VISUALIZING
THE ANALYSIS of ALGORITHMS

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Can graphics be
``more precise and revealing``
than mathematical formulae?

Ground rules

Graphical images can
* enhance understanding of technical concepts
* help identify or clarify technical goals
* reveal otherwise unnoticable characteristics

Advances in technology enable
creation of new types of images

Analysis of Algorithms researcher's toolkit
[GFs, asymptotics, special functions....]
TeX
Maple
C/C++
PostScript
Java

THEMES
Apply basic programming skills
Learn basic graphic design principles
Exploit mass-market technologies

``What tools would Euler be using?``

Graphic design for data visualization

GOAL: Communicate complex ideas with
clarity, precision, and efficiency

Ref: E. W. Tufte
The Visual Display
of Quantitative Information (1983)
Envisioning Information (1990)
Visual Explanations (1997)
Oriented towards statistical data sets
but basic principles are generally applicable

TUFTE: ``Graphical displays should
* show the data
* induce the viewer to think about substance
* avoid distorting what the data say
* present many numbers in a small place
* encourage comparison of different data pieces
* reveal the data at several levels of detail,
broad ovewrview to fine structure
* serve a clear purpose:
description, exploration, tabulation, decoration
* be closely integrated with statistical
and verbal description of data``

Tufte's books elaborate on these ideas
with extensive illustrative examples

Low-tech example

Basic divide-and-conquer recurrences

- study of mergesort
- properties of bitstrings
- arithmetic algorithms
- divide-and-conquer algorithms

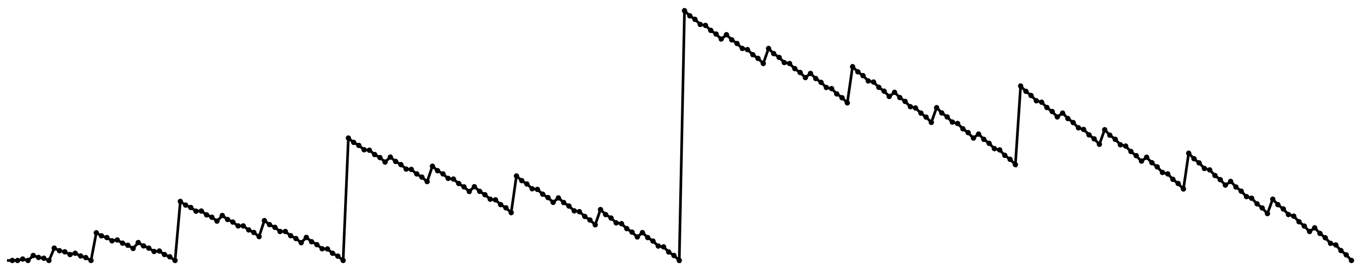
Ex: C program to print values of recurrence

```
#include <math.h>
#include <stdio.h>
void main(int argc, char* argv[])
 { int i, N, c[32], d[32];
   c[0] = 0; c[1] = 0;
   for (N = 2; N < 32; N++)
     {
       c[N] = 2*c[N/2] + N;
       d[N] = N*(log((float) N)/log(2.0))-c[N];
       printf("%2d %3d %3d ", N, c[N], d[N]);
       for (i = 0; i < d[N]; i++) printf(" ");
       printf("* ");
     }
 }
```

``Graphical`` version: loop to print spaces and *

``Higher tech``: use Postscript (stay tuned)

```
.   2  0 *
.   3  1  *
.   4  0 *
.   5  2   *
.   6  3    *
.   7  6       *
.   8  0 *
.   9  3    *
.  10  5      *
.  11  9          *
.  12  7        *
.  13 11            *
.  14 13              *
.  15 17                  *
.  16  0 *
.  17  4     *
.  18  7        *
.  19 11            *
.  20 10           *
.  21 15                *
.  22 18                   *
.  23 23                        *
.  24 14               *
.  25 19                    *
.  26 22                       *
.  27 27                            *
.  28 26                           *
.  29 31                                *
.  30 35                                    *
.  31 40                                         *
```



Aside: ancient low-tech example

Best case of quicksort
(with cutoff to insertion for small files)
Which partitioning values
minimize the total number of comparisons?

```
#include <stdio.h>
#include <stdlib.h>
void main(int argc, char* argv[])
 { int i, j, N = atoi(argv[1]), M = atoi(argv[2]);
   int c[1000], t[1000];
   for (i = 0; i < M; i++) c[i] = 0;
   for (i = M; i < N; i++)
     {
       for (j = 0; j < i; j++)
         t[j] = c[j] + c[i-j-1] + i;
       for (j = 0, c[i] = 1000000; j < i; j++)
         if (t[j] < c[i]) c[i] = t[j];
       for (j = 0; j < 40-i; j++) printf(" ");
       for (j = 0; j < i; j++)
         if (t[j] == c[i]) printf("* ");
                      else printf("  ");
       printf(" ");
     }
 }
```

Ref: Sedgewick, ``Quicksort,'' 1972

```
        *
       * *
        *
       * *
      * * *
       * *
        *
       * *
      * * *
     * * * *
    * * * * *
     * * * *
      * * *
       * *
        *
       * *
      * * *
     * * * *
    * * * * *
   * * * * * *
  * * * * * * *
 * * * * * * * *
* * * * * * * * *
 * * * * * * * *
  * * * * * * *
   * * * * * *
    * * * * *
     * * * *
      * * *
       * *
        *
       * *
      * * *
     * * * *
    * * * * *
   * * * * * *
  * * * * * * *
 * * * * * * * *
* * * * * * * * *
```

```
       * *
        *
       * *
      *   *
       * *
        *
       * *
      *   *
     * * * *
    *   *   *
     * * * *
      *   *
       * *
        *
       * *
      *   *
     * * * *
    *   *   *
   * * * * * *
  *   *   *   *
 * * * * * * * *
*   *   *   *   *
 * * * * * * * *
  *   *   *   *
   * * * * * *
    *   *   *
     * * * *
      *   *
       * *
        *
       * *
      *   *
     * * * *
    *   *   *
   * * * * * *
  *   *   *   *
 * * * * * * * *
*   *   *   *   *
```

```
          * * *
           * *
            *
           * *
          *   *
         *     *
          * * *
           * *
            *
           * *
          *   *
         *     *
        * * * * *
       *   * *   *
      *     *     *
       * * * * * *
        * *   * *
         *     *
          * * *
           * *
            *
           * *
          *   *
         *     *
        * * * * *
       *   * *   *
      *     *     *
     * * * * * * * *
    *   * *   * *   *
   *     *     *     *
  * * * * * * * * * * *
 *   * *   * *   * *   *
*     *     *     *     *
 * * * * * * * * * * * *
  * *   * *   * *   * *
   *     *     *     *
    * * * * * * * * *
```

```
     * * * *
      * * *
       * *
        *
       * *
      *   *
     *     *
    *       *
     * * * *
      * * *
       * *
        *
       * *
      *   *
     *     *
    *       *
   * * * * * *
  *   * * *   *
 *     * *     *
*       *       *
 * * * * * * * *
  * * *   * * *
   * *     * *
    *       *
     * * * *
      * * *
       * *
        *
       * *
      *   *
     *     *
    *       *
   * * * * * *
  *   * * *   *
 *     * *     *
*       *       *
```

Example: a familiar table of numbers

Binomial coefficients

```
.                     1
.                  1   1
.                1   2   1
.              1   3   3   1
.            1   4   6   4   1
.          1   5  10  10   5   1
.        1   6  15  20  15   6   1
.      1   7  21  35  35  21   7   1
.    1   8  28  56  70  56  28   8   1
.   1   9  36  84 126 126  84  36   9   1
```

Binomial distribution

```
.                  1
.              1/2   1/2
.           1/4   2/4   1/4
.        1/8   3/8   3/8   1/8
.    1/16  4/16  6/16  4/16  1/16
. 1/32  5/32 10/32 10/32  5/32  1/32
```

PostScript

Available in all modern computing environments
  [basic language in printing industry]
Has all basic components for our [modest] needs

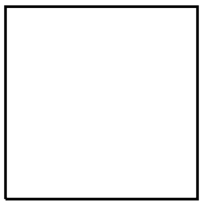
Postfix language, uses abstract stack machine
Ex: convert 9753 from hex to decimal (Horner alg)

```
9 16 mul 7 add 16 mul 5 add 16 mul 3 add
```

Coordinate system: rotate, translate, scale, ...
Turtle commands: moveto, lineto, rmoveto, rlineto,
Graphics commands: stroke, fill, ...
Arithmetic: add, sub, mul, div, ...
Stack commands: exch, dup, currentpoint, ...
Control: if, ifelse, while, for, ...
Define: /xx { ... } def

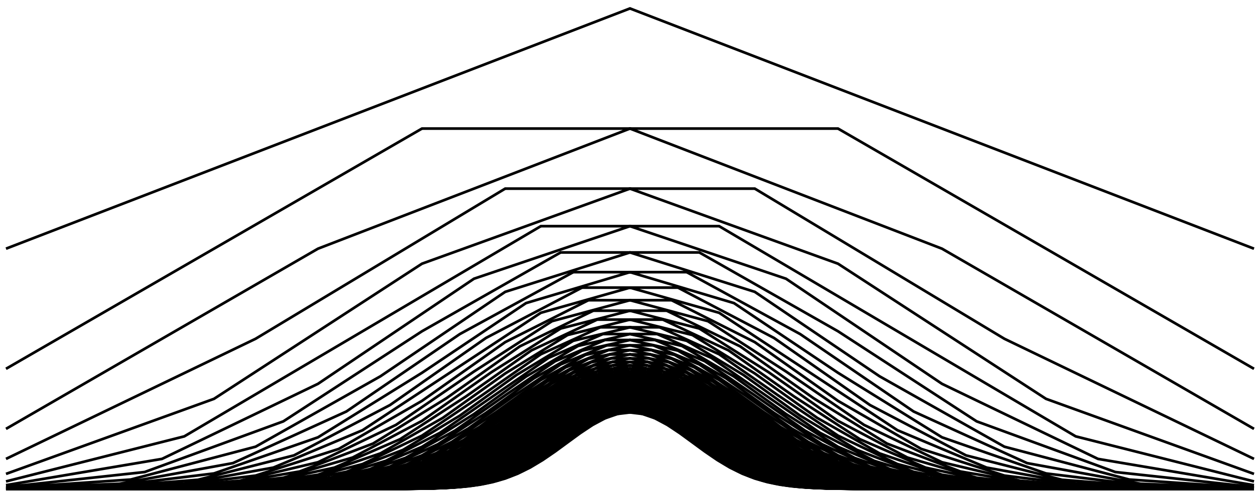
Everyone's first program: draw a box

```
%!
36 36 translate 0 0 moveto
0  72 rlineto  72 0 rlineto
0 -72 rlineto -72 0 rlineto
stroke
showpage
```



PostScript binomial distribution plot

```
%!
/inch { 72 mul } def
/Xsize 6.5 inch def /Ysize 4 inch def
/myscale
   { Ysize mul 2 N exp div exch
      N div Xsize mul exch } def
2 1 100
   { /N exch def
      newpath
      /Y 1 def
      0 Y myscale moveto
      1 1 N
         { /k exch def
            /Y Y N k sub 1 add mul k div def
            k Y myscale lineto
         } for
      stroke
   } for
showpage
```



Other familiar distributions

Graphic design ideas for binomial plot
- scale Y axis to [0, 1]
- scale X axis to [0, 1]
- superimpose all plots

Apply to numerous distributions
- Stirling numbers
- Catalan
- Eulerian
- size-cost in analysis of algorithms
  - Quicksort
  - tries
  - AVL trees
- ...

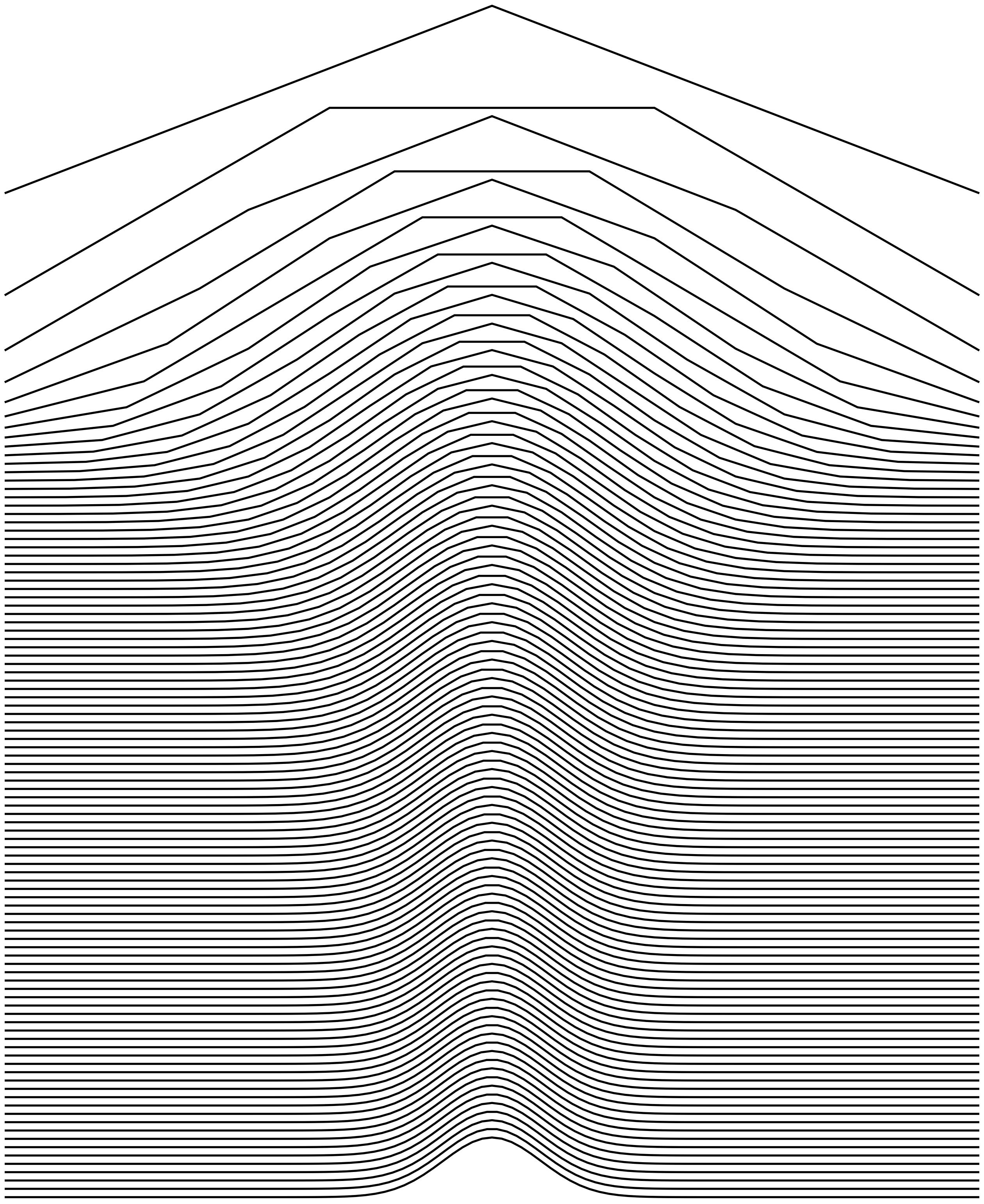
[cf. Flajolet-Sedgewick]

Trivial changes in programs
lead to striking differences in images

Ex: spread curves by a few points

See ASCII slides on talk web page
for Postscript code

Binomial distribution with spread curves



Cost of computation

How much time is required to produce a plot?

``Analysis of Algorithms`` problem
    [If we can't do this, who can!!]

Binomial plot: time proportional to N^2 ?
    [not really: depends on resolution]

Modern personal computer
    over 250 million operations per second

Million bits
    three-inch square image at 300 dpi

BOTTOM LINE
    now feasible to do 100s or 1000s of ops per bit

Note: EVERY ``image processing`` program does
    significant computation for each bit!

Aside: bitmap computational geometry algs
    now practical for huge problems

Caveat: printer may not be as fast as PC

Example: complex functions

Graphic design idea:

use color scale for absolute values

Postscript implementation

simple library of complex functions

compute each bit (!!)

Basic loop to plot each point

```
0 1 nY
   { /Y exch def
      0 1 nX
         { /X exch def
            X dx mul Y dy mul color pt
         } for
   } for
```

Graphics functions (scaling omitted)

```
/pt { sz 0 360 arc fill } def
/color
   {
      X Y scale /y exch def /x exch def f ABS
      dup MAX gt
         { pop 0.0 }
         { MAX div 1 exch sub}
      ifelse
      dup dup sethsbcolor
   } def
```

Complex functions (continued)

PostScript functions implement "complex" type

```
/Z { x y } def
/SUB
   { /d exch def /c exch def
      /b exch def /a exch def
      a c sub b d sub } def
/DIV
   { /d exch def /c exch def
      /b exch def /a exch def
      /dd c dup mul d dup mul add def
      a c mul b d mul add dd hackdiv
      b c mul a d mul sub dd hackdiv
   } def
/ABS { dup mul exch dup mul add sqrt } def
/f { 1 0 1 0 Z SUB DIV } def
```

Now can "plot" arbitrary complex functions

Examples from the analysis of algorithms

rational polynomial GFs

quicksort

tries

Catalan

...

Perspective

Three examples
- printing asterisks
- plotting curves
- set colors in bitmap

Characteristics
- easy to implement
- follow Tufte's principles
- expose essential algorithm-analysis concept

Why not use Maple, Mathematica plot packages?
- plus:
  - extensive built-in library
  - 3D rendering, etc., etc
- minus:
  - design inflexibility
  - graphics computation cost

Scratching the surface of design possibilities
- black-and-white
- grayscale
- color
- animation

Basic results broadly applicable