Impatiemment Attendu

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TABLE OF CONTENTS (continued)

Data Movement in Odd-Even Merging
by Robert Sedgewick ....

$$\zeta(2z,a) = \frac{1}{2} - a + z(2 \ln \Gamma(a) - \ln 2\pi) + O(z^2).$$

The expansion for $\Gamma(z)$ is well known (see [1, eq. 6.1.33]), the next two expansions are elementary, and the expansion for the ζ-function follows directly from [16, p. 271]. Multiplying these series together, we find the expansion at $z=0$ of the integrand in (23):

$$\frac{1}{4z^2 \ln 2} + \frac{1}{z}\left(\frac{1}{4} \lg j - \frac{\gamma}{4 \ln 2} - \frac{1}{4} + \lg \frac{\Gamma(1/4)^2}{2\pi}\right) + O(1). \qquad (24)$$

This gives the residue at $z=0$ (the coefficient of $1/z$).

To this we must add the residue at the other poles of $1/(4^z-1)$. The effect of these other terms is small (but not insignificant), and we shall encapsulate them in a single term,

$$\epsilon(j) \equiv \frac{2}{\ln 4} \sum_{k \neq 0} \Gamma\left(\frac{2k\pi i}{\ln 4}\right) j^{2k\pi i/\ln 4} \zeta\left(\frac{4k\pi i}{\ln 4}, \frac{1}{4}\right)$$

$$= \sum_{k \geq 1} (\xi_k \cos(k\pi \lg j) - \eta_k \sin(k\pi \lg j)),$$

where

$$\frac{2}{\ln 2} \Gamma\left(\frac{k\pi i}{\ln 2}\right) \zeta\left(\frac{2k\pi i}{\ln 2}, \frac{1}{4}\right) \equiv \xi_k + i\eta_k.$$

To finish the evaluation of our b_j and B_N we need to evaluate the Γ and ζ functions at these points along the imaginary axis (see [6, secs. 6.3 and 6.4]). Table 3 shows the values of ξ_k and η_k for $k=1,2,3$ computed in this way. The values get exceedingly small for larger k, as can be verified from the bounds (21) and (22).

Adding all the residues, we have, from (23):

$$b_j = \frac{1}{4} \lg j + \lg \frac{\Gamma(\frac{1}{4})^2}{2\pi} - \frac{1}{4} - \frac{\gamma}{4 \ln 2} + \epsilon(j) + O(j^{-1/2}). \qquad (25)$$

This leads to our final result.

Theorem 3. The average number of exchanges used by Batcher's odd-even merge for a random 2-ordered file of $2N$ elements is

$$B_N = \frac{1}{4} N \lg N + \left(\lg \frac{\Gamma(\frac{1}{4})^2}{2\pi} + \frac{1}{4} - \frac{\gamma+2}{4 \ln 2} + \delta(N)\right)N + O(\sqrt{N} \log N),$$

where $\delta(N)$ is an oscillatory function which is

k	ξ_k	η_k
1	.003704670+	.002500177+
2	.000001560+	-.000000832-
3	.000000001-	.000000002+

$$\xi_k + i\eta_k = \frac{2}{\ln 2}\Gamma\left(\frac{k\pi i}{\ln 2}\right)\zeta\left(\frac{k\pi i}{\ln 2}, \frac{1}{4}\right)$$

$\Gamma(\frac{1}{4})$ = 3.6256099082+

$\frac{1}{\ln 2}$ = 1.4426950408+

γ = 0.5772156649+

π = 3.1415926535+

Table 3. Values of constants

constant if N is a power of two, with $|\delta(N)| < .000680$, and $\delta(2^n) \approx .000317(-1)^n$. (The constant

$$\alpha \equiv \lg \frac{\Gamma(\frac{1}{4})^2}{2\pi} + \frac{1}{4} - \frac{\gamma+2}{4 \ln 2}$$

has the approximate value .385417224.)

Proof. From the discussion above, we need only substitute our result (25) for b_j into our eq. (14) for B_N and perform the summation. We have

$$B_N = \frac{1}{2} \frac{4^N}{\binom{2N}{N}} \sum_{1 \leq j < N} \frac{\binom{2j}{j}}{4^j}\left(\frac{1}{4} \lg j + \alpha + \frac{1}{2 \ln 2} + \epsilon(j) + O(j^{-1/2})\right).$$

The $\alpha + 1/(2 \ln 2)$ terms are easily taken care of: working backwards from (14), we see that they contribute $(\alpha + 1/(2 \ln 2))N$ to B_N. (The sum may be evaluated directly as an interesting application of several identities in Knuth [13]; a simple induction could also be used.)

For the other terms, we can remove the binomial coefficients with Stirling's approximation, as in the derivation of (16). We have

$$\frac{\binom{2j}{j}}{4^j} = \frac{1}{\sqrt{\pi j}} + O(j^{-3/2}), \quad \frac{4^N}{\binom{2N}{N}} = \sqrt{\pi N} + O(N^{-1/2}).$$

Therefore the $O(j^{-1/2})$ term sums to $O(\sqrt{N} \log N)$, and

$$\frac{1}{2} \frac{4^N}{\binom{2N}{N}} \sum_{1 \leq j < N} \frac{\binom{2j}{j}}{4^j} \frac{1}{4} \lg j = \frac{\sqrt{N}}{8} \sum_{1 \leq j < N} \frac{\lg j}{\sqrt{j}} + O(\sqrt{N})$$

$$= \frac{\sqrt{N}}{8} \int_1^N \frac{\lg x}{\sqrt{x}} dx + O(\sqrt{N})$$

$$= \frac{1}{4} N \lg N - \frac{1}{2 \ln 2} N + O(\sqrt{N}).$$

Here the second step follows from Euler-McLaurin summation (see, for example, [14, p. 110]) and the integral is evaluated by making the substitution $x = y^2$.

It remains to evaluate the oscillatory term

$$N\delta(N) = \frac{1}{2} \frac{4^N}{\binom{2N}{N}} \sum_{1 \leq j < N} \frac{\binom{2j}{j}}{4^j} \epsilon(j).$$

After substituting for $\epsilon(j)$, we proceed in the same way as we did for the $\lg j$ term. The result of using Stirling's approximation on the binomial coefficients and Euler-McLaurin summation on the resulting sums is

$$\delta(N) = \frac{1}{2\sqrt{N}} \sum_{k \geq 1} \left(\xi_k \int_1^N \frac{\cos k\pi \lg x}{\sqrt{x}} dx - \eta_k \int_1^N \frac{\sin k\pi \lg x}{\sqrt{x}} dx\right) + O(\sqrt{N}).$$

These integrals are elementary; the substitutions $x = y^2$, then $t = 2\pi k \lg y$ transform them into standard integrals (for example, [1, Eqs. 4.3.136, 4.3.137]) with the eventual result

$$\delta(N) = \sum_{k \geq 1} \frac{\sigma_k}{\sigma_k^2+1}\left(\xi_k(\sigma_k \cos \pi k \lg N + \sin \pi k \lg N) - \eta_k(\sigma_k \sin \pi k \lg N - \cos \pi k \lg N)\right)$$

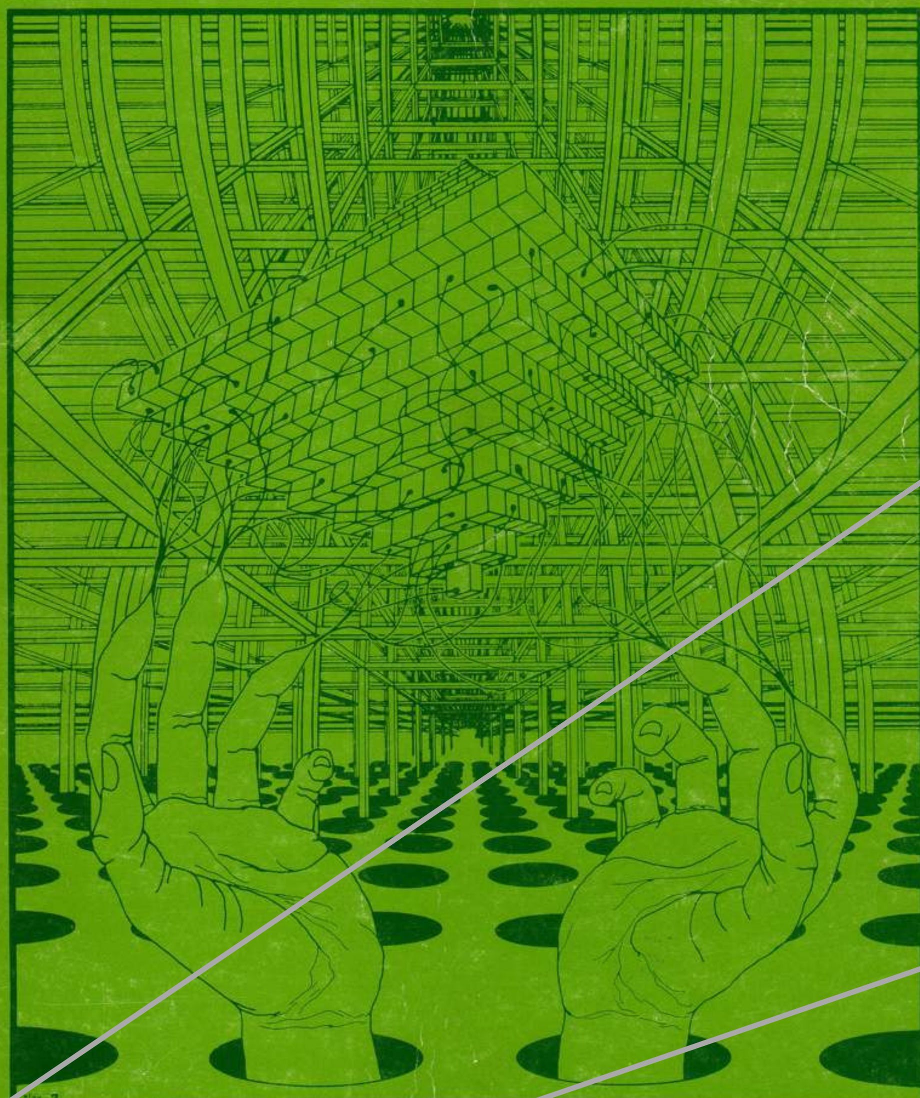
where σ_k is $(\ln 2/2\pi k)$. From this formula, we see that $\delta(N)$ is constant when N is a power of two, and has an oscillatory nature between powers of two. With the aid of Table 3 we can easily compute the stated values. □

$$\frac{2}{\ln 2} \Gamma\left(\frac{k\pi i}{\ln 2}\right) \zeta\left(\frac{2k\pi i}{\ln 2}, \frac{1}{4}\right)$$

18th Annual

Symposium on Foundations of Computer Science

(Formerly called the Annual Symposium on Switching and Automata Theory)



OCTOBER 31-NOVEMBER 2, 1977

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IEEE 77 CH1278-1 C

IEEE COMPUTER SOCIETY

INSTITUTE OF ELECTRICAL AND ELECTRONICS ENGINEERS

On the Average Number of Registers Required for Evaluating Arithmetic Expressions
by P. Flajolet, J. C. Raoult, and J. Vuillemin ....

We write $M_n = B_n + \frac{1}{n}\binom{2n}{n} I_n + o(B_n)$

with $$I_n = \int_0^\infty S_2(\lfloor t\sqrt{n}\rfloor)e^{-t^2}.H_3(t)dt \qquad (28)$$

Using Delange's formula (25),

$$U(x) = \int_0^x S_2(\lfloor t \rfloor)dt = \frac{1}{2}\ x\log_2 x + x\ F(\log_2 x) + h(x),$$

we compute I_n by parts :

$$I_n = [\frac{1}{\sqrt{n}} U(t\sqrt{n})e^{-t^2}H_3(t)]_0^\infty - \frac{1}{\sqrt{n}} \int_0^\infty U(t\sqrt{n})e^{-t^2}H_4(t)dt\ .$$

The first term reduces to zero for at least two reasons, and :

$$I_n = \frac{1}{\sqrt{n}}\int_0^\infty [\frac{1}{2}(t\sqrt{n})\log_2(t\sqrt{n})+(t\sqrt{n})F(\log_2(t\sqrt{n}))+h(t\sqrt{n})]e^{-t^2}H_4(t)dt.$$

Splitting in four :

$$I_n = -\frac{1}{2}\int_0^\infty t\ \log_2 t\ H_4(t)e^{-t^2}dt - \frac{1}{4}\log_2 n \int_0^\infty t\ H_4(t)e^{-t^2}dt$$

$$- \int_0^\infty tH_4(t)F(\frac{1}{2}\log_2 n+\log_2 t)e^{-t^2}dt - \frac{1}{\sqrt{n}}\int_0^\infty h(t\sqrt{n})H_4(t)e^{-t^2}dt$$

$$= -\frac{1}{2}I_1 - \frac{1}{4}\log_2 n\ \ I_2 - I_3 - \frac{1}{\sqrt{n}}\ I_4\ .$$

Each integral is computed in turn.

$$I_1 = \int_0^\infty t\ \log_2 t\ H_4(t)e^{-t^2}dt = -\frac{1}{\log 2}\int_0^\infty(\log t+1)H_3(t)e^{-t^2}dt$$

$$= -\frac{1}{\log 2}\int_0^\infty \log t(-8t^2+12t)e^{-t^2}dt - \frac{2}{\log 2}$$

$$= -\frac{1}{\log 2}\int_0^\infty \log t^2(-2t^2+3)e^{-t^2}d(t^2) - \frac{2}{\log 2}$$

$$= -\frac{1}{\log 2}\int_0^\infty \log x(3-2x)e^{-x}dx - \frac{2}{\log 2}$$

$$= -\frac{3}{\log 2}\int_0^\infty \log x\ e^{-x}dx + \frac{2}{\log 2}\int_0^\infty x\log x\ e^{-x}dx - \frac{2}{\log 2}$$

$$= -\frac{1}{\log 2}\int_0^\infty \log x\ e^{-x}dx\ .$$

This last integral is classical (see [Bu,33]) and its value is $-\gamma$, $\gamma \cong 0.577$ being Euler's constant.

Simple integrations by parts give $I_2 = -2$; as for I_4 , the function $h(t\sqrt{n})$ ranges between 0 and 1, thus I_4 has some definite value and

$$\frac{1}{\sqrt{n}}\ I_4 = o(1).$$

There just remains $I_3 = K(\log_4 n)$, where

$$K(u) = \int_0^\infty t\ H_4(t)\ F\ (u+\log_2 t)\ e^{-t^2}dt \qquad (29)$$

Since F is periodic with period 1 then so is K. The change of variable $x = u + \log_2 t$ shows that K is indefinitely differentiable.

We have thus proved :

<u>Theorem</u> : The average number $A_n = M_n/B_n$ satisfies

$$A_n = \log_4 n + D\ (\log_4 n) + o(1)$$

in which

$$D(u) = 1 - \frac{\gamma}{2\log 2} + K(u)\ ,$$

K(u) being the periodic function defined by (29).

Apart from the periodic term, one can, in principle, develop an asymptotic expansion as far as needed. H.Delange (private communication) has also computed the Fourier series of D :

$$D(t) = \sum_{k\in\mathbb{Z}} a_k\ e^{i2k\pi t}$$

and finds :

$$a_0 = \frac{1}{2} - \frac{\gamma+2}{2\log 2} + \log_2 \pi\ ,$$

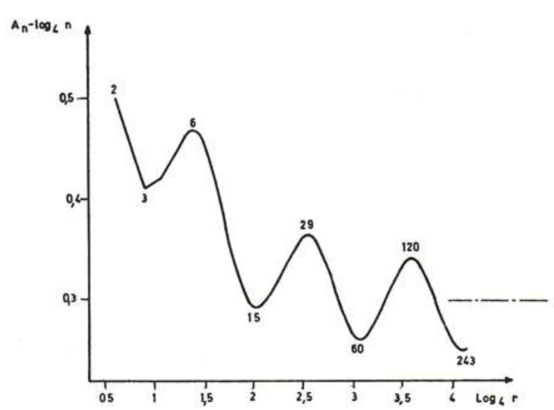
which is the mean value of D(t), and for $k \neq 0$,

$$a_k = \frac{2k\pi i - \log 2}{\log 2}\ \Gamma\left(\frac{k\pi i}{\log 2}\right)\zeta\left(\frac{2\ k\pi i}{\log 2}\right)$$

$$\frac{2k\pi i - \log 2}{\log 2}\ \Gamma\left(\frac{k\pi i}{\log 2}\right)\zeta\left(\frac{2\ k\pi i}{\log 2}\right)$$

where Γ and ζ are the classical gamma and Riemann's zeta functions.

The values of A_n have been machine computed for n ranging from 2 to 300 by means of formula (20). The results are plotted on figure 4, where the horizontal axis represents values of n in a logarithmic scale and the vertical axis represents values of $A_n - \log_4 n$.



<u>Figure 4</u>

The diagram shows an excellent agreement of A_n to $\log_4 n$ already for small values of n. It also shows the rapid convergence of $M_n - \log_4 n$ to the term $D(\log_4 n)$ centered around $a_0 = 0.292$, whose periodicity is apparent.

Genesis of “Analytic Combinatorics”

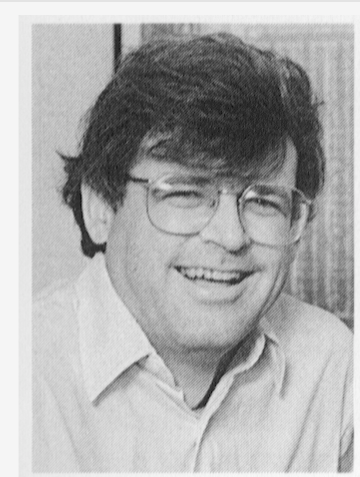
Late 1970s / early 1980s: optimism and opportunity

Knuth volumes 1-3

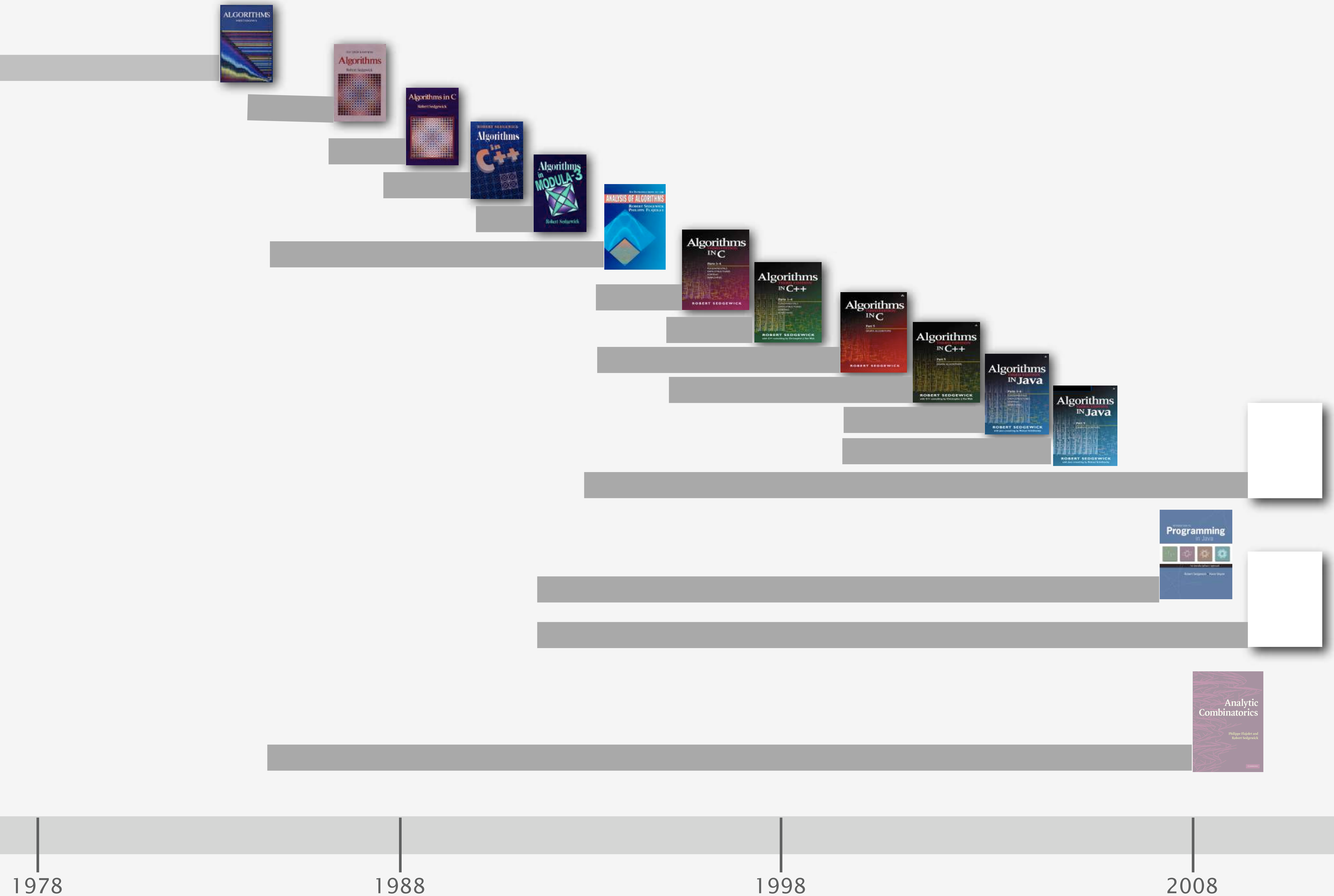
Search for generality

Algorithms for the masses

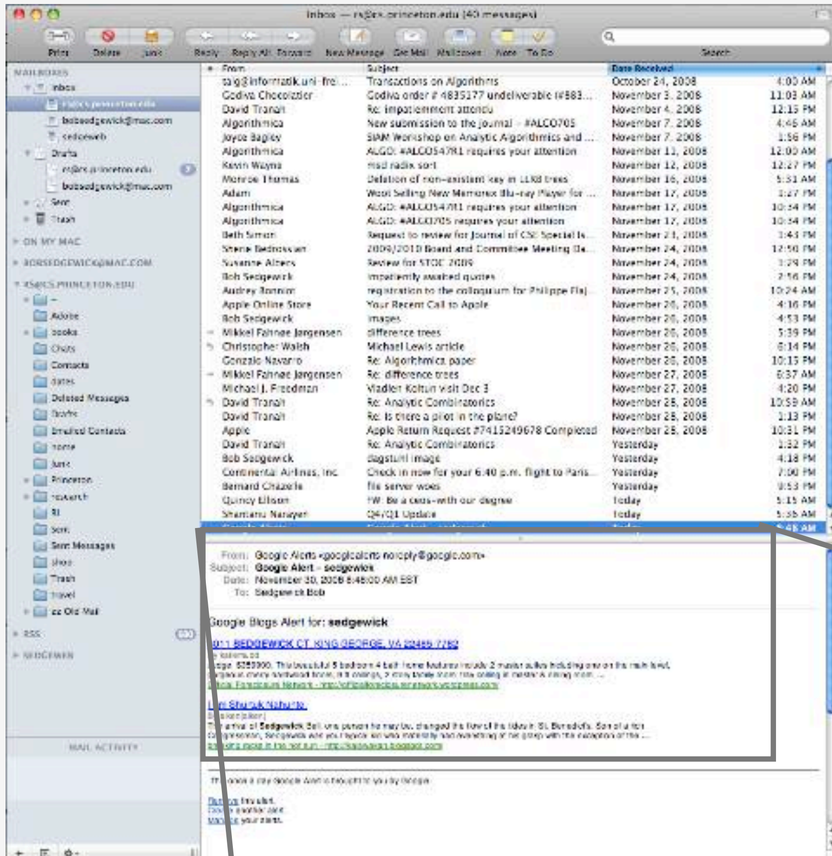
Teaching and research in AofA

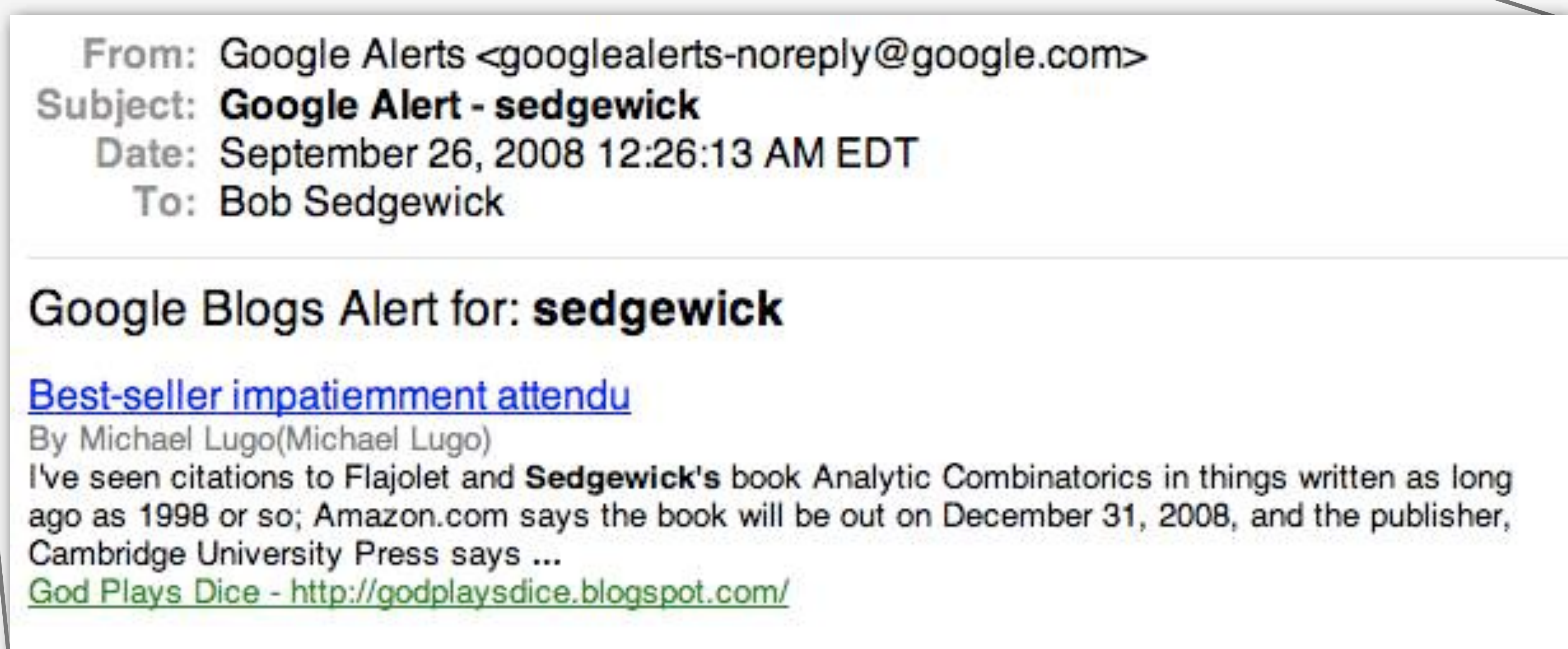


When will the book be available?



An alert: “best-seller impatiemment attendu”





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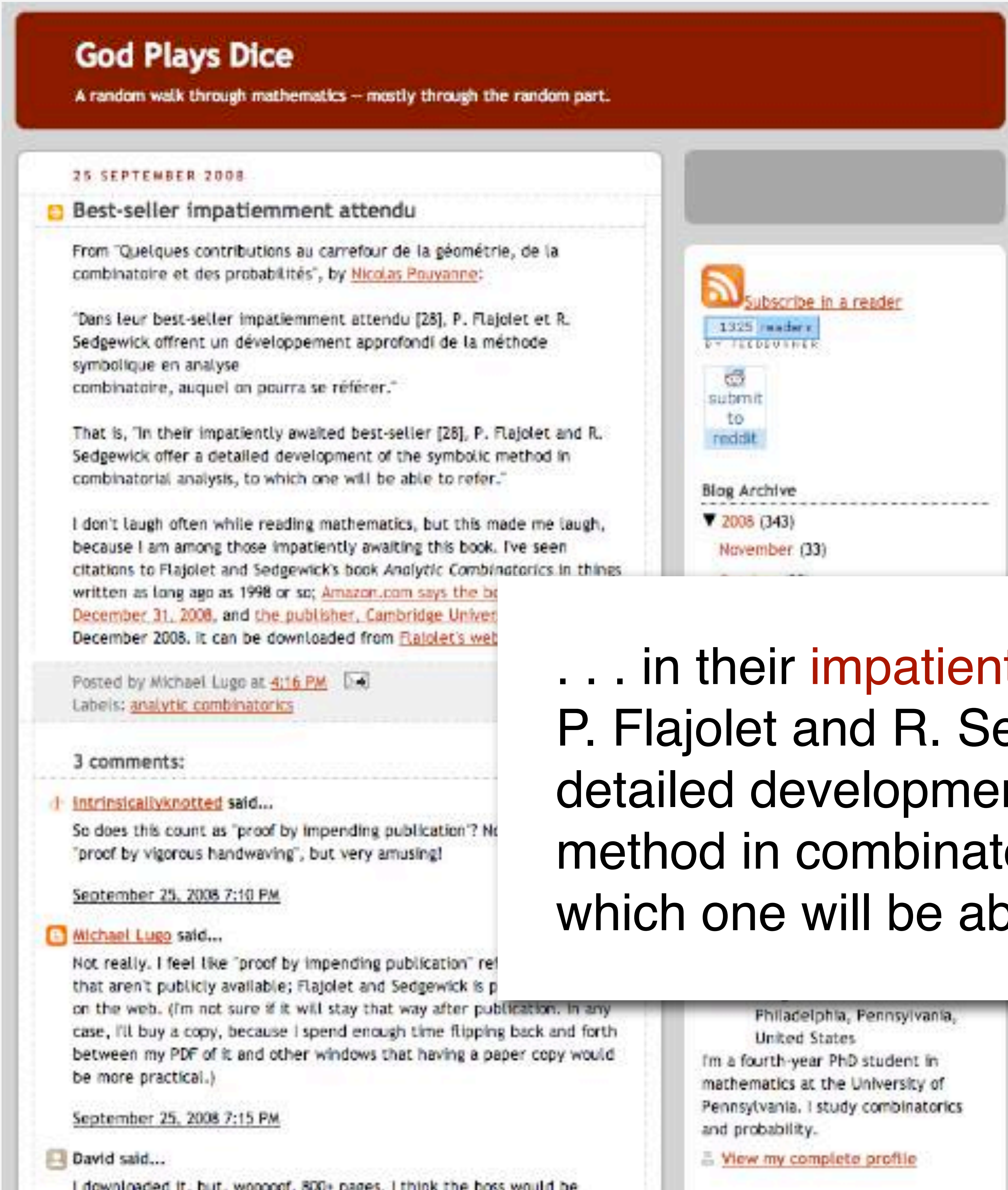


. . . in their impatiently awaited best-seller, P. Flajolet and R. Sedgewick offer a detailed development of the symbolic method in combinatorial analysis, to which one will be able to refer.

“Impatiently awaited” on the web

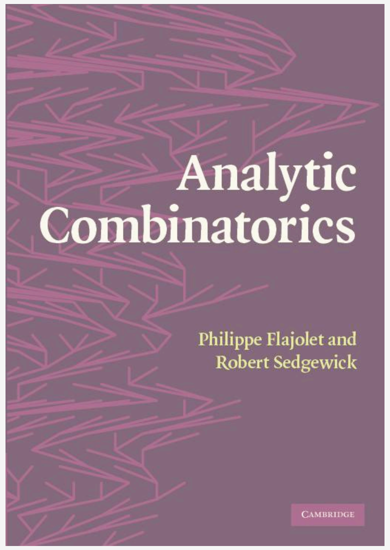
Prince of Persia

plasma display

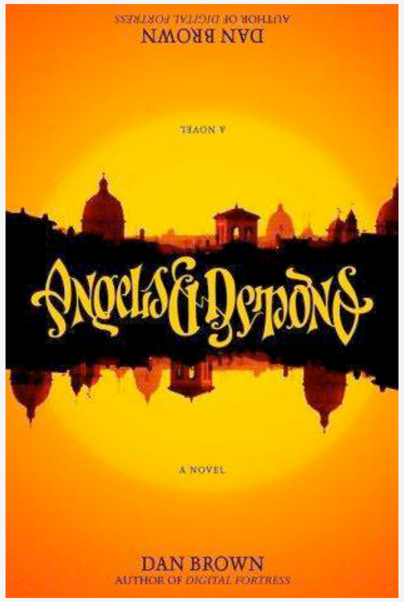
iPod

student visa

Berkeley Unix 4.4



Dan Brown novel



Heroes episode

“Impatiently awaited” in literature

Prince Andrew listened to the account of the opening of the Council of State, which he had so impatiently awaited and to which he had attached such importance

Leo Tolstoy, War and Peace

Custer and his cavalry contingent impatiently awaited marching orders.

Kingsley Bray, Crazy Horse: A Lakota Life

This done, he impatiently awaited the return of his companions.

Alexandre Dumas, The Count of Monte Cristo

A handsome young fellow like you does not obtain long leaves of absence from his mistress, and we were impatiently awaited at Paris, were we not ?

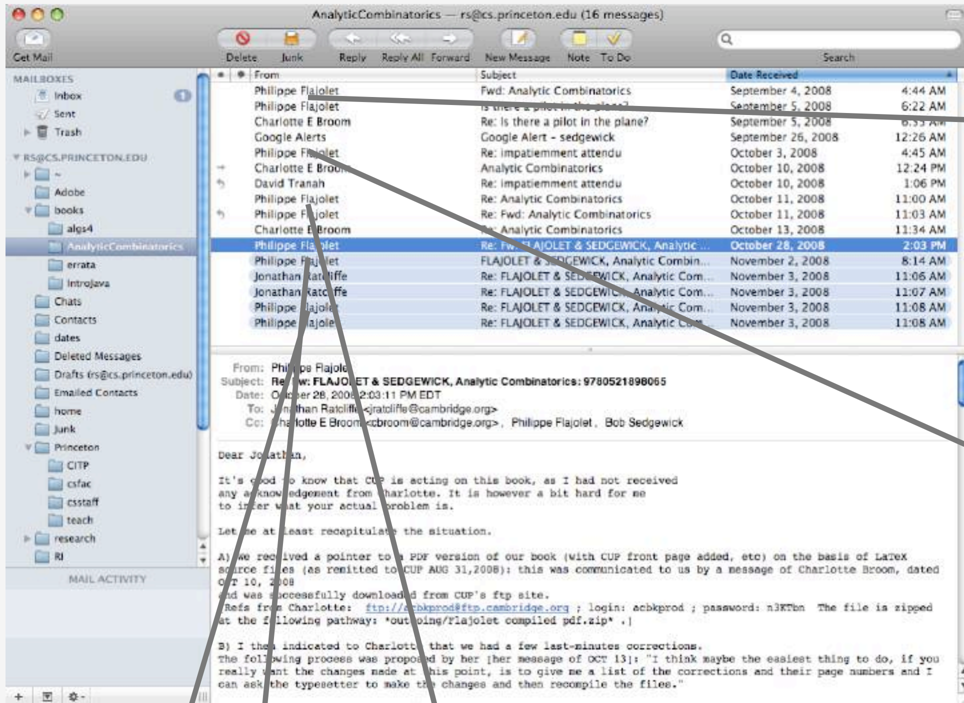
Alexandre Dumas, The Three Musketeers

He impatiently awaited her husband’s departure.

Guy de Maupassant, Bel Ami

Digression: Separated at birth??

Impatiemment attendu



Last Sunday I sent the message that follows, then a reminder mid-week, but haven't received an ACK. Did you get it? Was it properly addressed?

Is there a pilot in the plane?

At full speed on the highway ... but in reverse gear?

Where are we????????

Did some e-mail message get lost? Please at least ACK having received the message below, first sent OCT 28, 2008.

Patience

Dear Phillippe,

Apologies for not acknowledging, but I have been working around the clock to get this book to press which it did this morning. All of your corrections have been incorporated and we are on schedule for a stock date of 28th November.

Best Wishes,

[CUP staff member]

From: david.tranah@gmail.com
Subject: Re: Is there a pilot in the plane?
Date: November 28, 2008 1:12:56 PM EST
To: Philippe.Flajolet@inria.fr
Cc: rs@CS.Princeton.EDU, dtranah@cambridge.org

Dear Philippe and Bob

Just a quick note to say that I am looking forward to meeting you both again next week at the 60th birthday colloquium, and that we are on schedule to have books on display for SODA in NYC in January and at the annual meeting of the American Mathematical Society in Washington, also in January.

best wishes

david

It's on the web. Why publish a book?

The New York Times

OP-ED CONTRIBUTOR

How to Publish Without Perishing

By JAMES GLEICK
Published: November 29, 2008

As a technology, the book is like a hammer. That is to say, it is perfect: a tool ideally suited to its task. Hammers can be tweaked and varied but will never go obsolete. Even when builders pound nails by the thousand with pneumatic nail guns, every household needs a hammer.

....Now even modest titles have been granted a gift of unlimited longevity.

What should an old-fashioned book publisher do with this gift? Forget about cost-cutting and the mass market. Don't aim for instant blockbuster successes. You won't win on quick distribution, and you won't win on price. Cyberspace has that covered.

Go back to an old-fashioned idea: that a book, printed in ink on durable paper, acid-free for longevity, is a thing of beauty. Make it as well as you can. People want to cherish it.

Analytic Combinatorics

Philippe Flajolet and
Robert Sedgewick

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