On a conjecture of George Beck. II

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Abstract. In this paper, we give a combinatorial proof of a generating function identity concerning the sum of the smallest parts in the distinct partitions of n.

Keywords. Distinct partitions, sum of the smallest parts, combinatorial proof.

2010MSC. Primary 05A17; Secondary 11P84.

1. Introduction

As usual, a *partition* of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n.

In my previous paper [5], I proved the following conjecture due to George Beck:

Theorem 1.1. *The number of gap-free partitions (i.e. partitions with the difference between each consecutive parts being at most* 1*) of n is also the sum of the smallest parts in the distinct partitions (i.e. partitions with distinct parts) of n with an odd number of parts.*

Let $\mathscr{DO}$ denote the set of distinct partitions with an odd number of parts. The main idea in [5] is to use the differentiation technique to study to generating function of $\mathrm{sspt}_{\mathscr{DO}}(n)$, the sum of the smallest parts in the partitions of n in $\mathscr{DO}$. More precisely, I showed that

$$\sum_{n\geq 1}\mathrm{sspt}_{\mathscr{DO}}(n)q^n=\sum_{t\geq 1}\frac{q^t}{1-q^t}(-q)_{t-1}, \tag{1.1}$$

where and in what follows, we use the standard q-series notation

$$(a)_n=(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k),$$

and

$$(a)_\infty=(a;q)_\infty:=\prod_{k\geq 0}(1-aq^k).$$

In a subsequent paper [8], Yang also provided a combinatorial proof of Beck's conjecture.

Let $\mathscr{D}$ be the set of distinct partitions. Here we will not include the empty partition unless otherwise specified.

For any partition π, we denote by $|\pi|$ the sum of the parts of π, by $\Lambda(\pi)$ the largest part of π, by $\sigma(\pi)$ the smallest part of π, and by $\sharp(\pi)$ the number of parts of π.

The following general result is almost shown in [5] (in which the cases $z=\pm 1$ are proved).

Theorem 1.2. *It holds that*

$$\sum_{\pi\in\mathscr{D}}\sigma(\pi)z^{\sharp(\pi)}q^{|\pi|}=\sum_{t\geq 1}\frac{q^t}{1-q^t}\Big((-z)_t-1\Big). \tag{1.2}$$

The proof of this generating function identity can still be deduced by the differentiation technique. In fact, it is essentially the same as the proof of (2.3) in [5]. In this paper, we will focus on the combinatorial viewpoint.

2. A combinatorial approach

Our starting point is the following double counting argument, which appears to be able to be adapted to many types of partition sets.

For a nonnegative integer t, we define

$$\mathscr{D}_t:=\big\{\pi\in\mathscr{D}\ :\ \Lambda(\pi)\geq t+1 \text{ and } \Lambda(\pi)-\sigma(\pi)\leq t\big\}.$$

Now given any $\pi\in\mathscr{D}$, if $\pi\in\mathscr{D}_t$, then $\Lambda(\pi)-\sigma(\pi)\leq t\leq\Lambda(\pi)-1$ by the definition. Hence, π is exactly contained in the following $\sigma(\pi)$ partition sets: $\mathscr{D}_{\Lambda(\pi)-1}$, $\mathscr{D}_{\Lambda(\pi)-2}$, ..., $\mathscr{D}_{\Lambda(\pi)-\sigma(\pi)}$.

We therefore have

Theorem 2.1. *It holds that*

$$\sum_{\pi\in\mathscr{D}}\sigma(\pi)z^{\sharp(\pi)}q^{|\pi|}=\sum_{t\geq 0}\sum_{\pi\in\mathscr{D}_t}z^{\sharp(\pi)}q^{|\pi|}. \tag{2.1}$$

One immediately sees that the remaining task is to study the generating function for $\mathscr{D}_t$ with $t\geq 0$.

We remark that, for certain partition sets, if we only require the difference between the largest and smallest parts to be bounded by t, then the generating functions are studied in a series of papers [1–4, 6]. In particular, in my joint work with Yee [6], we provided a combinatorial approach that can be easily adapted to obtain the generating function for $\mathscr{D}_t$.

For convenience, we now consider the generating function for $\mathscr{D}_{t-1}$ with $t\geq 1$.

Let $\mathscr{B}_t$ be the set of partition pairs (μ,ν) where μ is nonempty and its parts all have size t, and ν is a nonempty distinct partition with 0 being allowed as a part and the largest part being at most $t-1$. For example,

$$\big((5,5,5,5,5),\ (4,2,1,0)\big)\in\mathscr{B}_5.$$

Furthermore, we use $|(\mu,\nu)|$ to denote $|\mu|+|\nu|$.

For $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ in $\mathscr{D}_{t-1}$, we put $s=\lfloor\pi_\ell/t\rfloor$ with $\lfloor x\rfloor$ being the conventional floor function. We also let k be the positive integer such that $\pi_k\geq(s+1)t$ and $\pi_{k+1}<(s+1)t$. If there is no such k, then we let $k=0$.

Now we construct a map $\phi_t:\mathscr{D}_{t-1}\to\mathscr{B}_t$ by

$$\begin{aligned}\phi_t\colon &(\pi_1,\pi_2,\ldots,\pi_\ell)\\ &\mapsto\Big((\underbrace{t,t,t,\ldots,t}_{\substack{s(\ell-k)+(s+1)k\\ \text{times}}}),\ \big(\pi_{k+1}-st,\ldots,\pi_\ell-st,\pi_1-(s+1)t,\ldots,\pi_k-(s+1)t\big)\Big).\end{aligned}$$

Note that the second subpartition can be treated as $(\pi_1,\pi_2,\ldots,\pi_\ell)$ reduced modulo t, cyclically permuted such that they are weakly decreasing.

Similar to Theorem 2.1 of [6], we have

Lemma 2.2. *ϕ_t is a weight preserving map from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Furthermore, the number of parts is preserved by the second subpartition of the image.*

Proof. Let $(\mu,\nu)=\phi_t(\pi)$. We first show that μ is nonempty. Since $\pi\in\mathscr{D}_{t-1}$, we have $\pi_1\geq(t-1)+1=t$. Hence we take out at least one t from π_1 to form μ, which implies that μ is not empty.

On the other hand, we know that π is a distinct partition. Since $\pi_1-\pi_\ell\leq t-1<t$, $s=\lfloor\pi_\ell/t\rfloor$, and $\pi_k\geq(s+1)t>\pi_{k+1}$, we have

$$t>\pi_{k+1}-st>\cdots>\pi_\ell-st>\pi_1-(s+1)t>\cdots>\pi_k-(s+1)t.$$

Note that $\pi_k-(s+1)t$ could be 0 since π_k could be $(s+1)t$. Hence ν satisfies the conditions. It follows that $(\mu,\nu)\in\mathscr{B}_t$.

At last, it is obvious from the definition of ϕ_t that $|\phi_t(\pi)|=|\pi|$ and $\sharp(\nu)=\sharp(\pi)$. □

The rest of the argument is different to that in [6]. We shall show

Lemma 2.3. *ϕ_t is invertible.*

Proof. Let $(\mu,\nu)\in\mathscr{B}_t$. Let the number of t in μ be $r\geq1$ and let $\nu=(\nu_1,\nu_2,\ldots,\nu_\ell)$. Now we write $r=m\ell+r^*$ with $m\geq0$ and $0\leq r^*\leq\ell-1$ being integers. We construct the inverse $\phi_t^{-1}:\mathscr{B}_t\to\mathscr{D}_{t-1}$ as follows.

$$\phi_t^{-1}\colon(\mu,\nu)\mapsto(\nu_{\ell-r^*+1}+(m+1)t,\ldots,\nu_\ell+(m+1)t,\nu_1+mt,\ldots,\nu_{\ell-r^*}+mt).$$

We now show that the image is in $\mathscr{D}_{t-1}$. Recall that $0\leq\nu_\ell<\cdots<\nu_1\leq t-1$.

If $r^*\neq0$, since $\nu_\ell+t>\nu_1$, we have

$$\nu_{\ell-r^*+1}+(m+1)t>\cdots>\nu_\ell+(m+1)t>\nu_1+mt>\cdots>\nu_{\ell-r^*}+mt.$$

Notice that $\nu_{\ell-r^*+1}+(m+1)t\geq t$. We further notice that $\nu_{\ell-r^*}$ is not the smallest part of ν, and hence $\nu_{\ell-r^*}>0$. At last, we have $(\nu_{\ell-r^*+1}+(m+1)t)-(\nu_{\ell-r^*}+mt)=t-(\nu_{\ell-r^*}-\nu_{\ell-r^*+1})\leq t-1$. Hence in this case the image is in $\mathscr{D}_{t-1}$.

If $r^*=0$, then $m\geq1$ since $r\geq1$. We have $\nu_1+mt>\cdots>\nu_\ell+mt>0$ and $\nu_1+mt\geq t$. We also have $(\nu_1+mt)-(\nu_\ell+mt)=\nu_1-\nu_\ell\leq t-1$. Hence the image is also in $\mathscr{D}_{t-1}$.

From the definition of ϕ_t and ϕ_t^{-1}, it is apparent that $\phi_t^{-1}(\phi_t(\pi))=\pi$. Hence ϕ_t is invertible. □

Example 2.1. For the partition sets $\mathscr{D}_4$ and $\mathscr{B}_5$, we have

$$(9,7,6,5)\ \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}}\ \big((5,5,5,5),\ (4,2,1,0)\big)$$

and

$$(10,9,7,6)\ \underset{\phi_5^{-1}}{\overset{\phi_5}{\rightleftharpoons}}\ \big((5,5,5,5,5),\ (4,2,1,0)\big).$$

It follows from Lemmas 2.2 and 2.3 that ϕ_t is a bijection from $\mathscr{D}_{t-1}$ to $\mathscr{B}_t$. Hence, for $t\geq1$,

$$\sum_{\pi\in\mathscr{D}_{t-1}}z^{\sharp(\pi)}q^{|\pi|}=\sum_{(\mu,\nu)\in\mathscr{B}_t}z^{\sharp(\nu)}q^{|\mu|+|\nu|}. \tag{2.2}$$

The generating function for $\mathscr{B}_t$ is easy to get:

$$\sum_{(\mu,\nu)\in\mathscr{B}_t}z^{\sharp(\nu)}q^{|\mu|+|\nu|}=\frac{q^t}{1-q^t}\Big((-z)_t-1\Big), \tag{2.3}$$

where $q^t/(1-q^t)$ comes from the first subpartition whereas $(-z)_t - 1$ comes from the second subpartition.

Consequently, we have

Theorem 2.4. *For $t \geq 1$, it holds that*

$$\sum_{\pi \in \mathscr{D}_{t-1}} z^{\sharp(\pi)} q^{|\pi|} = \frac{q^t}{1-q^t}\Big((-z)_t - 1\Big). \tag{2.4}$$

Together with (2.1), we have

$$\sum_{\pi \in \mathscr{D}} \sigma(\pi) z^{\sharp(\pi)} q^{|\pi|} = \sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((-z)_t - 1\Big),$$

which completes the proof of Theorem 1.2.

3. Closing remarks

As I showed in [5], (1.1) follows since

$$\sum_{n\geq 1} \mathrm{sspt}_{\mathscr{D}\mathscr{O}}(n) q^n = \frac{1}{2} \sum_{\pi\in\mathscr{D}} \sigma(\pi)\Big(1^{\sharp(\pi)} - (-1)^{\sharp(\pi)}\Big) q^{|\pi|}.$$

Note that the $z = 1$ and $z = -1$ cases of (1.2) respectively correspond to (2.3) and (2.2) in [5].

I was also pointed out by Dazhao Tang that another conjecture of Beck, which is proposed in [7, A237665], can be proved in the same way.

Conjecture 3.1 (Beck)**.** *The number of gap-free partitions of n with at least two different parts is also the sum of the smallest parts in the distinct partitions of n with an even number of parts.*

Theorem 3.2. *Conjecture 3.1 is true.*

Proof. Let $\mathrm{sspt}_{\mathscr{D}\mathscr{E}}(n)$ denote the sum of the smallest parts in the distinct partitions of n with an even number of parts. We have

$$\begin{aligned}\sum_{n\geq 1} \mathrm{sspt}_{\mathscr{D}\mathscr{E}}(n) q^n &= \frac{1}{2} \sum_{\pi\in\mathscr{D}} \sigma(\pi)\Big(1^{\sharp(\pi)} + (-1)^{\sharp(\pi)}\Big) q^{|\pi|}\\ &= \sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((-q)_{t-1} - 1\Big).\end{aligned}$$

We next observe that the conjugates of gap-free partitions with at least two different parts are partitions with at least two different parts where only the largest part may repeat.

Let $\mathrm{gf}_2(n)$ count the number of gap-free partitions of n with at least two different parts. We have

$$\begin{aligned}\sum_{n\geq 1} \mathrm{gf}_2(n) q^n &= \sum_{t\geq 1} \frac{q^t}{1-q^t}(-q)_{t-1} - \sum_{t\geq 1} \frac{q^t}{1-q^t}\\ &= \sum_{t\geq 1} \frac{q^t}{1-q^t}\Big((-q)_{t-1} - 1\Big).\end{aligned}$$

Here $\sum_{t\geq 1} q^t(-q)_{t-1}/(1-q^t)$ is the generating function of partitions where only the largest part may repeat, and $\sum_{t\geq 1} q^t/(1-q^t)$ is the generating function of partitions with only one different part.

We conclude that $\mathrm{gf}_2(n) = \mathrm{sspt}_{\mathscr{DE}}(n)$. $\square$

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