Supplementary material for ”Effect of an orientation-dependent non-linear grain fluidity on bulk directional enhancement factors”

Nicholas M. Rathmann[1], Chrstine S. Hvidberg[1], Aslak Grinsted[1], David A. Lilien[1], Dorthe Dahl-Jensen[1]

[1]*Niels Bohr Institute, University of Copenhagen, Denmark*
Correspondence: Nicholas M. Rathmann <rathmann@nbi.ku.dk>

This supplementary material provides the derivation of both the transversely isotropic rheology (Supplementary A) and the spectral grain rotation model (Supplementary B) used in the main article. The notation follows that introduced in the main article.

SUPPLEMENTARY A: TRANSVERSELY ISOTROPIC RHEOLOGY

Starting from the flow rule of steady-state creep theory

$$\dot{\boldsymbol{\epsilon}} = \frac{\partial W(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}, \tag{A1}$$

where $\dot{\boldsymbol{\epsilon}} = (\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathsf{T}})/2$ is the strain-rate tensor and $W(\boldsymbol{\sigma})$ is the creep potential, constitutive equations may be constructed by demanding that W is unchanged under relevant coordinate (symmetry) transformations, $\mathbf{Q}$, in the sense that $W(\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^{\mathsf{T}}) = W(\boldsymbol{\sigma})$. Objectivity implies that W must depend on the stress-tensor invariants of the symmetry transformations. In the case of transverse isotropy, five unique stress-tensor invariants exist under coordinate transformations that leave the symmetry axis $\mathbf{m}$ unchanged (Naumenko and Altenbach, 2007):

$$I_1 = \mathrm{tr}(\boldsymbol{\sigma}), \quad I_2 = \mathrm{tr}(\boldsymbol{\sigma}^2), \quad I_3 = \mathrm{tr}(\boldsymbol{\sigma}^3),$$
$$I_4 = \boldsymbol{\sigma}\cdot\cdot\,\mathbf{mm}, \quad I_5 = \boldsymbol{\sigma}^2\cdot\cdot\,\mathbf{mm}.$$

Consider expressing the dependence of W on $I_1,\cdots,I_5$ in terms of an effective stress, $\sigma_{\mathrm{E}}(I_1,\cdots,I_5)$, such that $W(\boldsymbol{\sigma}) = W(\sigma_{\mathrm{E}}(I_1,\cdots,I_5))$. If one posits that the relationship between the effective stress and effective strain rate is a power law corresponding to a Norton–Bailey creep potential, implying $\partial W(\sigma_{\mathrm{E}})/\partial\sigma_{\mathrm{E}} = A\sigma_{\mathrm{E}}^n = \dot{\epsilon}_{\mathrm{E}}$, then (A1) becomes

$$\dot{\boldsymbol{\epsilon}} = \frac{\partial W(\sigma_{\mathrm{E}})}{\partial\sigma_{\mathrm{E}}}\frac{\partial\sigma_{\mathrm{E}}}{\partial\boldsymbol{\sigma}} = \frac{A}{2}\sigma_{\mathrm{E}}^{n-1}\frac{\partial\sigma_{\mathrm{E}}^2}{\partial\boldsymbol{\sigma}}, \tag{A2}$$

where the chain rule was used twice.

In the general-most case, the functional form of $\sigma_{\mathrm{E}}(I_1,\cdots,I_5)$ is taken to be a linear combination of products of $I_1,\cdots,I_5$ resulting in first-, second- and third-order dependence on $\boldsymbol{\sigma}$, with square and cubic roots taken of the second- and third-order terms, respectively (Naumenko and Altenbach, 2007). Disregarding nonclassical effects (no third-order dependencies on $\boldsymbol{\sigma}$) and requiring conformity with Glen–Nye's law in the isotropic limit (no first-order dependencies on $\boldsymbol{\sigma}$), the effective stress becomes

$$\sigma_{\mathrm{E}}^2 = \lambda_1 I_1^2 + \lambda_2 I_2 + \lambda_3 I_1 I_4 + \lambda_4 I_4^2 + \lambda_5 I_5, \tag{A3}$$

where $\lambda_1,\cdots,\lambda_5$ are free material parameters.

By imposing incompressibility, a reduction of the number of free material parameters is possible. Let us adopt the usual decomposition

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - p\mathbf{I}, \tag{A4}$$

where $\boldsymbol{\tau}$ is the deviatoric stress, $p = -\mathrm{tr}(\boldsymbol{\sigma})/d$ is the pressure, and $d = \dim(\mathbf{u})$ is the dimensionality of the problem.

Applying the stress decomposition to (A2) requires in turn applying it to $\sigma_{\mathrm{E}}^{n-1}$ and $\partial\sigma_{\mathrm{E}}^2\big/\partial\boldsymbol{\sigma}$. In the case of σ_{E}^2, one finds

$$\sigma_{\mathrm{E}}^2 = F(p) + \lambda_2 I_2(\boldsymbol{\tau}) + \lambda_4 I_4^2(\boldsymbol{\tau}) + \lambda_5 I_5(\boldsymbol{\tau}), \tag{A5}$$

where terms depending on p have been grouped into $F(p)$:

$$F(p) = -(d\lambda_3 + 2\lambda_4 + 2\lambda_5)\,pI_4(\boldsymbol{\tau}) + \left(d^2\lambda_1 + d\lambda_2 + d\lambda_3 + \lambda_4 + \lambda_5\right)p^2. \tag{A6}$$

In the case of $\partial\sigma_{\mathrm{E}}^2\big/\partial\boldsymbol{\sigma}$, the derivatives of the invariants must be calculated, yielding

$$\frac{\partial I_1}{\partial\boldsymbol{\sigma}} = \mathbf{I}, \quad \frac{\partial I_1^2}{\partial\boldsymbol{\sigma}} = 2I_1\mathbf{I}, \quad \frac{\partial I_2}{\partial\boldsymbol{\sigma}} = 2\boldsymbol{\sigma},$$
$$\frac{\partial I_4}{\partial\boldsymbol{\sigma}} = \mathbf{mm}, \quad \frac{\partial I_4^2}{\partial\boldsymbol{\sigma}} = 2I_4\mathbf{mm}, \quad \frac{\partial I_5}{\partial\boldsymbol{\sigma}} = \{\boldsymbol{\sigma},\mathbf{mm}\},$$

where the anti-commutator is defined as

$$\{\boldsymbol{\sigma},\mathbf{mm}\} = \boldsymbol{\sigma}\cdot\mathbf{mm} + \mathbf{mm}\cdot\boldsymbol{\sigma}. \tag{A7}$$

It follows using (A4) that

$$\frac{\partial\sigma_{\mathrm{E}}^2}{\partial\boldsymbol{\sigma}} = G(p) + 2\lambda_2\boldsymbol{\tau} + \lambda_3 I_4(\boldsymbol{\tau})\mathbf{I} + 2\lambda_4 I_4(\boldsymbol{\tau})\mathbf{mm} + \lambda_5\{\boldsymbol{\tau},\mathbf{mm}\}, \tag{A8}$$

where terms depending on p have been grouped into $G(p)$:

$$G(p) = -(d\lambda_3 + 2\lambda_4 + 2\lambda_5)\,p\mathbf{mm} - (2d\lambda_1 + 2\lambda_2 + \lambda_3)\,p\mathbf{I}. \tag{A9}$$

Incompressibility implies no dependence on the pressure terms $F(p)$ and $G(p)$, and each parenthesis must therefore vanish in (A6) and (A9). Solving the resulting set of equations, it follows that λ_3 is constrained by $\lambda_3 = -2(\lambda_4+\lambda_5)/d$, and the rheology upon combining (A5) and (A8) with (A2), becomes (Johnson, 1977)

$$\dot{\boldsymbol{\epsilon}} = \eta^{-1}\left(\lambda_2\boldsymbol{\tau} - \frac{\lambda_4+\lambda_5}{d}I_4\mathbf{I} + \lambda_4 I_4\mathbf{mm} + \frac{\lambda_5}{2}\{\boldsymbol{\tau},\mathbf{mm}\}\right), \quad \text{(A10)}$$

$$\eta^{-1} = A\left(\lambda_2 I_2 + \lambda_4 I_4^2 + \lambda_5 I_5\right)^{(n-1)/2}, \quad \text{(A11)}$$

where $I_i = I_i(\boldsymbol{\tau})$ is implied.

Interpretation of material parameters

The rheology (A10)–(A11) may be posed in form that is more relevant to glaciology by expressing λ_2, λ_4 and λ_5 in terms of an isotropic rate factor (A) and enhancement factors of the longitudinal and shear components of $\dot{\boldsymbol{\epsilon}}$ w.r.t. $\mathbf{m}$. If a factor of $\lambda_2^{(n+1)/2}$ is absorbed into A, calculating the longitudinal and shear components w.r.t. $\mathbf{m}$, gives

$$\dot{\epsilon}_{mm} = \eta^{-1}\left(1 + (1-1/d)\left(\lambda_4/\lambda_2 + \lambda_5/\lambda_2\right)\right)\tau_{mm}, \quad \text{(A12)}$$

$$\dot{\epsilon}_{mt} = \eta^{-1}\left(1 + (\lambda_5/\lambda_2)/2\right)\tau_{mt}, \quad \text{(A13)}$$

where $\mathbf{t} \perp \mathbf{m}$, which suggests defining the two enhancement factors as

$$E_{mm} = 1 + (1-1/d)\left(\lambda_4/\lambda_2 + \lambda_5/\lambda_2\right), \quad \text{(A14)}$$

$$E_{mt} = 1 + (\lambda_5/\lambda_2)/2. \quad \text{(A15)}$$

Expressing λ_4/λ_2 and λ_5/λ_2 in terms of E_{mm} and E_{mt}, the rheology (A10)–(A11) takes the form

$$\dot{\boldsymbol{\epsilon}} = \eta^{-1}\Bigg(\boldsymbol{\tau} - \frac{E_{mm}-1}{d-1}I_4\mathbf{I} + \left[\frac{d\,(E_{mm}+1)-2}{d-1} - 2E_{mt}\right]I_4\mathbf{mm} + (E_{mt}-1)\{\boldsymbol{\tau},\mathbf{mm}\}\Bigg), \quad \text{(A16)}$$

$$\eta^{-1} = A\Bigg(I_2 + \left[\frac{d\,(E_{mm}+1)-2}{d-1} - 2E_{mt}\right]I_4^2 + 2\,(E_{mt}-1)\,I_5\Bigg)^{(n-1)/2}. \quad \text{(A17)}$$

In the main text, we consider the three-dimensional problem ($d = 3$) with a vertical symmetry axis ($\mathbf{m} = \hat{\mathbf{z}}$). For this special case, (A16) becomes

$$\dot{\boldsymbol{\epsilon}}(\mathbf{m}=\hat{\mathbf{z}}) = \eta^{-1}\begin{pmatrix} \tau_{xx} + \frac{1}{2}(1-E_{mm})\tau_{zz} & \tau_{xy} & E_{mt}\tau_{xz} \\ \tau_{xy} & \tau_{yy} + \frac{1}{2}(1-E_{mm})\tau_{zz} & E_{mt}\tau_{yz} \\ E_{mt}\tau_{xz} & E_{mt}\tau_{yz} & E_{mm}\tau_{zz} \end{pmatrix}. \quad \text{(A18)}$$

Notice that shear strain rates are not enhanced in the plane of isotropy.

Inverse rheology

Posing the rheology (A10)–(A11) in a closed inverse form, $\boldsymbol{\tau}(\dot{\boldsymbol{\epsilon}})$, is algebraically tedious compared to inverting the Glen–Nye isotropic law. While inverting the isotropic law amounts to solving one equation with one unknown, inverting (A10)–(A11) requires solving an anticommutator matrix equation followed by three equations with three unknowns due to the existence of three invariants.

Starting out by collecting the terms in (A10) that depend tensorially on $\boldsymbol{\tau}$, yields

$$\lambda_2\boldsymbol{\tau} + \frac{\lambda_5}{2}\{\boldsymbol{\tau},\mathbf{mm}\} = \eta(\boldsymbol{\tau})\dot{\boldsymbol{\epsilon}} - \lambda_4 I_4(\boldsymbol{\tau})\mathbf{mm} + \frac{\lambda_4+\lambda_5}{d}I_4(\boldsymbol{\tau})\mathbf{I},$$

which is an anticommutator matrix equation with respect to $\boldsymbol{\tau}$. Its solution requires vectorizing each term by the stacking columns according to

$$\mathcal{V}(X_{ij}) = (X_{11}, X_{21}, X_{31}, X_{12}, \cdots, X_{33}),$$

giving

$$\mathbf{P}_{d^2}\cdot\mathcal{V}(\boldsymbol{\tau}) = \eta(\boldsymbol{\tau})\mathcal{V}(\dot{\boldsymbol{\epsilon}}) - \lambda_4 I_4(\boldsymbol{\tau})\mathcal{V}(\mathbf{mm}) + \frac{\lambda_4+\lambda_5}{d}I_4(\boldsymbol{\tau})\mathcal{V}(\mathbf{I}), \quad \text{(A19)}$$

where

$$\mathbf{P}_{d^2} = \lambda_2\mathbf{I}_{d^2} + \frac{\lambda_5}{2}\left(\mathbf{mm}\otimes\mathbf{I}_d + \mathbf{I}_d\otimes\mathbf{mm}\right)$$

is a $d^2\times d^2$ matrix, $\mathbf{I}_d$ is the $d\times d$ identity, and $\otimes$ is the generalized outer product (Kronecker product). Applying the inverse $\mathbf{P}_{d^2}^{-1}$ to an arbitrary vectorized symmetric matrix, $\mathcal{V}(\mathbf{X})$, gives

$$\mathbf{P}_{d^2}^{-1}\cdot\mathcal{V}(\mathbf{X}) = \mathcal{V}(\mathbf{X}) + \frac{\lambda_5}{\lambda_5+2}\left(\frac{\lambda_5}{\lambda_5+1}I_4(\mathbf{X})\mathcal{V}(\mathbf{mm}) - \mathcal{V}(\{\mathbf{X},\mathbf{mm}\})\right). \quad \text{(A20)}$$

Applying the inverse $\mathbf{P}_{d^2}^{-1}$ to both sides of (A19) using (A20) and subsequently reverting the vectorization gives

$$\boldsymbol{\tau} = \eta(\boldsymbol{\tau})\left(\dot{\boldsymbol{\epsilon}} + \frac{\lambda_5}{\lambda_5+2}\left(\frac{\lambda_5}{\lambda_5+1}I_4(\dot{\boldsymbol{\epsilon}})\mathbf{mm} - \{\dot{\boldsymbol{\epsilon}},\mathbf{mm}\}\right)\right) - \frac{\lambda_5}{\lambda_5+1}\left(\frac{\lambda_4}{\lambda_5} + \frac{\lambda_4+\lambda_5}{d}\right)I_4(\boldsymbol{\tau})\mathbf{mm} + \frac{\lambda_4+\lambda_5}{d}I_4(\boldsymbol{\tau})\mathbf{I}. \quad \text{(A21)}$$

The rheology (A21) is, however, not in a closed form due to the right-hand side dependencies on $\boldsymbol{\tau}$, which requires expressing the three unknowns $I_2(\boldsymbol{\tau})$, $I_4(\boldsymbol{\tau})$, and $I_5(\boldsymbol{\tau})$ in terms of $I_2(\dot{\boldsymbol{\epsilon}})$, $I_4(\dot{\boldsymbol{\epsilon}})$, and $I_5(\dot{\boldsymbol{\epsilon}})$. By invoking the definition $I_4(\dot{\boldsymbol{\epsilon}}) = \dot{\boldsymbol{\epsilon}}\cdot\cdot\,\mathbf{mm}$, and requiring that the rate of energy dissipation (both total and the contribution due to deformation along $\mathbf{m}$) is identical in both the forward and inverse rheology, the unknowns may be determined by solving (not expanded for brevity)

$$I_4(\dot{\boldsymbol{\epsilon}}) = \dot{\boldsymbol{\epsilon}}(\boldsymbol{\tau})\cdot\cdot\,\mathbf{mm},$$

$$\boldsymbol{\tau}(\dot{\boldsymbol{\epsilon}})\cdot\cdot\,\dot{\boldsymbol{\epsilon}} = \boldsymbol{\tau}\cdot\cdot\,\dot{\boldsymbol{\epsilon}}(\boldsymbol{\tau}),$$

$$(\boldsymbol{\tau}(\dot{\boldsymbol{\epsilon}})\cdot\dot{\boldsymbol{\epsilon}})\cdot\cdot\,\mathbf{mm} = (\boldsymbol{\tau}\cdot\dot{\boldsymbol{\epsilon}}(\boldsymbol{\tau}))\cdot\cdot\,\mathbf{mm},$$

where $\boldsymbol{\tau}(\dot{\boldsymbol{\epsilon}})$ is given by (A21) and $\dot{\boldsymbol{\epsilon}}(\boldsymbol{\tau})$ is given by (A10). Upon writing the material parameters in terms of the enhancement factors (A14)–(A15), the inverse rheology finally becomes

$$\boldsymbol{\tau} = \eta\Bigg(\dot{\boldsymbol{\epsilon}} - \frac{E_{mm}^{-1} - 1}{d-1} I_4 \mathbf{I} + \left[\frac{d\left(E_{mm}^{-1}+1\right)-2}{d-1} - 2E_{mt}^{-1}\right] I_4 \mathbf{mm} + \left(E_{mt}^{-1} - 1\right)\{\dot{\boldsymbol{\epsilon}}, \mathbf{mm}\}\Bigg), \quad \text{(A22)}$$

$$\eta = A^{-1/n}\Bigg(I_2 + \left[\frac{d\left(E_{mm}^{-1}+1\right)-2}{d-1} - 2E_{mt}^{-1}\right] I_4^2 + 2\left(E_{mt}^{-1}-1\right) I_5\Bigg)^{(1-n)/2n}, \quad \text{(A23)}$$

where $I_i = I_i(\dot{\boldsymbol{\epsilon}})$ is implied.

SUPPLEMENTARY B: SPECTRAL GRAIN ROTATION

Vector identities allow the continuous grain rotation model, $\dot{n}(\theta,\phi) = -\boldsymbol{\nabla}\cdot(n(\theta,\phi)\dot{\mathbf{c}}(\theta,\phi))$, to be written as

$$\dot{n}(\theta,\phi) = -n(\theta,\phi)\boldsymbol{\nabla}\cdot\dot{\mathbf{c}}(\theta,\phi) - \dot{\mathbf{c}}(\theta,\phi)\cdot\boldsymbol{\nabla}n(\theta,\phi), \quad \text{(B1)}$$

where the gradient and divergence operators act on S^2. In spherical coordinates, the identity

$$\dot{\mathbf{c}}(\theta,\phi) = \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin(\theta)\hat{\boldsymbol{\phi}} \quad \text{(B2)}$$

applies, where

$$\hat{\mathbf{r}}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$
$$\hat{\boldsymbol{\theta}}(\theta,\phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),$$
$$\hat{\boldsymbol{\phi}}(\theta,\phi) = (-\sin\phi, \cos\phi, 0).$$

Inserting (B2) into (B1), yields

$$\dot{n}(\theta,\phi) = -Rn(\theta,\phi), \quad \text{(B3)}$$

where R is the linear operator

$$R = \dot{\theta}\cot(\theta) + \frac{\partial\dot{\theta}}{\partial\theta} + \frac{\partial\dot{\phi}}{\partial\phi} + \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi}. \quad \text{(B4)}$$

If $n(\theta,\phi)$ is expanded in terms of spherical harmonics, it is convenient to adopt the bra-ket notation by writing the fabric state in terms of the state vector ($n = n(\theta,\phi)$ and $Y_l^m = Y_l^m(\theta,\phi)$ assumed implicit)

$$|n\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} n_l^m |Y_l^m\rangle, \quad \text{(B5)}$$

and hence

$$|\dot{n}\rangle = -R|n\rangle. \quad \text{(B6)}$$

The rate-of-change of the expansion coefficients, $\dot{n}_l^m$, then follow from calculating the overlap integral

$$\dot{n}_l^m = \langle Y_l^m|\dot{n}\rangle = \sum_{l'=0}^{\infty}\sum_{m=-l'}^{l'} \langle Y_l^m|R|Y_{l'}^{m'}\rangle n_{l'}^{m'}. \quad \text{(B7)}$$

Determining the matrix elements $\langle Y_l^m|R|Y_{l'}^{m'}\rangle$ requires specifying the angular velocities $\dot{\theta}$ and $\dot{\phi}$ in (B4). Equating the discrete grain rotation model, $\dot{\mathbf{c}} = \boldsymbol{\omega}\cdot\mathbf{c} - (\dot{\boldsymbol{\epsilon}}\cdot\mathbf{c} - \mathbf{cc}\cdot\dot{\boldsymbol{\epsilon}}\cdot\mathbf{c})$, with (B2), and forming the inner product with $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$, respectively, gives

$$\dot{\theta} = \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\omega}\cdot\mathbf{c} - \hat{\boldsymbol{\theta}}\cdot\dot{\boldsymbol{\epsilon}}\cdot\mathbf{c}, \quad \text{(B8)}$$

$$\dot{\phi}\sin\theta = \hat{\boldsymbol{\phi}}\cdot\boldsymbol{\omega}\cdot\mathbf{c} - \hat{\boldsymbol{\phi}}\cdot\dot{\boldsymbol{\epsilon}}\cdot\mathbf{c}. \quad \text{(B9)}$$

Before inserting (B8)–(B9) into (B4), considerable notational simplicity may be achieved in the final result by expressing $\dot{\boldsymbol{\epsilon}}$ and $\boldsymbol{\omega}$ in terms of the expansion coefficients $\dot{\epsilon}_l^m$ and ω_l^m of the quadric surfaces $\dot{\boldsymbol{\epsilon}}\cdot\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}$ and $\boldsymbol{\omega}\cdot\cdot\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}$, respectively, defined as

$$\dot{\epsilon}_l^m = \int_{S^2} \dot{\boldsymbol{\epsilon}}\cdot\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}\left(Y_l^m\right)^* \mathrm{d}\Omega,$$
$$\omega_l^m = \int_{S^2} \boldsymbol{\omega}\cdot\cdot\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\left(Y_l^m\right)^* \mathrm{d}\Omega,$$

which evaluate exactly to (i.e. higher wave-number coefficients vanish)

$$\dot{\epsilon}_0^0 = \frac{2}{3}\sqrt{\pi}\,\mathrm{tr}(\dot{\boldsymbol{\epsilon}}), \quad \dot{\epsilon}_2^0 = -\frac{2}{3}\sqrt{\frac{\pi}{5}}\left(\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy} - 2\dot{\epsilon}_{zz}\right),$$
$$\dot{\epsilon}_2^{-1} = 2\sqrt{\frac{2\pi}{15}}\left(\dot{\epsilon}_{xz} + i\dot{\epsilon}_{yz}\right), \quad \dot{\epsilon}_2^1 = -\left(\dot{\epsilon}_2^{-1}\right)^*,$$
$$\dot{\epsilon}_2^{-2} = \sqrt{\frac{2\pi}{15}}\left(\dot{\epsilon}_{xx} - \dot{\epsilon}_{yy} + 2i\dot{\epsilon}_{xy}\right), \quad \dot{\epsilon}_2^2 = \left(\dot{\epsilon}_2^{-2}\right)^*,$$

and

$$\omega_1^0 = \sqrt{\frac{4\pi}{3}}\omega_{xy}, \quad \omega_1^{-1} = \sqrt{\frac{2\pi}{3}}\left(\omega_{yz} - i\omega_{xz}\right), \quad \omega_1^1 = -\left(\omega_1^{-1}\right)^*.$$

Given the above quadric expansion coefficients, $\dot{\boldsymbol{\epsilon}}$ and $\boldsymbol{\omega}$ may be rewritten as (exactly)

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \times \begin{pmatrix} \dot{\epsilon}_2^{-2} + \dot{\epsilon}_2^2 - \sqrt{\frac{2}{3}}\dot{\epsilon}_2^0 & -i\left[\dot{\epsilon}_2^{-2} - \dot{\epsilon}_2^2\right] & \dot{\epsilon}_2^{-1} - \dot{\epsilon}_2^1 \\ & -\dot{\epsilon}_2^{-2} - \dot{\epsilon}_2^2 - \sqrt{\frac{2}{3}}\dot{\epsilon}_2^0 & -i\left[\dot{\epsilon}_2^{-1} + \dot{\epsilon}_2^1\right] \\ \text{sym.} & & 2\sqrt{\frac{2}{3}}\dot{\epsilon}_2^0 \end{pmatrix} \quad \text{(B10)}$$

and

$$\boldsymbol{\omega} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\begin{pmatrix} 0 & \sqrt{2}\omega_1^0 & i\left[\omega_1^{-1} + \omega_1^1\right] \\ & 0 & \omega_1^{-1} - \omega_1^1 \\ \text{anti sym.} & & 0 \end{pmatrix}. \quad \text{(B11)}$$

Inserting (B8)–(B9) with (B10)–(B11) into (B4), it follows from long but arithmetically straight-forward calculations that

$$R = \mathbf{Y}\cdot\left(\mathbf{g}_0 + \mathbf{g}_z L_z + \mathbf{g}_- L_- + \mathbf{g}_+ L_+\right), \quad \text{(B12)}$$

where

$$\mathbf{Y} = \left(Y_0^0, Y_2^{-2}, Y_2^{-1}, Y_2^0, Y_2^1, Y_2^2\right),$$
$$\mathbf{g}_0 = 3\left(0, \dot{\epsilon}_2^{-2}, \dot{\epsilon}_2^{-1}, \dot{\epsilon}_2^0, \dot{\epsilon}_2^1, \dot{\epsilon}_2^2\right),$$
$$\mathbf{g}_z = \left(-i\sqrt{3}\omega_1^0, -\dot{\epsilon}_2^{-2}, 0, 0, 0, \dot{\epsilon}_2^2\right),$$
$$\mathbf{g}_- = \frac{1}{2}\left(\sqrt{\frac{5}{6}}\dot{\epsilon}_2^{-1} - \sqrt{6}i\omega_1^{-1}, 0, \dot{\epsilon}_2^{-2}, \sqrt{\frac{2}{3}}\dot{\epsilon}_2^{-1}, \sqrt{\frac{3}{2}}\dot{\epsilon}_2^0, 2\dot{\epsilon}_2^1\right),$$
$$\mathbf{g}_+ = \frac{1}{2}\left(\sqrt{\frac{5}{6}}\dot{\epsilon}_2^1 + \sqrt{6}i\omega_1^1, 2\dot{\epsilon}_2^{-1}, \sqrt{\frac{3}{2}}\dot{\epsilon}_2^0, \sqrt{\frac{2}{3}}\dot{\epsilon}_2^1, \dot{\epsilon}_2^2, 0\right).$$

The angular momentum operators L_z and $L_\pm$ are defined as

$$L_z |Y_l^m\rangle = m |Y_l^m\rangle ,$$
$$L_\pm |Y_l^m\rangle = \sqrt{(l \mp m)(l \pm m + 1)}\, |Y_l^{m\pm 1}\rangle .$$

Reaching the above result requires invoking the recurrence relations (identities)

$$\frac{\partial |Y_l^m\rangle}{\partial \phi} = i L_z |Y_l^m\rangle ,$$
$$2\frac{\partial |Y_l^m\rangle}{\partial \theta} = e^{-i\phi} L_+ |Y_l^m\rangle - e^{i\phi} L_- |Y_l^m\rangle ,$$
$$-2\cot(\theta) L_z |Y_l^m\rangle = e^{-i\phi} L_+ |Y_l^m\rangle + e^{i\phi} L_- |Y_l^m\rangle .$$

Notice that because **Y** depends only on Y_l^m of even l, an initially antipodally symmetric distribution will remain antipodally symmetric.

Finally, we note that calculating the matrix elements in (B7) using (B12) involves integrals over triple products of Y_l^m. Such integrals may be evaluated by leveraging the mutual orthogonality of Y_l^m, which requires reducing the triple products to sums over products between two harmonics using the contraction rule (expanding the product of any two spherical harmonics in terms of a spherical harmonic series)

$$Y_{l'}^{m'} Y_{l''}^{m''} = \sqrt{\frac{(2l'+1)(2l''+1)}{4\pi}} \times \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-1)^m \sqrt{2l+1} \begin{pmatrix} l' & l'' & l \\ m' & m'' & -m \end{pmatrix} \begin{pmatrix} l' & l'' & l \\ 0 & 0 & 0 \end{pmatrix} Y_l^m , \tag{B13}$$

where the Winger $3j$ symbols determine which harmonics are selected.

Structure tensors

Provided that the c-axis distribution $n(\theta,\phi)$ is expanded in terms of spherical harmonics, calculating the corresponding structure tensors $\langle \mathbf{c}^k \rangle$ is a matter of expressing $\mathbf{c}^k$ in terms of spherical harmonics, too (Advani and Tucker, 1987).

Consider an arbitrary c-axis

$$\mathbf{c} = \mathbf{c}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{B14}$$

characterized by the two angles θ and ϕ. Expressing $\mathbf{c}(\theta,\phi)$ in terms of $Y_l^m(\theta,\phi)$ gives

$$\mathbf{c} = \sqrt{\frac{2\pi}{3}} \left(Y_1^{-1} - Y_1^1, i\left[Y_1^{-1} + Y_1^1\right], \sqrt{2} Y_1^0 \right). \tag{B15}$$

Calculating the dyad $\mathbf{c}^2$ using (B15) results in products between Y_1^m that may be re-expressed as a sum over spherical harmonics using the contraction rule (B13):

$$\mathbf{c}^2 = \frac{\sqrt{4\pi}}{3} Y_0^0 \mathbf{I} + \sqrt{\frac{2\pi}{15}} \times \begin{pmatrix} Y_2^{-2} + Y_2^2 - \sqrt{\frac{2}{3}} Y_2^0 & i\left[Y_2^{-2} - Y_2^2\right] & Y_2^{-1} - Y_2^1 \\ & -Y_2^{-2} - Y_2^2 - \sqrt{\frac{2}{3}} Y_2^0 & i\left[Y_2^{-1} + Y_2^1\right] \\ \text{sym.} & & 2\sqrt{\frac{2}{3}} Y_2^0 \end{pmatrix}. \tag{B16}$$

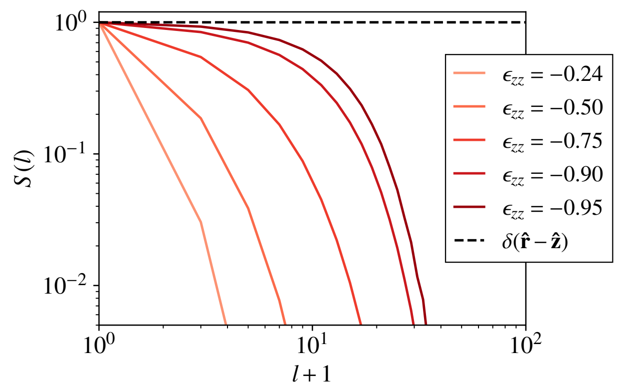

Fig. B1. Power spectrum of $n(\theta,\phi)$ as $\epsilon_{zz} \to -1$ (unconfined vertical pure shear).

Calculating $\mathbf{c}^4$, in turn, follows by repeating the procedure w.r.t. the outer product $\mathbf{c}^2\mathbf{c}^2$. For $\mathbf{c}^k$, the largest occurring wavenumber modes are Y_k^m. Notice that successive applications of the contraction rule produces very large tensors for $k \geq 4$ that we used a symbolic solver to determine.

The k-th order structure tensor is defined as

$$\langle \mathbf{c}^k \rangle_{n(\theta,\phi)} = \frac{\int_{S^2} \mathbf{c}^k n(\theta,\phi)\,\mathrm{d}\Omega}{\int_{S^2} n(\theta,\phi)\,\mathrm{d}\Omega}. \tag{B17}$$

The numerator therefore consists of integrals over products of Y_l^m that are easily evaluated by noting their mutual orthogonality, $\int_{S^2} Y_l^m (Y_{l'}^{m'})^* \,\mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$, and $(Y_l^m)^* = (-1)^m Y_l^{-m}$. The denominator is simply $n_0^0\sqrt{4\pi}$. In this way, the entries of $\langle \mathbf{c}^k \rangle$ are linear combinations of n_l^m for $l \leq k$.

Regularization

As $n(\theta,\phi)$ evolves and becomes anisotropic, the coefficients n_l^m associated with high wavenumber modes (large l and m, and thus small-scale structure) must increase in magnitude relative to the low wavenumber coefficients (small l and m). If the series (B5) is truncated at $l = L$, then $l > L$ modes cannot evolve, and the truncated solution will reach an unphysical quasi-steady state (not shown). To prevent this, regularization must be introduced. Applying Laplacian diffusion to the expansion,

$$\dot{n}(\theta,\phi) = \nu \nabla^2 n(\theta,\phi), \tag{B18}$$

is a useful approach that conveniently allows the growth of high wavenumber modes to be disproportionately damped depending on a diffusion coefficient, ν. The associated rate-of-change of n_l^m is

$$\dot{n}_l^m = \langle Y_l^m | \dot{n} \rangle = -\nu l(l+1) n_l^m . \tag{B19}$$

The value of ν must be adjusted depending on L: if ν is too large then the high wavenumber coefficients do not evolve and hence small-scale structure can not be represented, while if ν too small the solution eventually evolves in the undesirable manner described above. For the simulations presented in the main text, $L = 40$ and $\nu = 0.5 \times 10^{-2}$ were used.

As the fabric strengthens due to unconfined vertical pure shear ($\epsilon_{zz} \to -1$), Fig. B1 shows the corresponding power spectrum of

$n(\theta, \phi)$, defined as

$$S(l) = \frac{1}{2l+1} \sum_{m=-l}^{l} \left| n_l^m \right|^2. \tag{B20}$$

With increasing fabric strength the high wavenumber components (large l) become non-negligible, but with increasing l these components are also disproportionally dampened by the regularization. As a reference, the power spectrum of $n(\theta, \phi) = \delta(\hat{\mathbf{r}} - \mathbf{m})$ is plotted too (black dashed line), which is the ideal limit of the unregularized and untruncated solution as $\epsilon_{zz} \to -1$.

REFERENCES

Advani SG and Tucker CL (1987) The Use of Tensors to Describe and Predict Fiber Orientation in Short Fiber Composites. *Journal of Rheology*, **31**(8), 751–784 (doi: 10.1122/1.549945)

Johnson A (1977) Creep characterization of transversely-isotropic metallic materials. *Journal of the Mechanics and Physics of Solids*, **25**(2), 117–126, ISSN 0022-5096 (doi: https://doi.org/10.1016/0022-5096(77)90007-2)

Naumenko K and Altenbach H (2007) *Modeling of creep for structural analysis*. Springer Science & Business Media, ISBN 978-3-540-70834-6 (doi: https://doi.org/10.1007/978-3-540-70839-1)