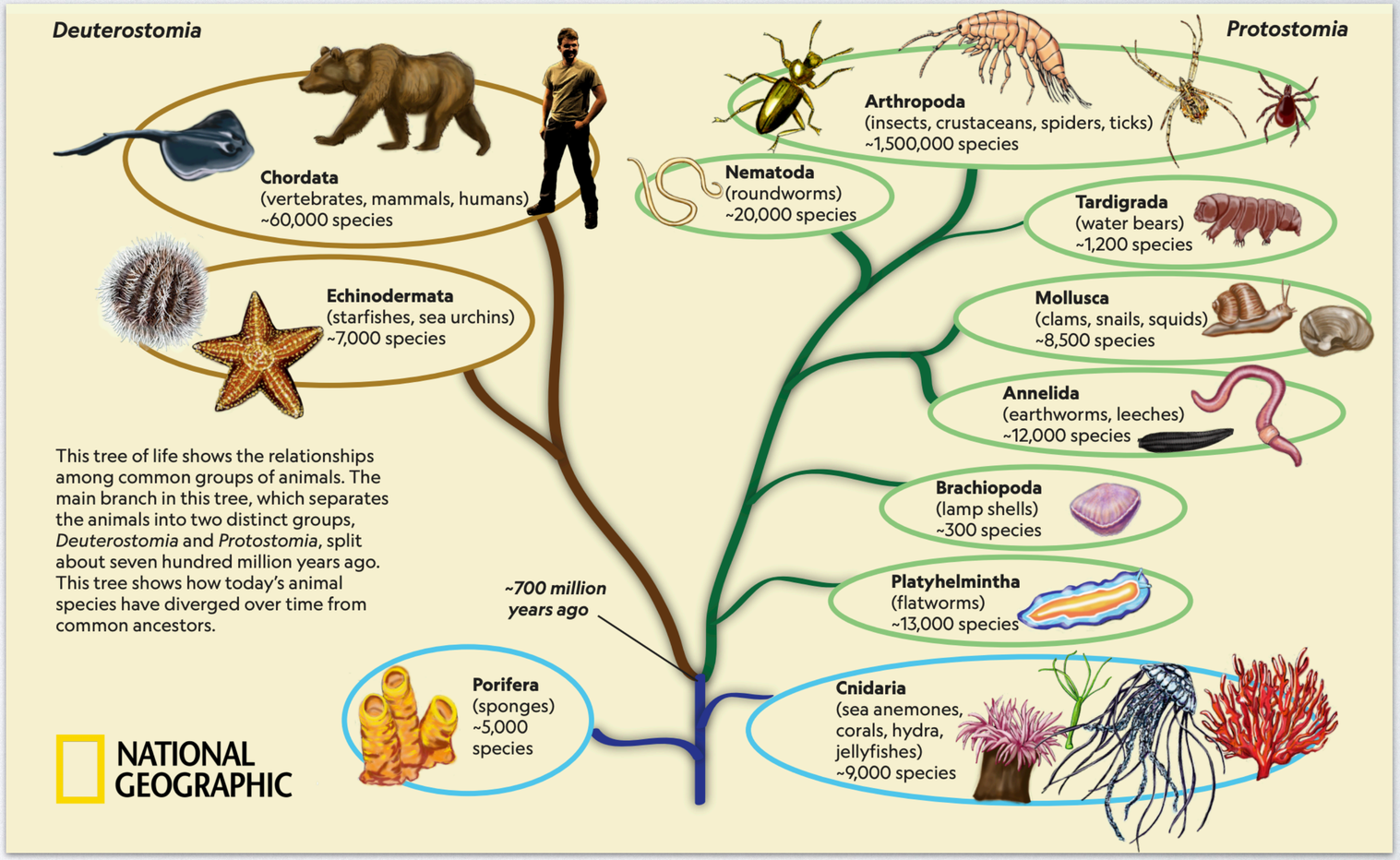


Image source: Kochenderfer, et al. “Next-Generation Airborne Collision Avoidance System.” Lincoln Laboratory Journal (2012).

Application: Statistical Phylogenetics

- Describe statistical models
- Automated reasoning
- Apply Bayesian inference to infer evolutionary history
- Solve previously intractable problems



Ronquist, et al. "Universal probabilistic programming offers a powerful approach to statistical phylogenetics." Communications Biology (2021).
Image source: https://www.nationalgeographic.org/media/tree-life/.

Probabilistic Programs

- True randomness
- A distribution on execution paths
- Probabilistic nondeterminism

```
if
| prob(1/3) → cc := 1
| prob(1/3) → cc := 2
| prob(1/3) → cc := 3
fi
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Demonic Programs

- Dijkstra's **Guarded Command Language** (GCL)
- A set of execution paths
- Demonic nondeterminism

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pc :∈ {1,2,3}
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Example: Monty Hall

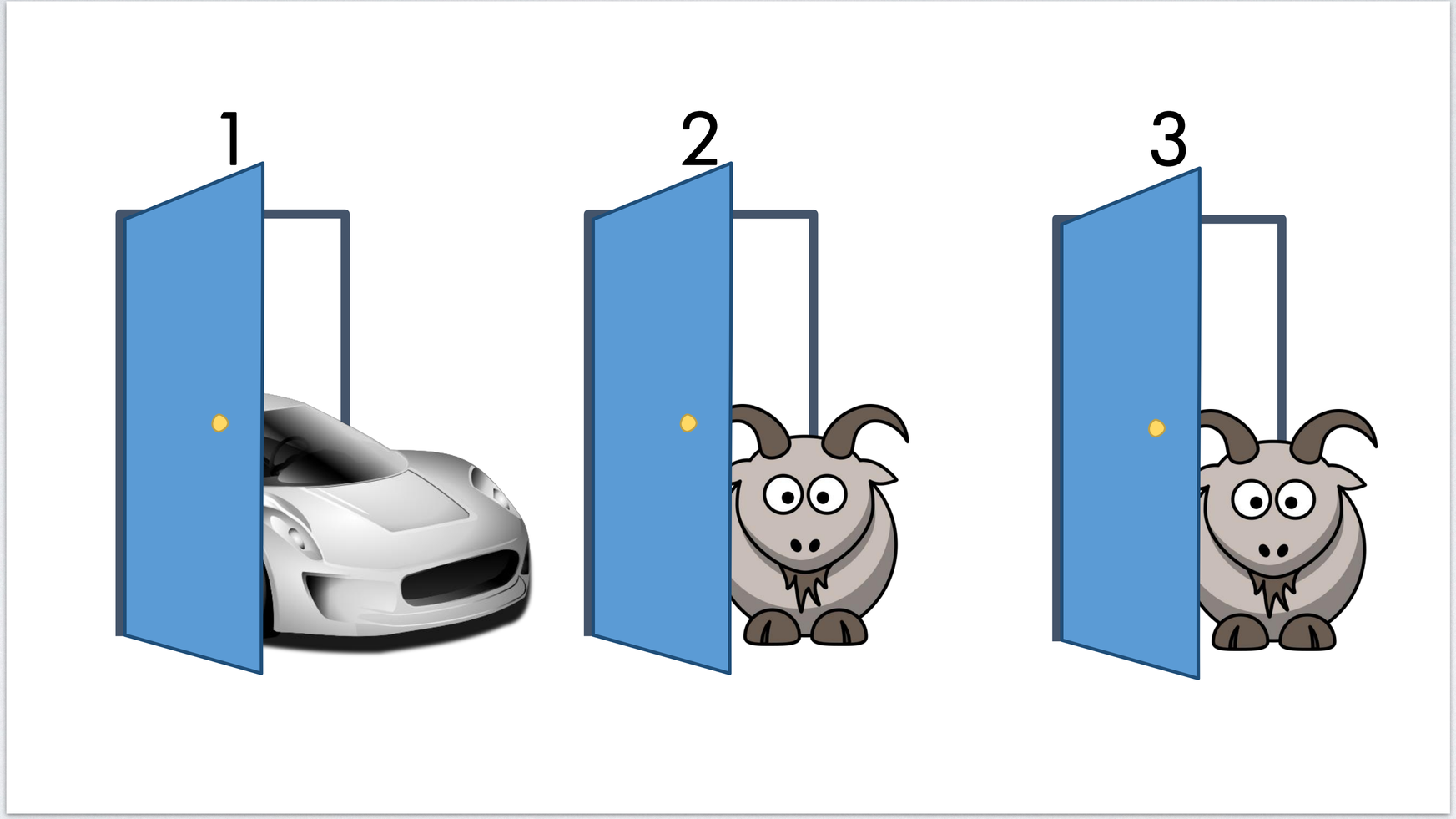

Image source: Maria Gorinova's Advances in Programming Languages (Guest Lecture) slides on Probabilistic Programming.

Example: Monty Hall

- McIver and Morgan's **probabilistic Guarded Command Language** (pGCL)
- Combine two forms of nondeterminism:
  - Probabilistic
  - Demonic
- What is the probability that `cc` equals to `pc`, if we switch to another door?
  - "Demons" minimize the probability

```
pc :∈ {1,2,3};
cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3);
ac :∈ {1,2,3} \ {pc,cc};
if switch then
  cc :∈ {1,2,3} \ {cc,ac}
fi
```

Reasoning about pGCL Programs

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- Dijkstra's GCL
  - Weakest pre-condition calculus
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$$\{$$
$$1/3 * [1 \geq 2] + 1/3 * [2 \geq 2] + 1/3 * [3 \geq 2]$$
$$= 2/3$$
$$\}$$

```
cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3)
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$$\{\ [cc \geq 2]\ \}$$

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$$\{\ 1/3 * [1 \geq 2] + 1/3 * [2 \geq 2] + 1/3 * [3 \geq 2] = 2/3\ \}$$

```
cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3)
```

$$\{\ [cc \geq 2]\ \}$$

$$\{\ \min([1 \geq 2], [2 \geq 2], [3 \geq 2]) = 0\ \}$$

```
pc :∈ {1,2,3}
```

$$\{\ [pc \geq 2]\ \}$$

Example: Monty Hall

$\{\ [\neg switch] * 1/3 + [switch] * 2/3\ \}$

```
pc :∈ {1,2,3};
```

$\{$

$[\neg switch]/3 * ([pc = 1] + [pc = 2] + [pc = 3]) + [switch]/3 * ([pc \neq 1] + [pc \neq 2] + [pc \neq 3])$

$\}$

```
cc :∈ (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3);
```

$\{\ [\neg switch] * [cc = pc] + [switch] * [cc \neq pc]\ \}$

```
ac :∈ {1,2,3} \ {pc,cc};
```

$\{\ [\neg switch] * [cc = pc] + [switch] * [\{cc, pc, ac\} = \{1,2,3\}]\ \}$

```
if switch then
  cc :∈ {1,2,3} \ {cc,ac}
fi
```

$\{\ [cc = pc]\ \}$

How to automate such quantitative reasoning about probabilistic programs?

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Examples

What is the probability that an assertion holds?

Is there any expectation invariant for a probabilistic loop?

What is the expected time complexity of a program?

Towards a compositional and flexible framework for program analysis of probabilistic programs

Semantics: Support different semantics for nondeterminism

Representation: Extend pGCL with unstructured control-flow

Algorithm: Solve program analyses in a non-iterative way

Our Papers

- Di Wang, Jan Hoffmann, Thomas Reps (2018). **PMAF: An Algebraic Framework for Static Analysis of Probabilistic Programs**. In *PLDI'18*.

- Di Wang, Jan Hoffmann, Thomas Reps (2019). **A Denotational Semantics for Low-Level Probabilistic Programs with Nondeterminism**. In *MFPS'19*.

- Di Wang, Thomas Reps (2023). **Newtonian Program Analysis of Probabilistic Programs**. *Working Paper*.

Denotational Semantics

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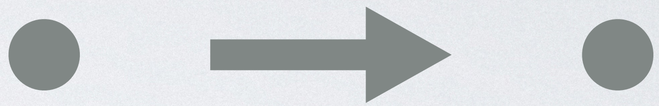
Standard: $State \rightarrow State$



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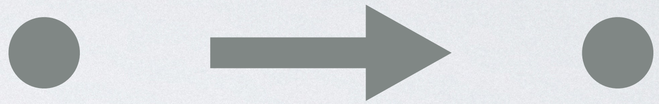
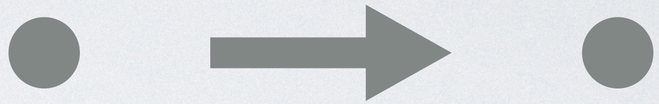


GCL: $State \rightarrow \wp(State)$

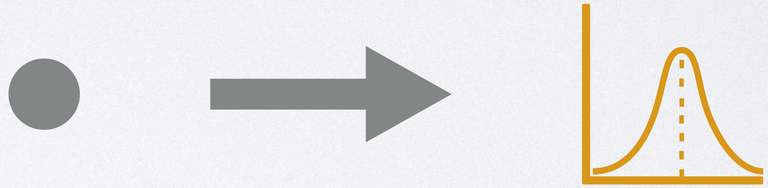
Denotational Semantics

Standard: $State \to State$



GCL: $State \to \wp(State)$

Probabilistic: $State \to \mathbb{D}(State)$



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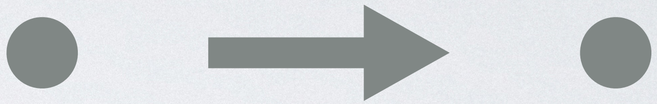

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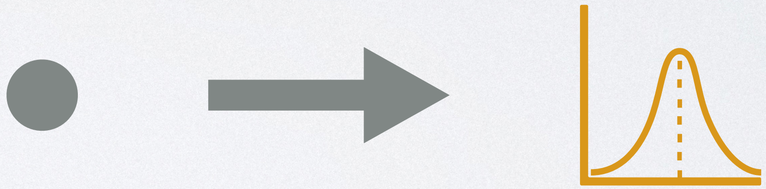
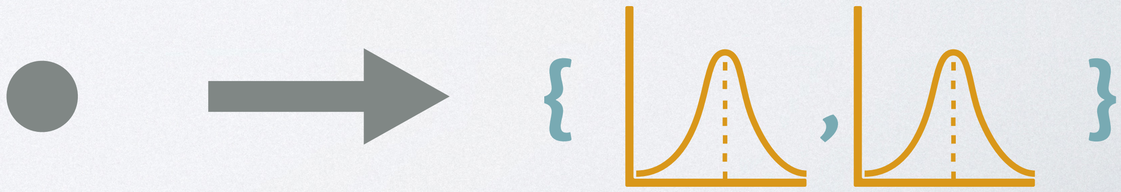


pGCL: $State \to \wp(\mathbb{D}(State))$



Different Models for Combining Nondeterminism

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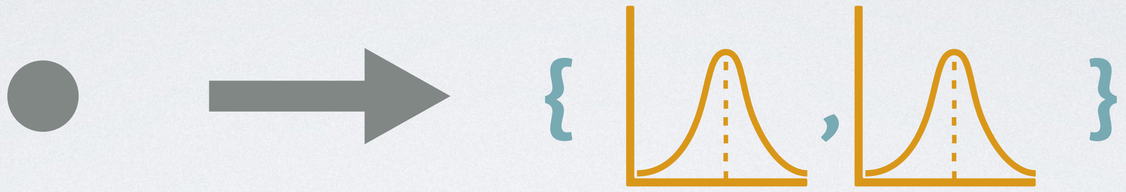

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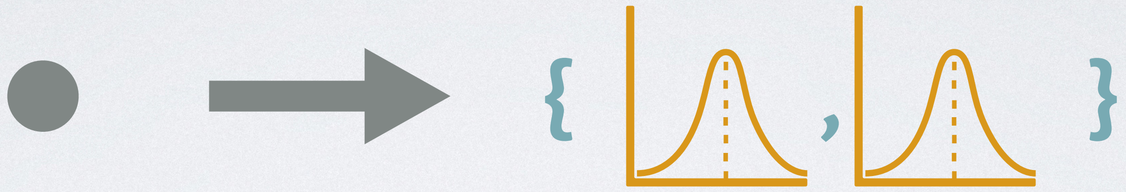
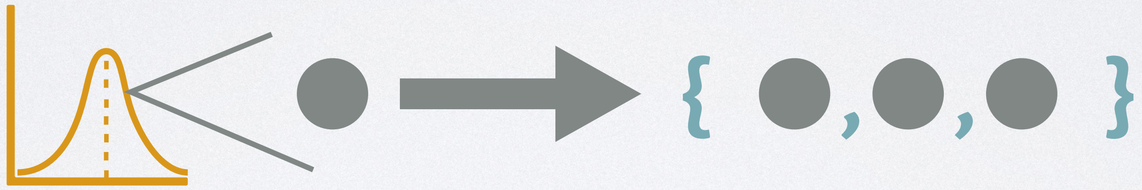


- Cousot's **Probabilistic Abstract Interpretation** (PAI): $\mathbb{D}(State \to \wp(State))$



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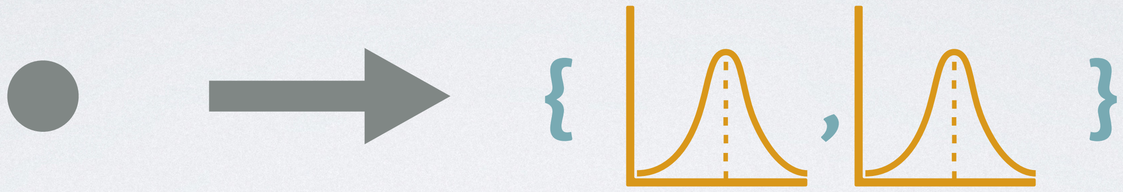

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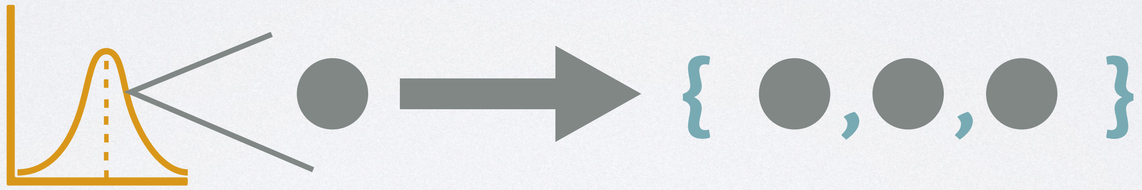
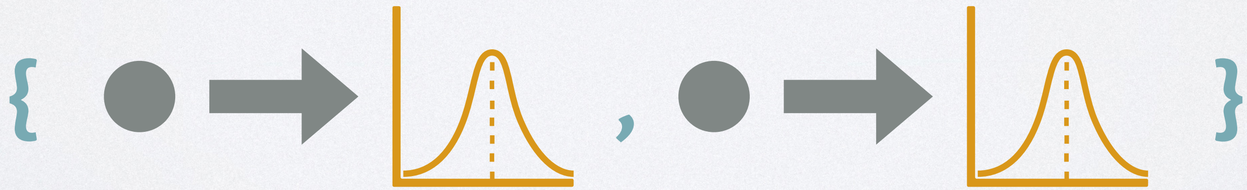


- **Compile-time/Relational** Nondeterminism: $\wp(State \to \mathbb{D}(State))$



Our Approach: Markov Algebras

- Key observation: Probabilistic programs have **multiple confluence operations**

$$\left\langle M, \sqsubseteq, \otimes, {}_{\phi}\oplus, \sqcap, \bot, \underline{1} \right\rangle$$

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Program state transformers
form a complete partial order

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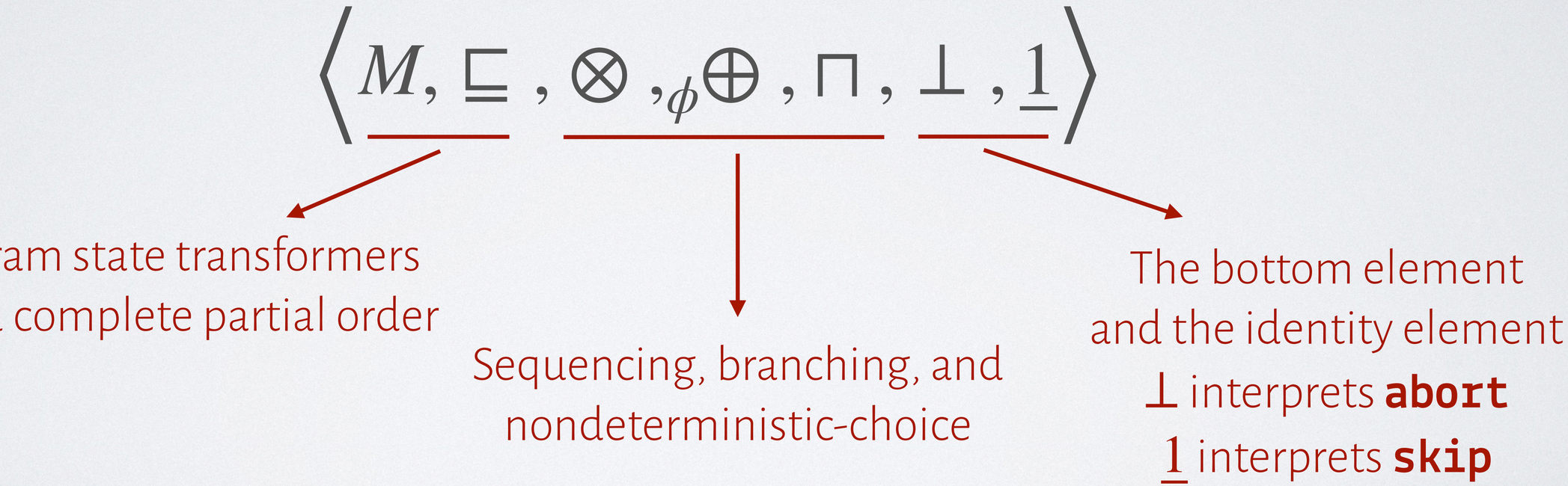

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$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$
$$a \otimes \underline{1} = \underline{1} \otimes a = a$$
$$a {}_{\phi}\oplus b = b {}_{\overline{\phi}}\oplus a$$
$$a \sqcap a = a$$
$$\dots$$

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$$\mu S \,.\, ((x \coloneqq x{+}1 \oplus_{1/2} x \coloneqq x{-}1) \otimes S)_{[x>0]} \oplus \mathbf{skip}$$

Towards a compositional and flexible framework for program analysis of probabilistic programs

☑ **Semantics:** Markov Algebras for Multiple Kinds of Confluence

Representation: Extend pGCL with unstructured control-flow

Algorithm: Solve program analyses in a non-iterative way

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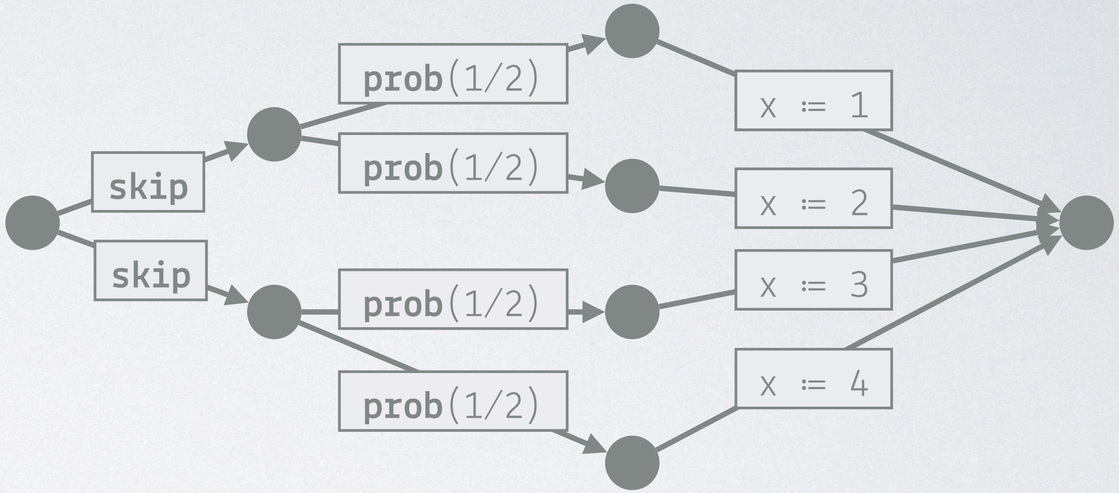

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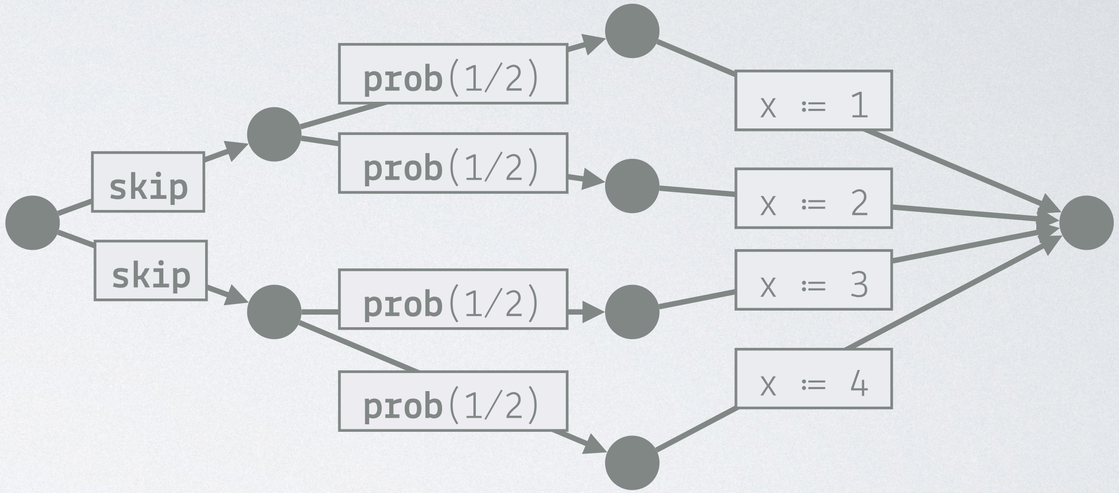

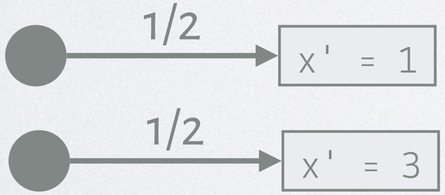

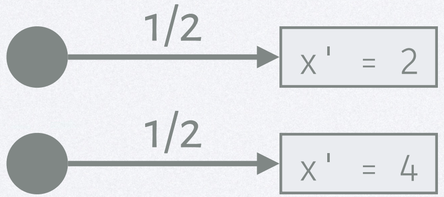

Probabilities sum up to 2!

Our Approach: Control-flow Hyper-graphs

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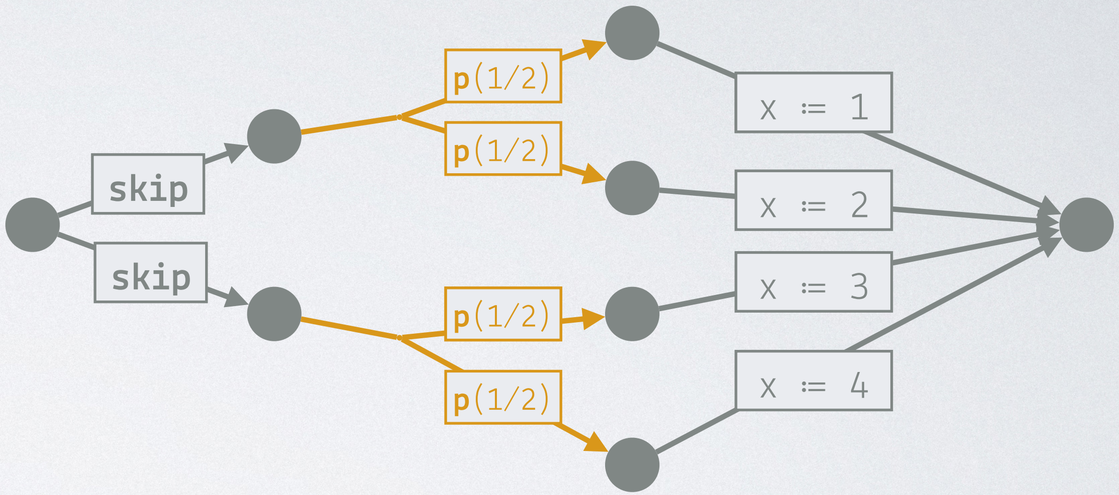

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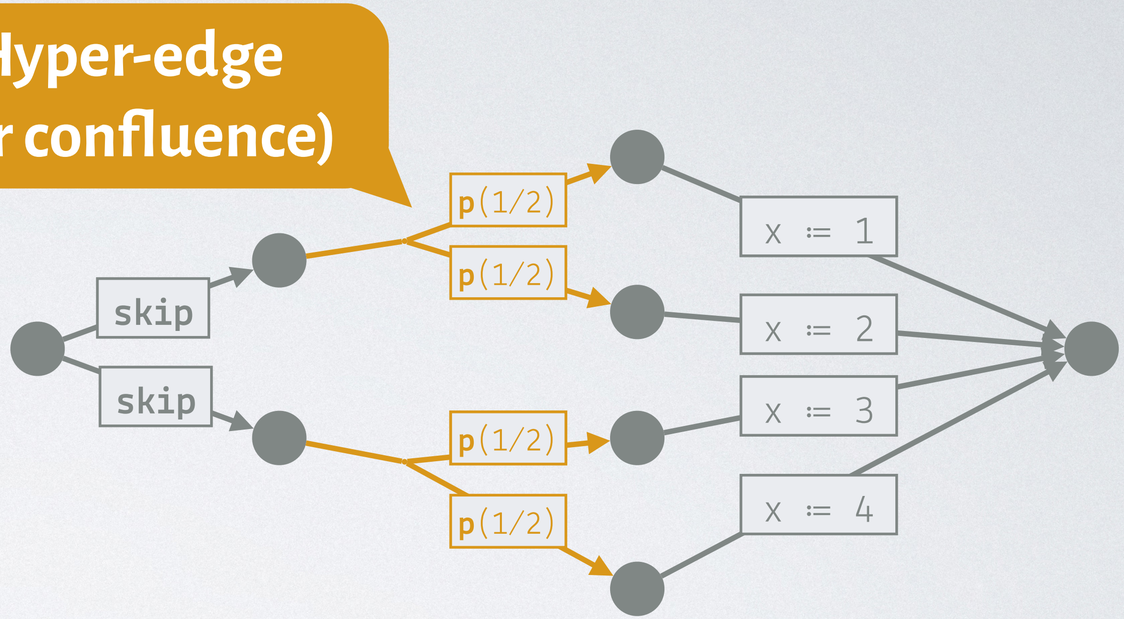

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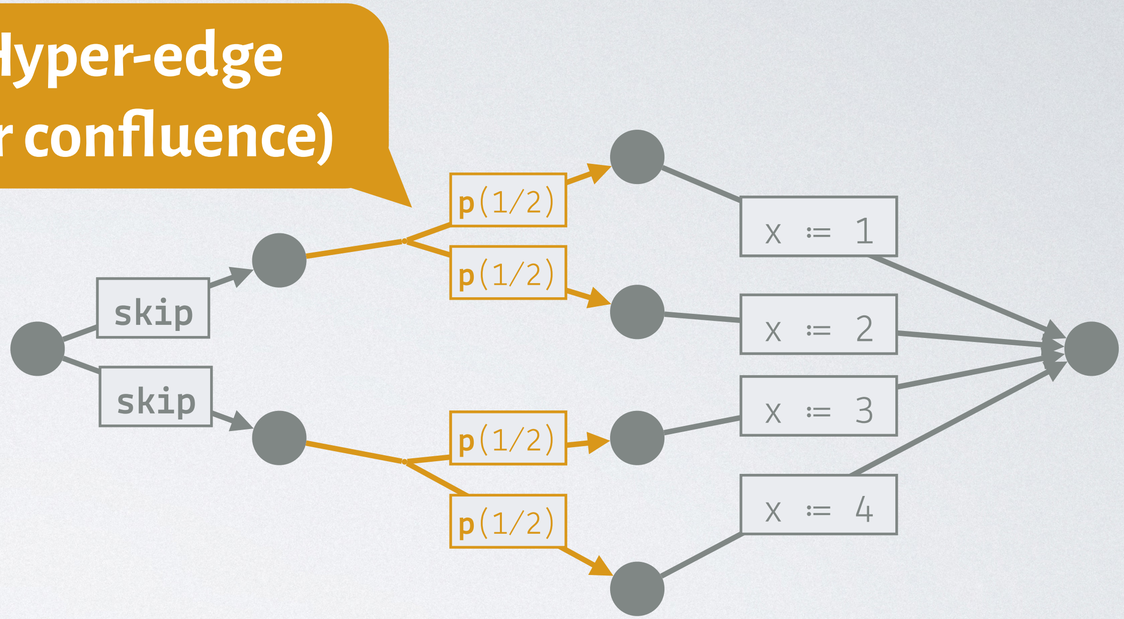

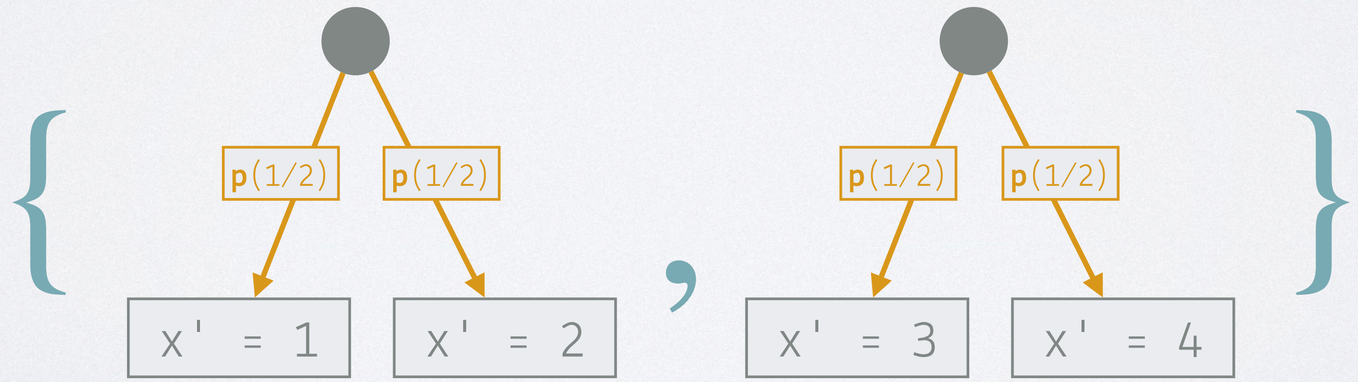

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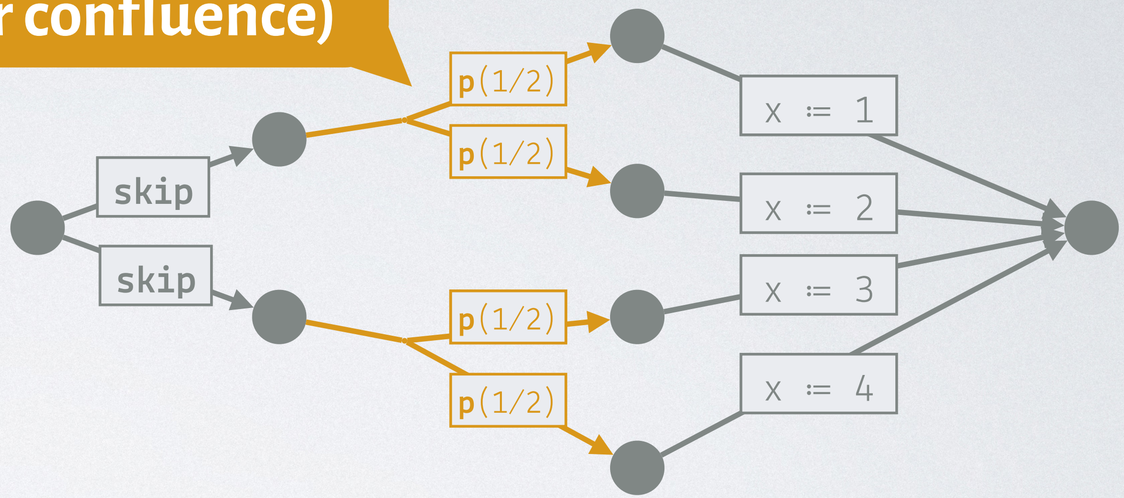
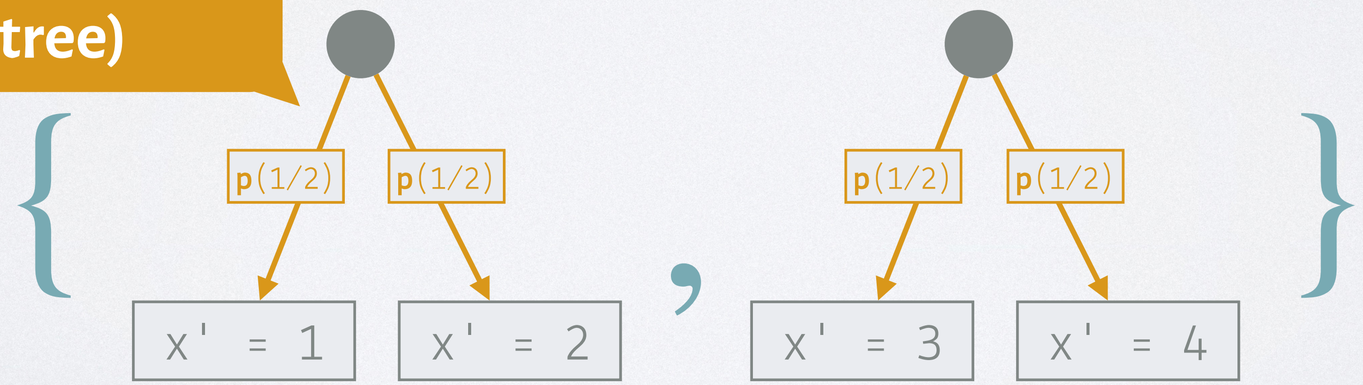
Hyper-edge (for confluence)



Hyper-path (like a tree)



Hyper-paths are Infinite Trees!

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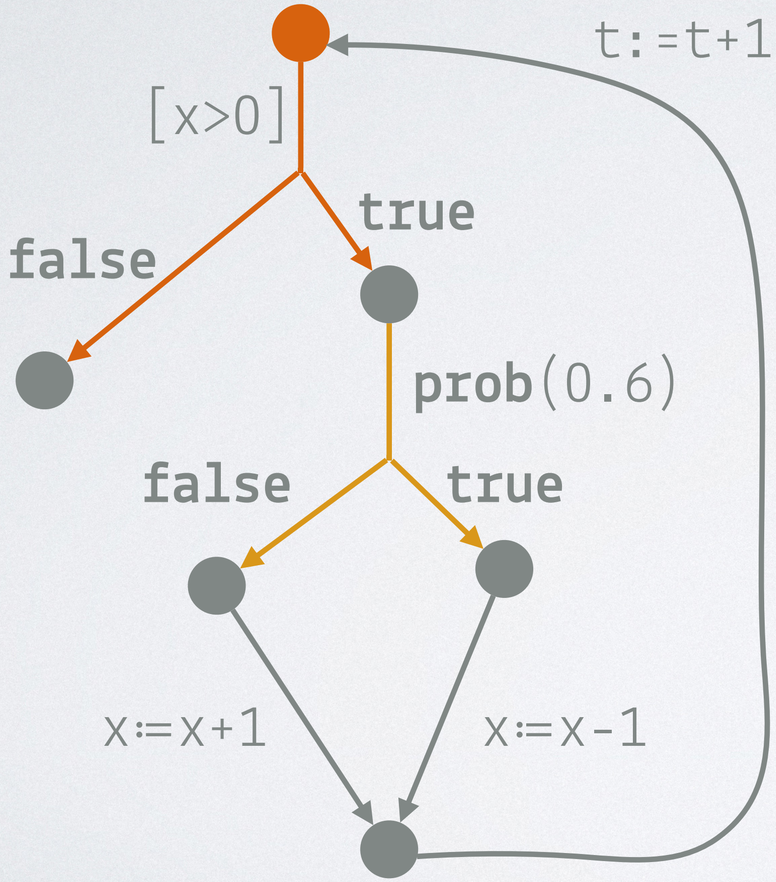

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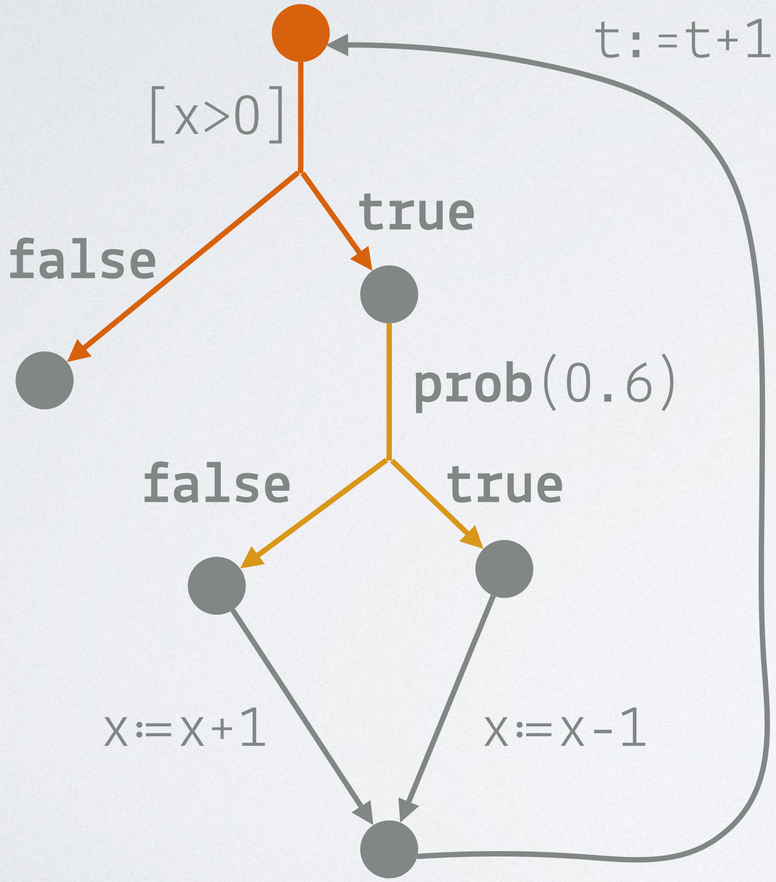
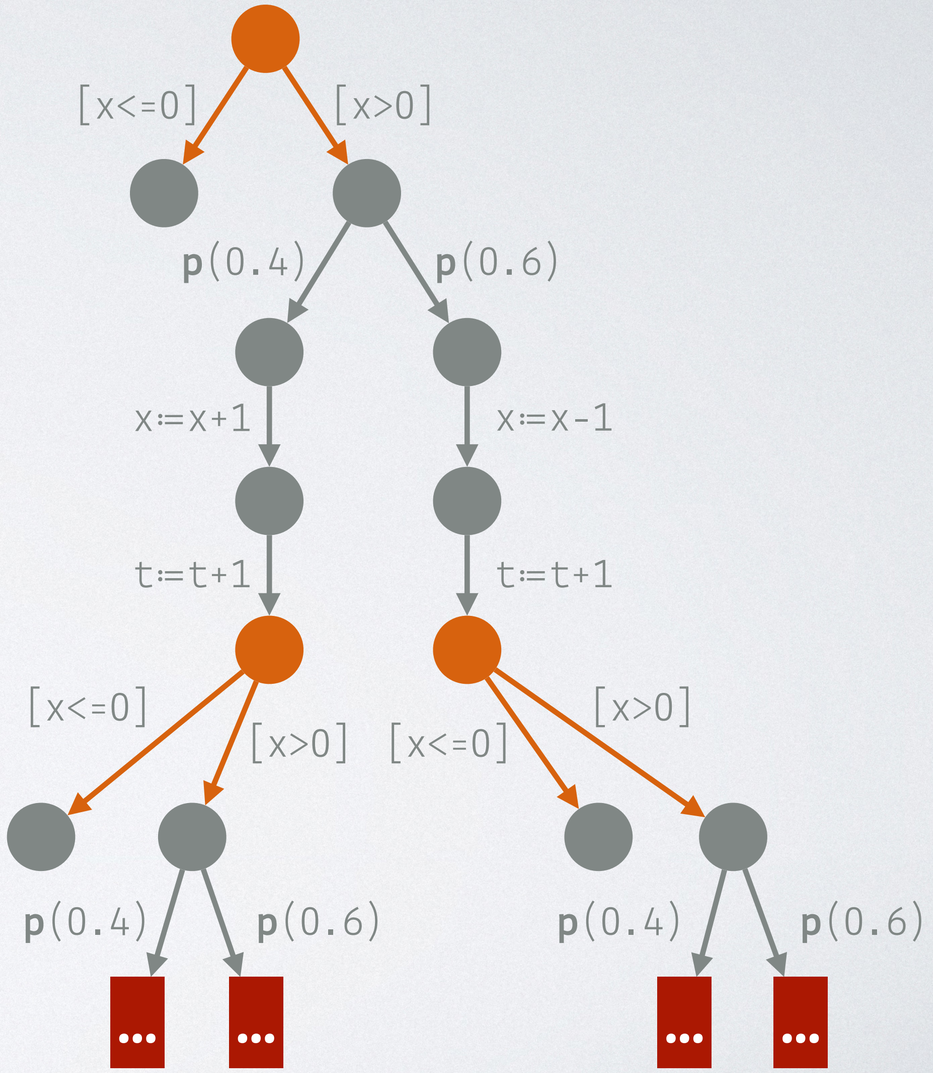
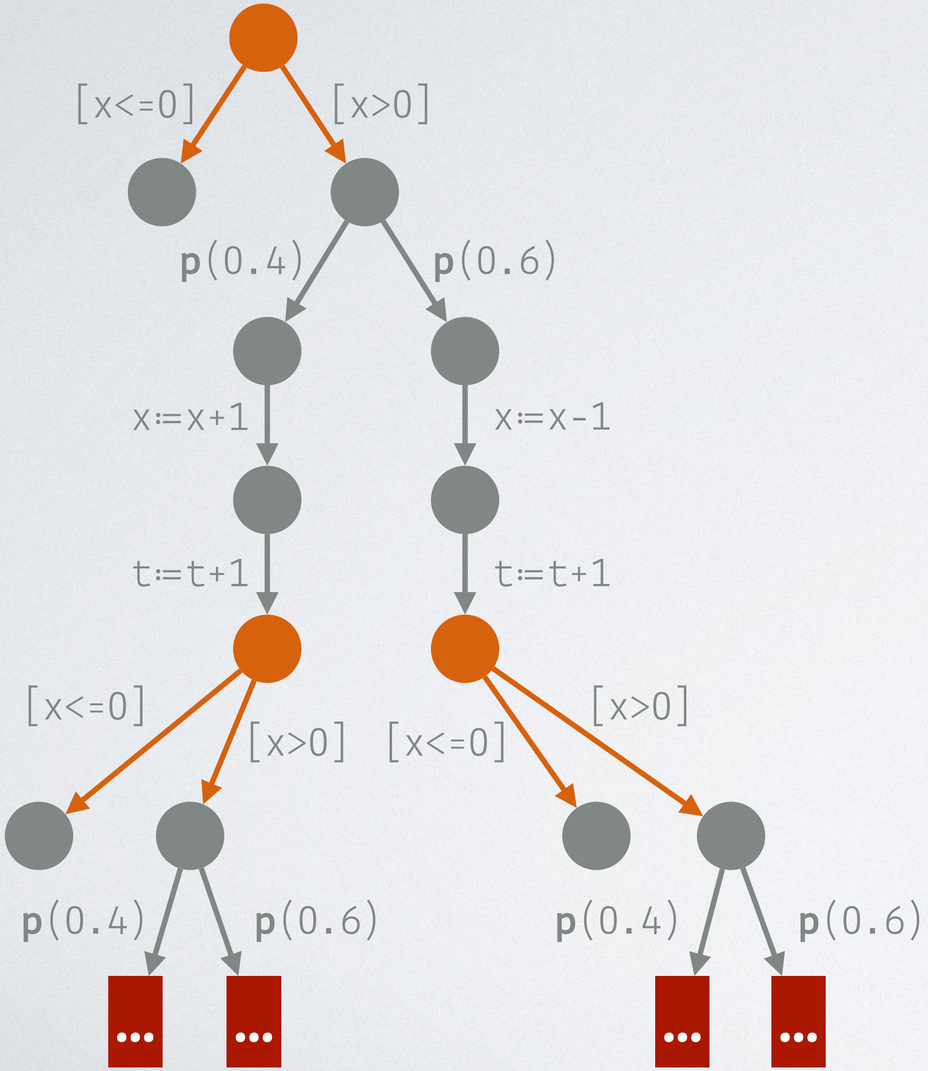




Regular Infinite-tree Expressions



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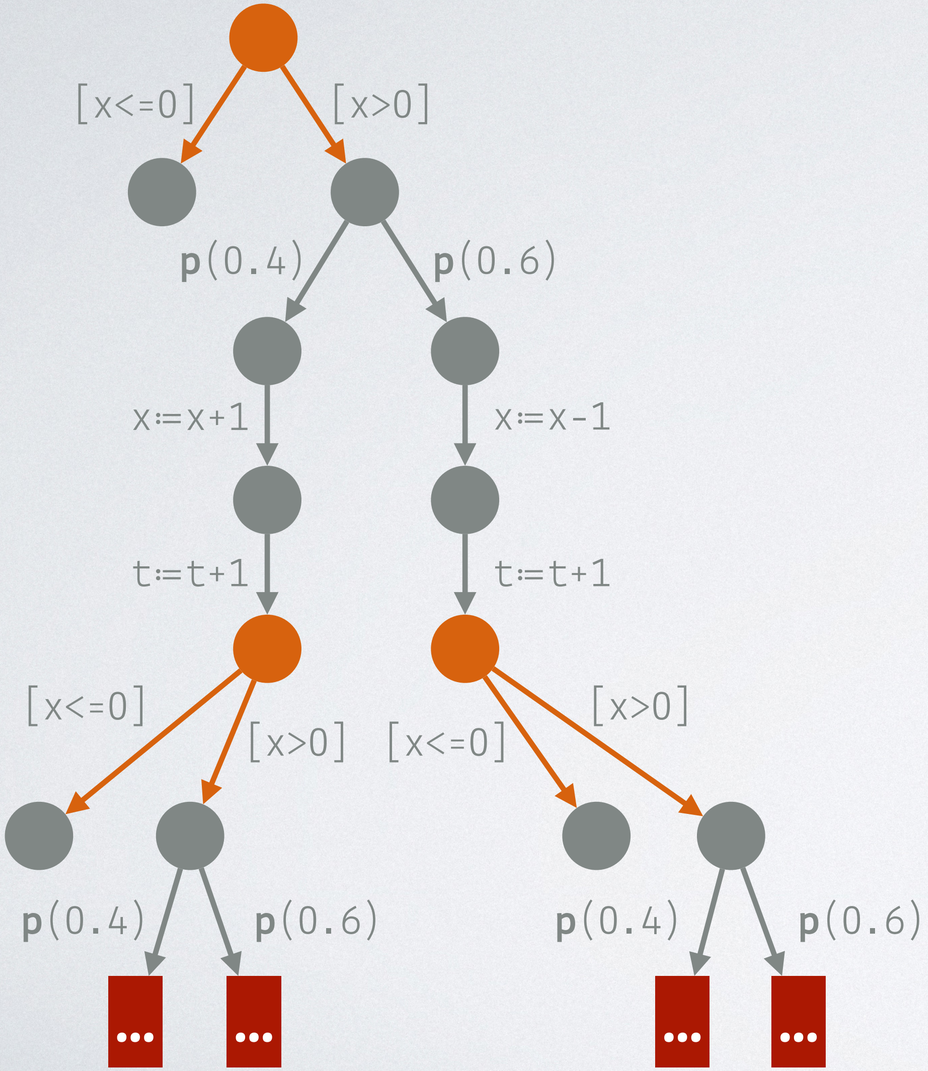

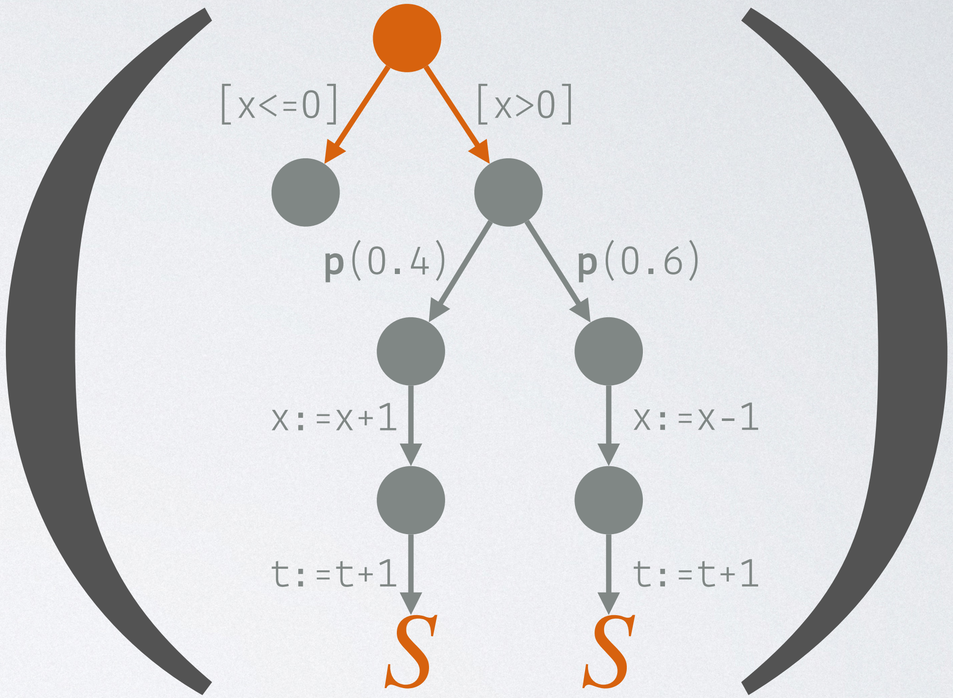

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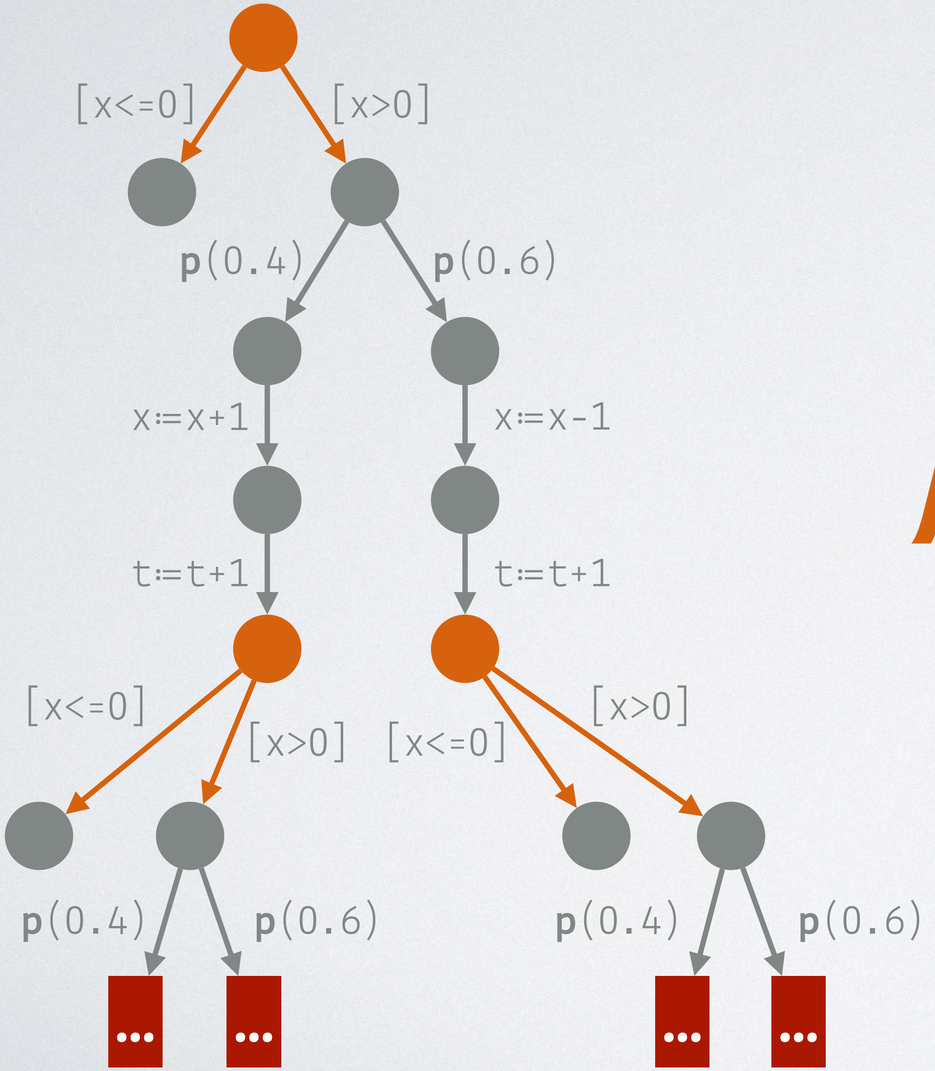
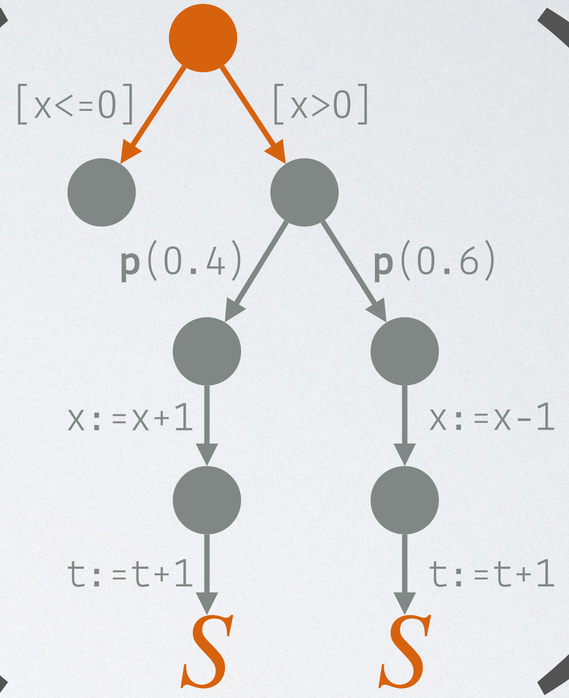


$$\mu S . \left(\right.$$



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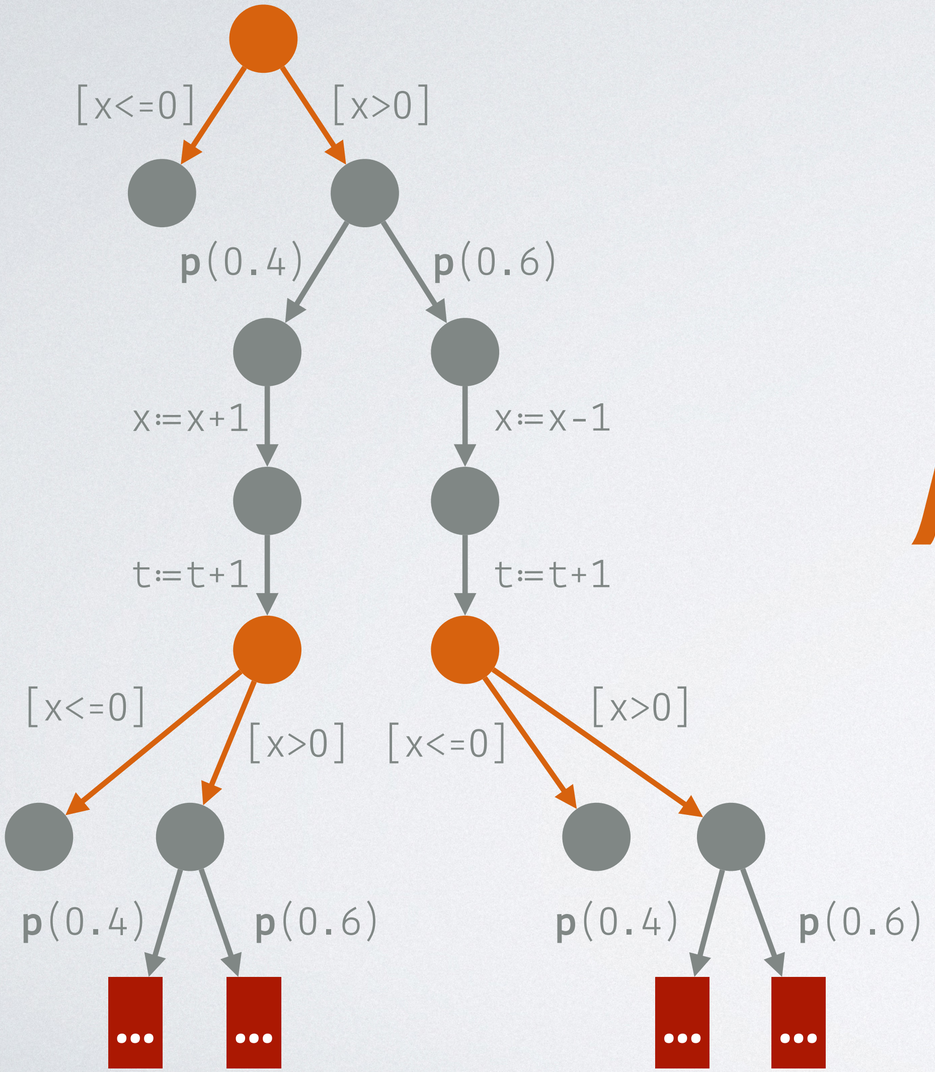

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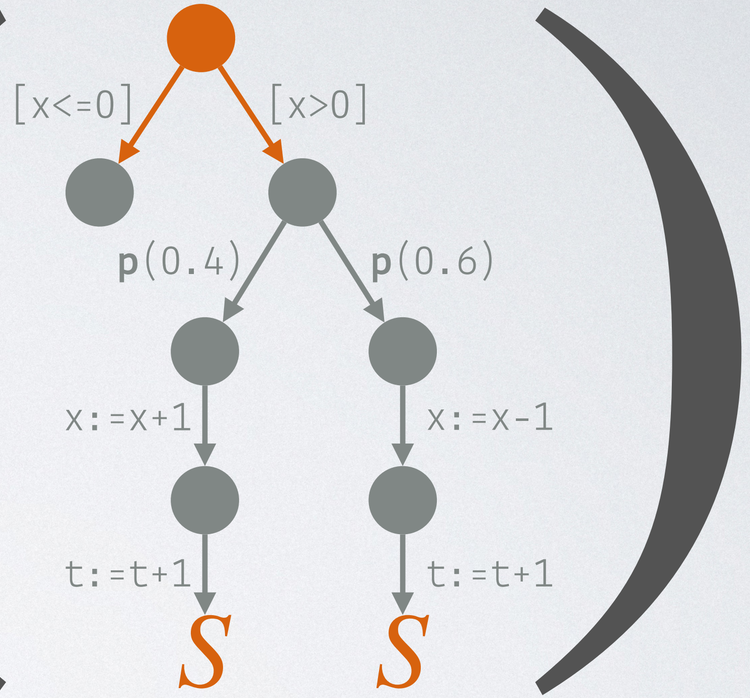

$$\mu S . \begin{pmatrix} \mathrm{cond}[x > 0](\mathrm{prob}[0.6](\mathrm{seq}[x := x - 1](\mathrm{seq}[t := t + 1](S)); \\ \mathrm{seq}[x := x + 1](\mathrm{seq}[t := t + 1](S))); \\ \underline{1}) \end{pmatrix}$$

Towards a compositional and flexible framework for program analysis of probabilistic programs

☑ **Semantics:** Markov Algebras for Multiple Kinds of Confluence

☑ **Representation:** Control-flow Hyper-graphs and Tree Expressions

Algorithm: Solve program analyses in a non-iterative way

Iterative Program Analysis

```
while prob(2/3) do
   x := x + 1
od
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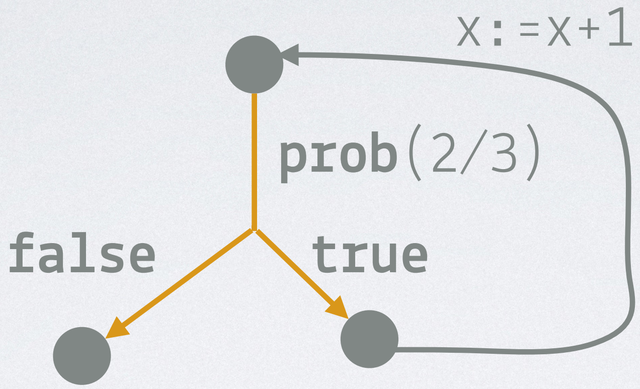

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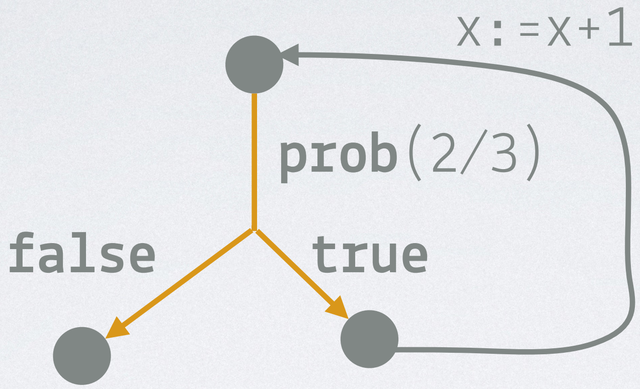

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  - Sequencing: $r_1 \otimes r_2 \triangleq r_1 + r_2$
  - Branching: $r_{1p} \otimes r_2 \triangleq p * r_1 + (1-p) * r_2$

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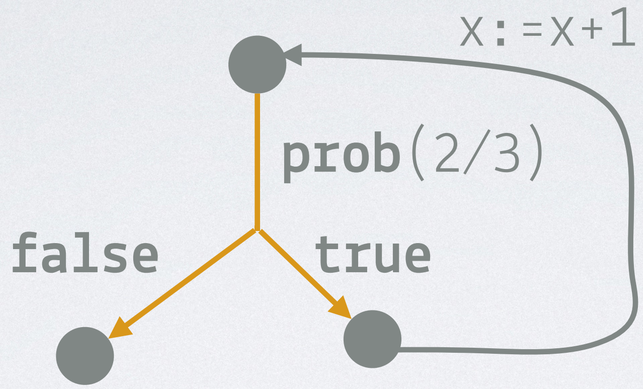

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$$\kappa^{(0)} = 0$$
$$\kappa^{(1)} = 2/3 * (1 + \kappa^{(0)}) + 1/3 * 0 = 2/3$$
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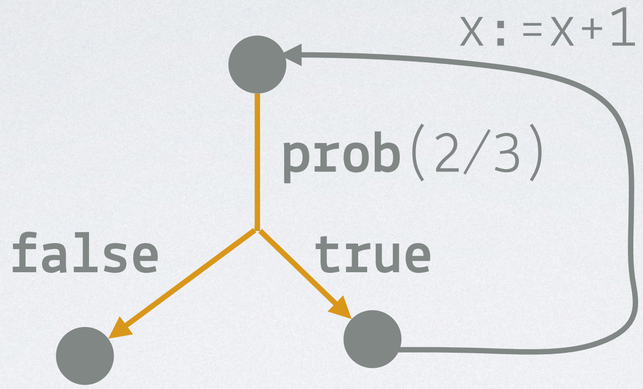

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Need ∞ iterations to converge!

Non-iterative Program Analysis

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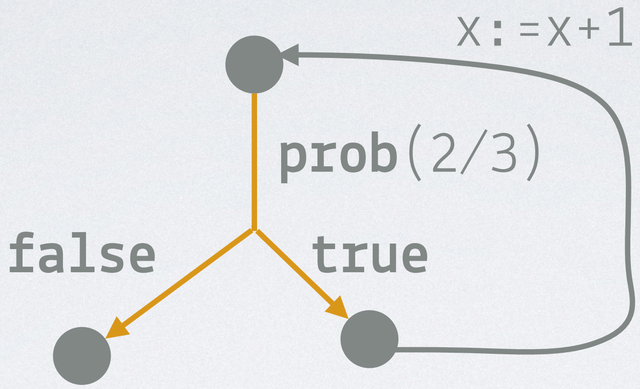

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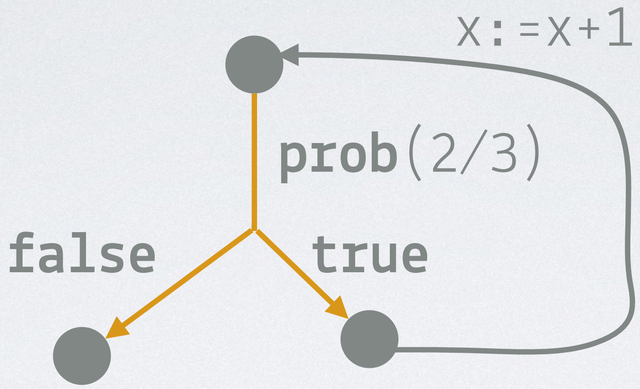

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This analysis is equivalent to solve

$$s = 2/3 * (1 + s) + 1/3 * 0,$$

which can be solve analytically:

$$s = 2$$

No need for iteration!

Inter-procedural Analysis

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proc X begin
  if prob(1/3) then
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  else
    call X;
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$$p = 1/3 * 1 + 2/3 * (p * p)$$

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Newtons's method!

Newton's Method Converges Faster!

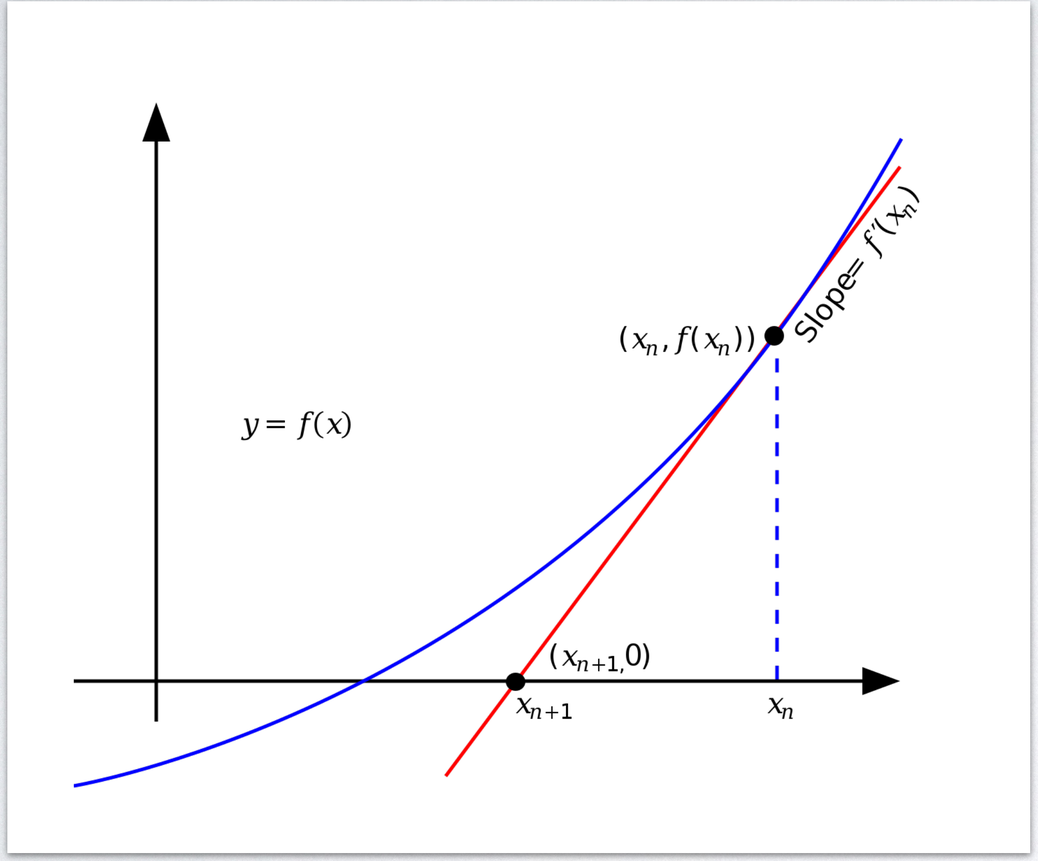

Image source: https://commons.wikimedia.org/wiki/File:Newton_iteration.svg.

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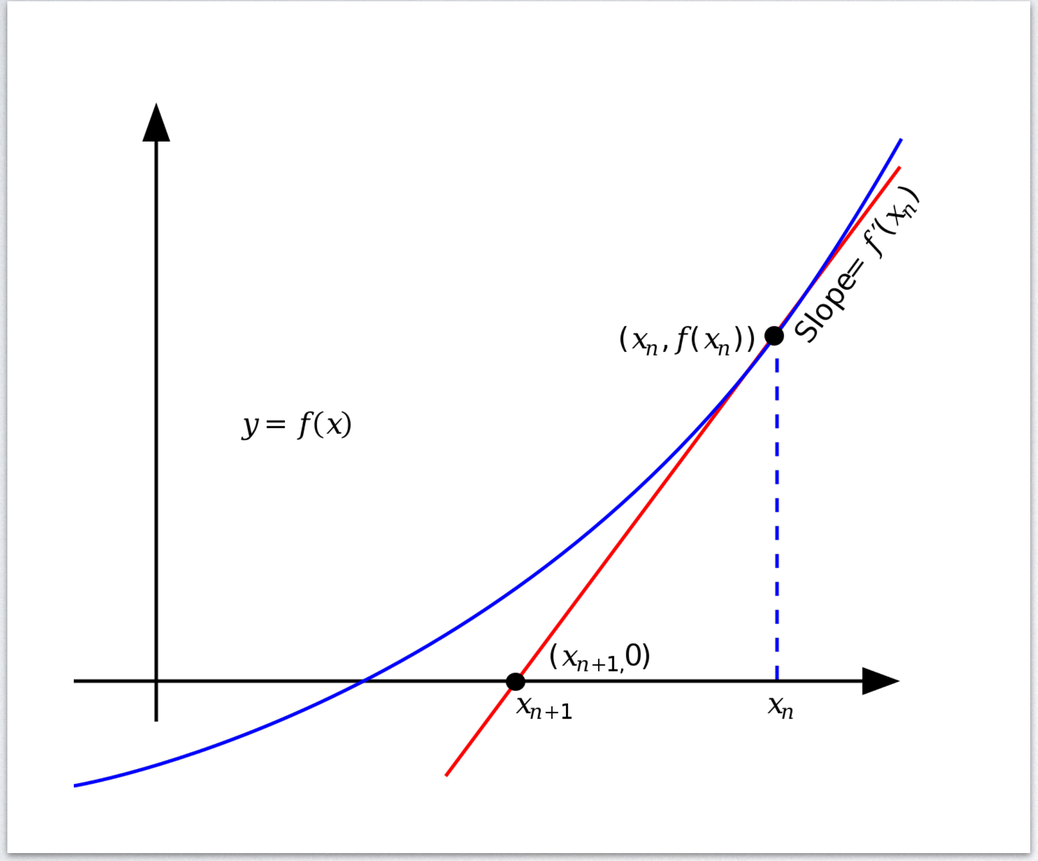

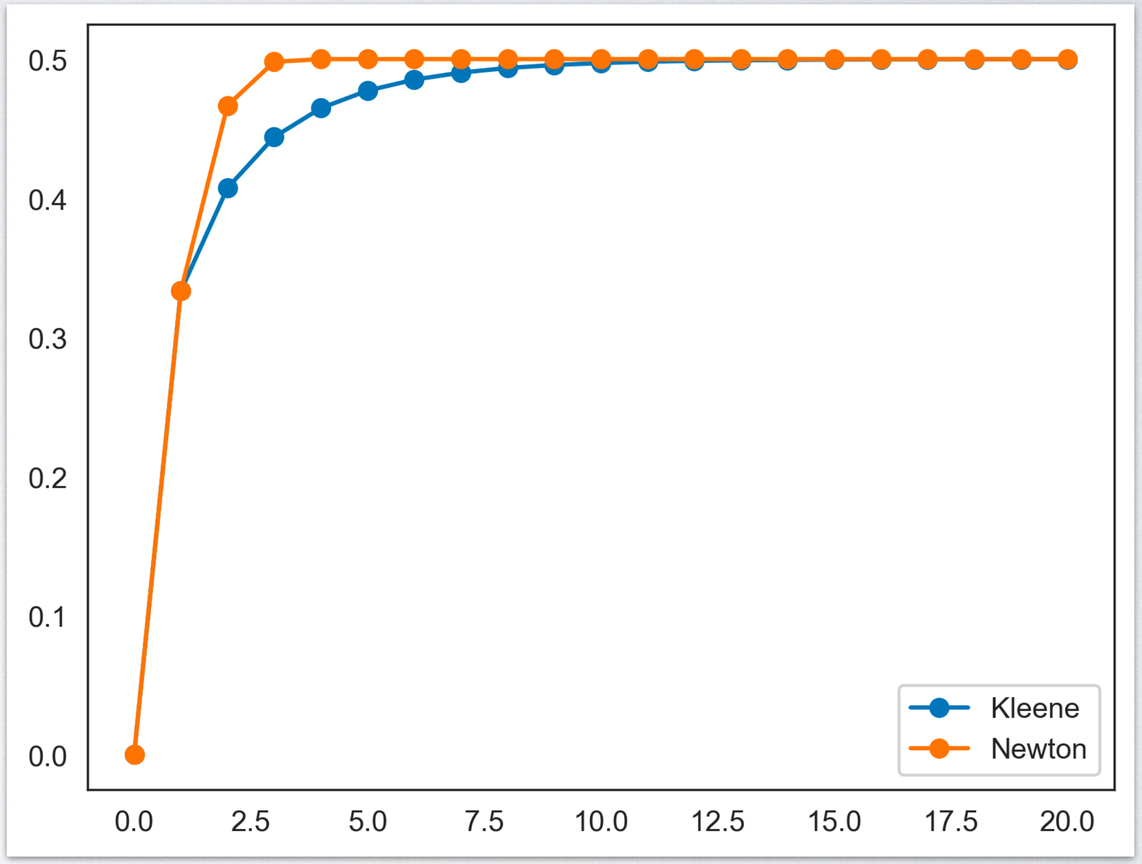

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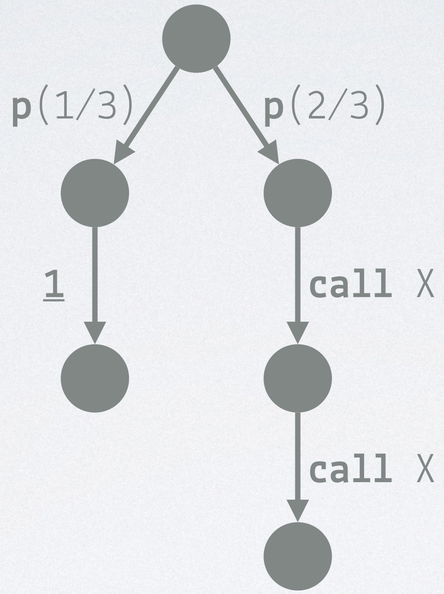

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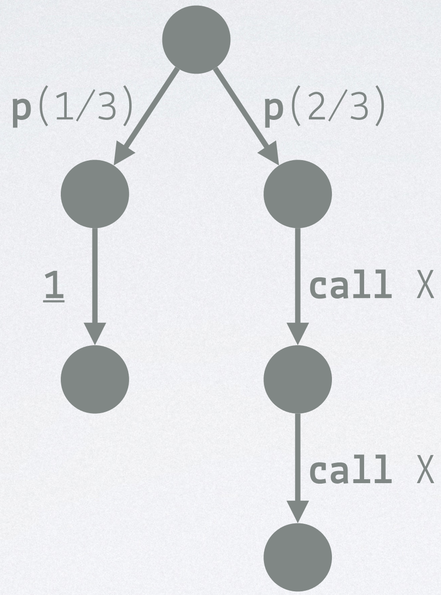
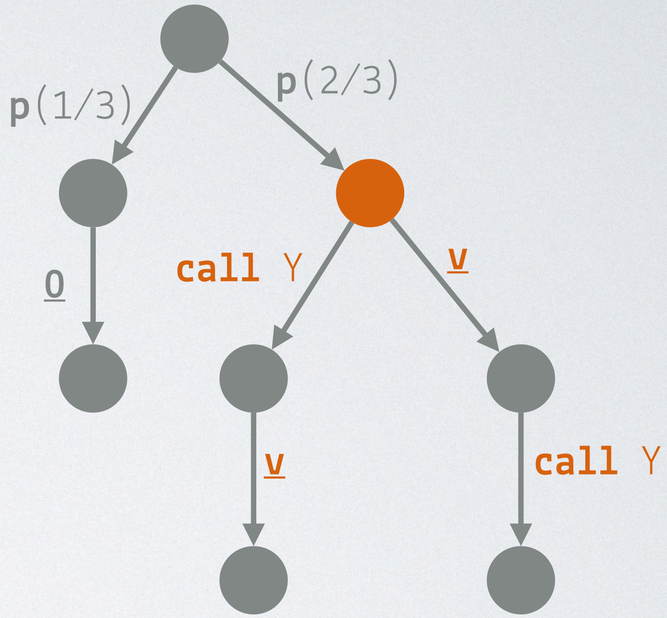


Differentiate



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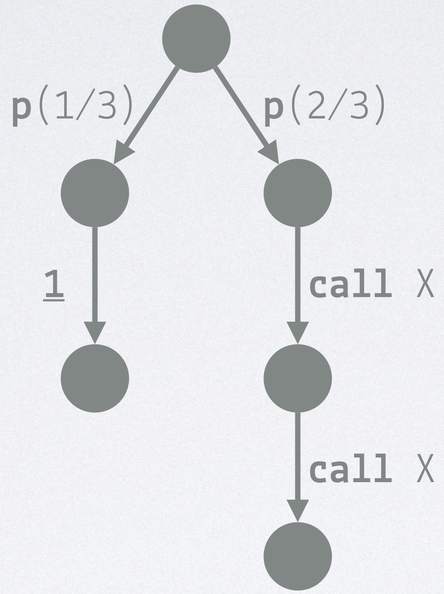

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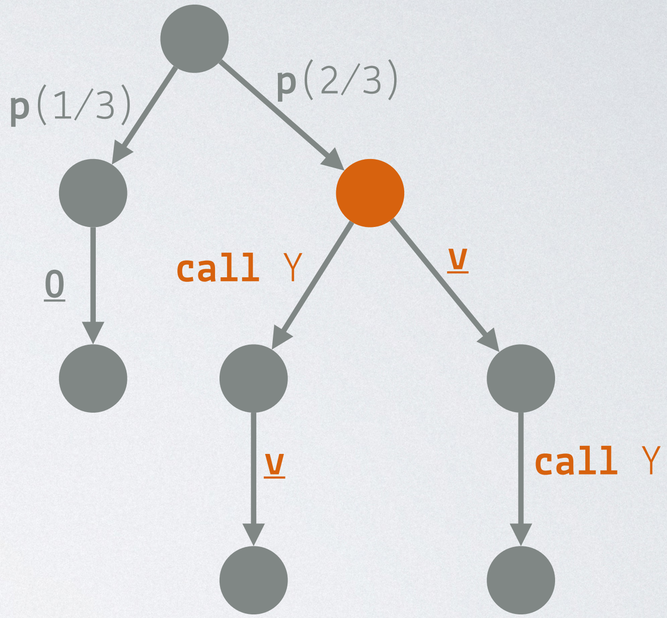

Every root-to-leaf path contains **at most one call**!

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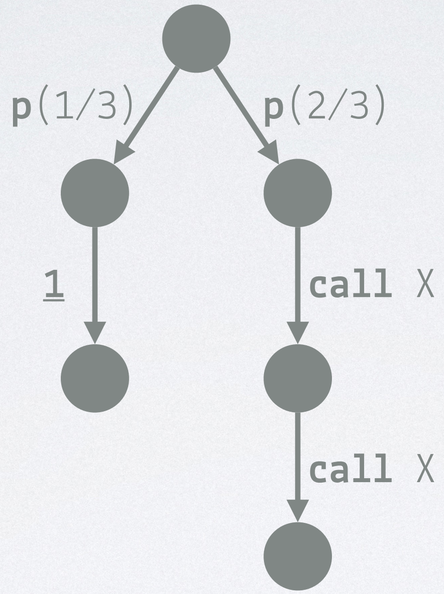

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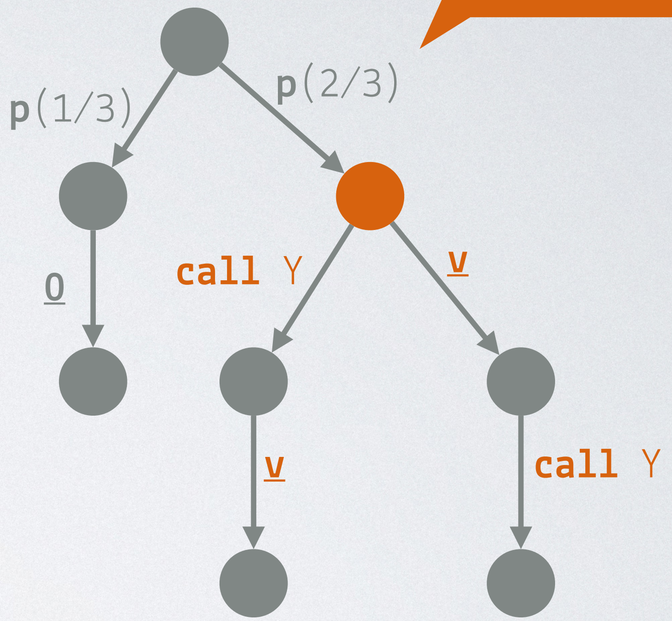

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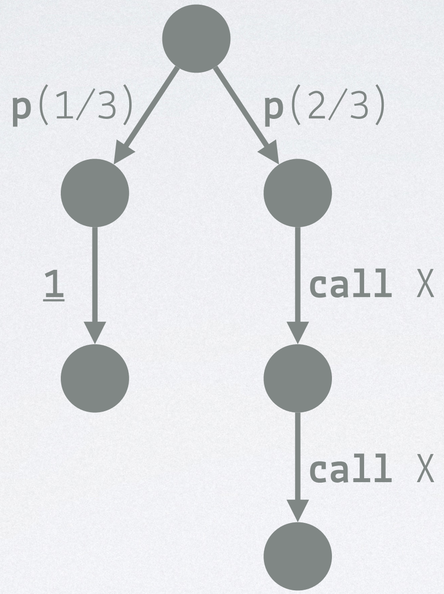

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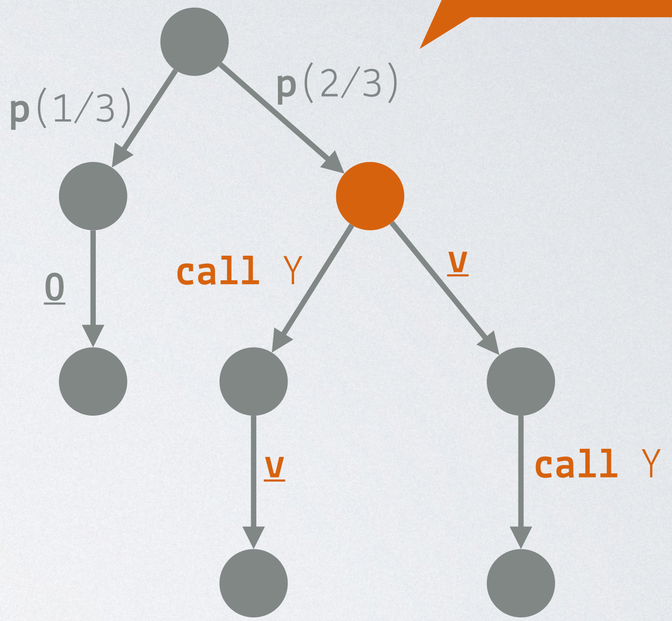

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$$\frac{d(f * g)}{dx} = \frac{df}{dx} * g + f * \frac{dg}{dx}$$

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Our Approach: NPA for Pre-Markov Algebras

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Equation System

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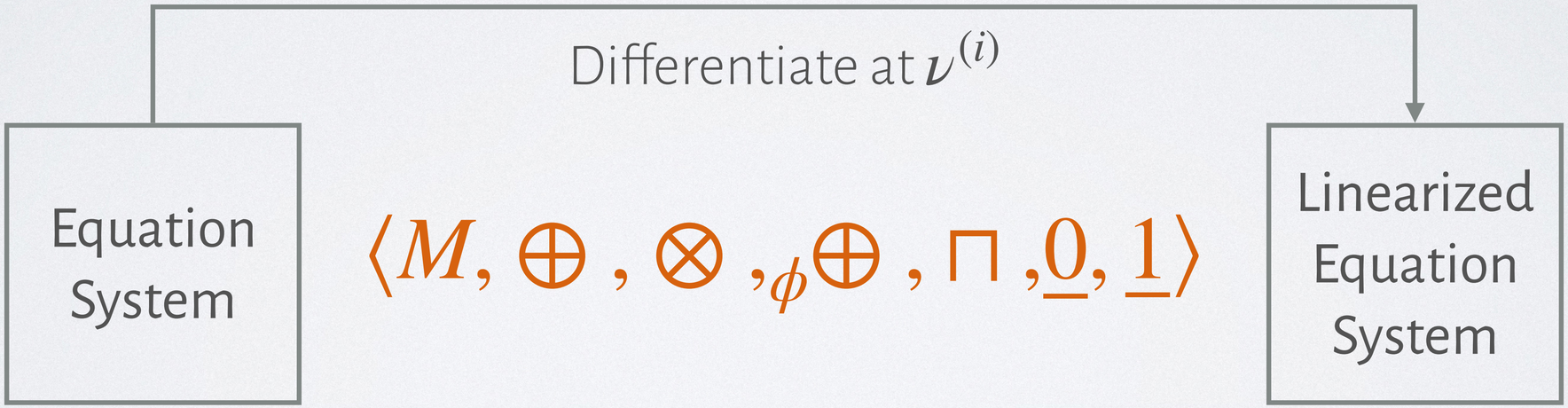

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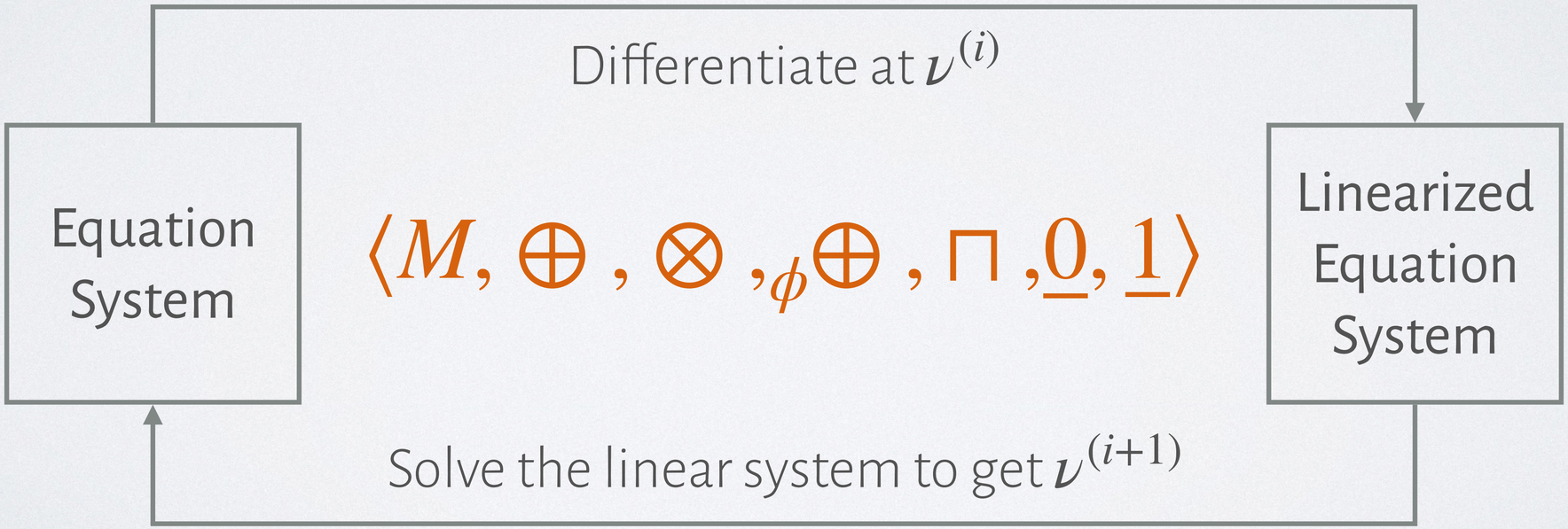
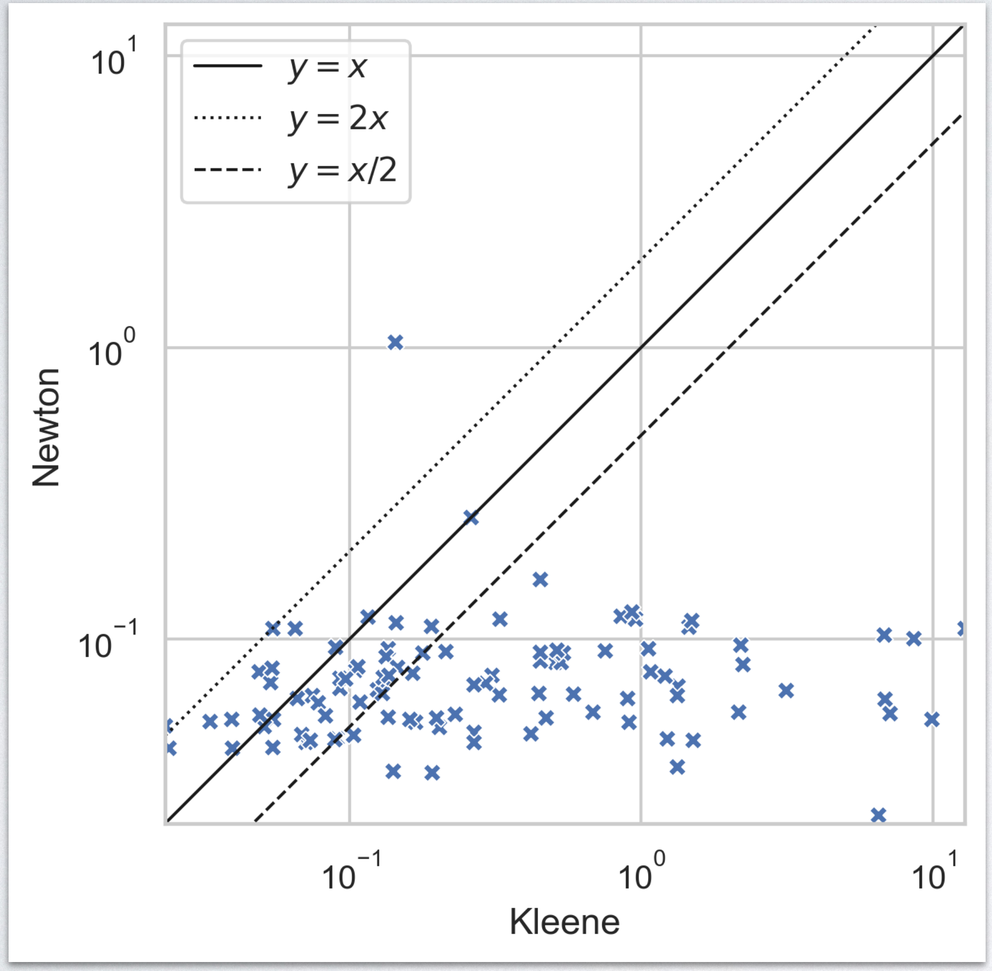
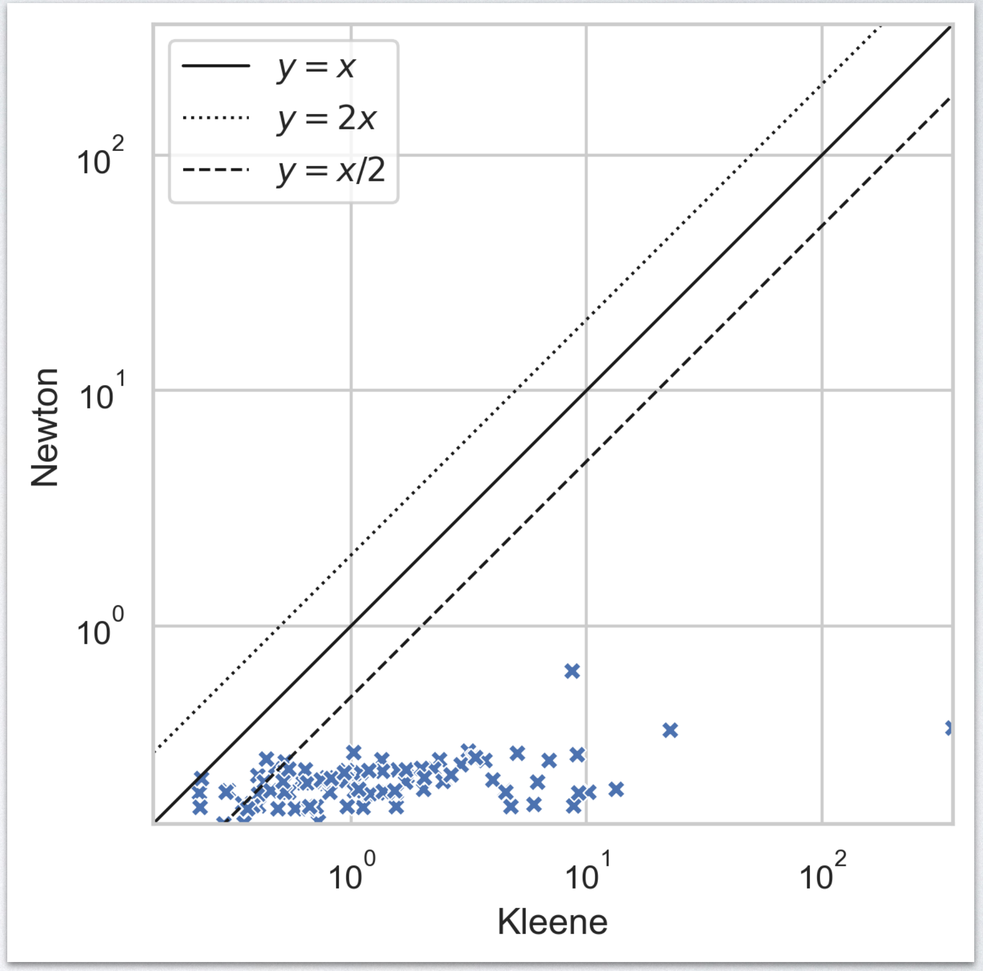


Preliminary Evaluation





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