INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

The Average Case Analysis of Algorithms: Mellin Transform Asymptotics

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N° 2956

Août 1996

PROGRAMME 2

Rapport de recherche

1996

ISSN 0249-6399

THE AVERAGE CASE ANALYSIS OF ALGORITHMS: *Mellin Transform Asymptotics*

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Abstract. *This report is part of a series whose aim is to present in a synthetic way the major methods of "analytic combinatorics" needed in the average–case analysis of algorithms. It reviews the use of Mellin-Perron formulæ and of Mellin transforms in this context. Applications include: divide-and-conquer recurrences, maxima finding, mergesort, digital trees and plane trees.*

L'ANALYSE EN MOYENNE D'ALGORITHMES: *La transformation de Mellin*

Résumé. Ce rapport fait partie d'une série dont le but est de présenter de manière unifiée les principales méthodes de "combinatoire analytique" utiles à l'analyse d'algorithmes. Il y est décrit l'utilisation des formules de Mellin-Perron et de la transformation de Mellin dans ce contexte. Les applications comprennent: les récurrences diviser-pour-régner, la recherche de maxima, le tri-fusion, ainsi que les arbres digitaux et les arbres plans.

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Foreword

This report is part of a projected series whose aim is to present in a synthetic way the major methods and models in the average–case analysis of algorithms. It belongs to a collection of reports relative to "*Analytic Combinatorics*" that comprises the following sections:

I. Counting and generating functions.

II. Complex asymptotics and generating functions.

III. Saddle point asymptotics.

IV. Mellin transform asymptotics.

V. Functional equations.

VI. Multivariate asymptotics.

Parts I, II, and III consist of 6 chapters that have been issued as INRIA Research Reports 1888 (116 pages, 1993), 2026 (100 pages, 1993), and 2376 (55 pages, 1994).

The present report constitutes Section IV. It is devoted to complex asymptotics methods based on Mellin transforms. It consists of one chapter (numbered consecutively after those of Parts I, II, III):

7. Mellin Transform Asymptotics.

Acknowledgements. The work of Philippe Flajolet was supported in part by the Long Term Research Project of the European Union ALCOM-IT (# 20244).

The authors are grateful to Tsutomu Kawabata for a careful scrutiny of the paper and for detailed technical comments.

Chapter 7

Mellin Transform Asymptotics

Die Theorie der reziproken Funktionen und Integrale ist ein centrales Gebiet, welches manche anderen Gebiete der Analysis miteinander verbindet.

— Hjalmar Mellin

This chapter presents a collection of closely related methods for the asymptotic analysis of sums that arise in combinatorial problems and have a number-theoretic flavour. Such sums involve coefficients either related to the multiplicative structure of integers (the number of divisors in the analysis of the expected height of plane trees), to the binary representation of integers (the number of 1-digits for the best case of the sorting method known as "mergesort"), or to the powers of 2 (the Bernoulli splitting process and the analysis of digital trees also known as "tries" in computing applications.

Typically, what is required there is to estimate asymptotically sums of more or less standard functions weighted by coefficients that fluctuate rather wildly, the resulting estimates themselves showing sometimes traces of such oscillatory behaviour.

The Mellin transform is a classical integral transform closely related to the Laplace and Fourier transforms. It establishes an explicit mapping between the asymptotic expansions of a function near zero and infinity on the one hand, and the set of singularities of the transform in the complex plane on the other hand. At the same time, it transforms a general class of sums, called harmonic sums, into a tractable factored form. Regarding applications in combinatorics and the analysis of algorithms, the power of Mellin transform methods derives in an essential way from the combination of these two features.

There is actually a whole galaxy of methods related to Mellin transform. A simple type that is especially close to basic analytic number theory deals with the analysis of coefficients of Dirichlet series, and we start our exposition with examples falling into this category. An important application is the exact asymptotics of divide-and-conquer recurrences that arise in connection with one of the most fruitful paradigm of algorithm design. This is a subject that we discuss in some detail and Mellin related techniques are especially instrumental in analyzing the fractal component that is often present in such algorithms.

Mellin transforms largely originate in number theory, going back to Riemann's celebrated memoir on the distribution of prime numbers. They have found applications in the theory of functions, as initially showed by Mellin (see [33] for a biographical notice with references), and in various areas of applied mathematics. As should become apparent in this chapter, Mellin transform are also part of the arsenal of asymptotic methods for discrete mathematics and the analysis of algorithms.

This chapter is based on a series of papers dealing with Mellin transform asymptotics in analytic combinatorics [15, 16, 17, 22].

7.1 Dirichlet series and coefficient formulae

Many applications in combinatorial analysis, discrete probability, and the analysis of algorithms involve quantities of an arithmetic nature. For instance, the analysis of the expected height of plane trees involves the divisor function $d(k)$ (the number of divisors of integer k), register allocation and some sorting networks lead to quantities related to the binary representation of integers, like $v_2(k)$ (the exponent of 2 in the prime number decomposition of integer k) or $\nu(k)$ (the number of 1-digits in the binary representation of k), etc.

Algebra of Dirichlet series. In situations involving arithmetic quantities, asymptotic estimates are best performed by using Dirichlet generating functions rather than ordinary or exponential generating functions. A clear treatment of the elementary aspects discussed in this section can be found in Apostol's book [2].

Definition 7.1 *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers. The* Dirichlet generating function, DGF *in short, of the sequence is the formal sum*

$$\alpha(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

The simplest Dirichlet series is the zeta function of Riemann,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where the sum defines an analytic function of s in the half-plane $\Re(s) > 1$. Other examples of Dirichlet series are

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \qquad \sum_{k=1}^{\infty} \frac{1}{(2^k)^s} = \frac{2^{-s}}{1 - 2^{-s}},$$

which defines the "alternating" zeta function, and the DGF of the characteristic function of powers of 2 for $\Re(s) > 0$.

Given three DGFs, $\alpha(s), \beta(s), \gamma(s)$ with coefficients a_n, b_n, c_n, sum and product relations translate over coefficients as follows,

$$\begin{aligned} \alpha(s) = \beta(s) + \gamma(s) \quad &\Longrightarrow \quad a_n = b_n + c_n \\ \alpha(s) = \beta(s) \cdot \gamma(s) \quad &\Longrightarrow \quad a_n = \sum_{d \mid n} b_d c_{n/d}, \end{aligned}$$

where the sum is over the integers $d \geq 1$ that divide n, a property written $d \mid n$. The relation that corresponds to the product is called the multiplicative convolution or the Dirichlet convolution of coefficients. From it, we see for instance that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s), \qquad \sum_{n=1}^{\infty} \frac{v_2(n)}{n^s} = \frac{\zeta(s)}{1 - 2^{-s}}.$$

In the same vein, one has the famous product formula of Euler for $\zeta(s)$:

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \tag{7.1}$$

where the product ranges over all primes p. The identity (7.1) is easily checked by distributing the products in

$$\prod_p \frac{1}{1-p^{-s}} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right),$$

and it is logically equivalent to the property that every integer decomposes uniquely as a product of prime powers.

An interesting application of Dirichlet convolutions is to the famous *Moebius inversion* relations. Define $\mu(n)$, the Moebius function, by

$$\mu(p_1^{\alpha_1}\cdots p_r^{\alpha_r}) = \begin{cases} (-1)^r & \text{if } \alpha_1 = \cdots = \alpha_r = 1 \\ 0 & \text{if some } \alpha_j \geq 2. \end{cases}$$

and $\mu(1) = 1$. Then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

The relation $\alpha(s) = \beta(s)\zeta(s)$ is equivalent to $a_n = \sum_{d \mid n} b_d$. Solving for b_n is achieved by solving for $\beta(s)$ which gives $\beta(s) = \alpha(s)\zeta^{-1}(s)$. Expressing the Dirichlet convolution in turn yields the Moebius inversion relation:

$$a_n = \sum_{d \mid n} b_d \qquad \Longrightarrow \qquad b_n = \sum_{d \mid n} a_d \, \mu(\frac{n}{d}).$$

Moebius inversion is in particular useful in dealing with ordinary generating functions. The infinite functional equation

$$\sum_{d=1}^{\infty} f(z^d) = g(z),$$

with $g(z) = O(z)$ at $z = 0$, admits the formal solution

$$f(z) = \sum_{d=1}^{\infty} \mu(d) g(z^d),$$

as is directly obtained by applying Moebius inversion to the induced relation on coefficients. As an application, let the class $\mathcal{G}$ be the multiset construction (as defined in Chapter 1) applied to $\mathcal{F}$. Then,

$$G(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} F(z^m)\right),$$

or, taking logarithms and applying Moebius inversion on coefficients,

$$F(z) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log G(z^d).$$

This yields an explicit enumeration for the component class $\mathcal{F}$ when the function (OGF) of the multiset class $\mathcal{G}$ is known. A typical instance is the counting according to degree of the class $\mathcal{F} \subset \mathcal{G} = GF(q)[x]$ of irreducible (monic) polynomials over a finite field for which $G(z) = (1 - qz)^{-1}$: the number of irreducible polynomials of degree n is

$$F_n = \frac{q^n}{n} \sum_{d \mid n} \mu(d) q^{n/d}.$$

Admissibility of the unlabelled cycle construction, as stated in Chapter 1, obeys similar principles.

Exercise 1. Find the DGFs of $\log(n)$, of $p(n)$ and of $p(n)q(\log n)$ with p, q arbitrary polynomials.

Exercise 2. For DGFs $\alpha_j(s) = \sum a_{j,n} n^{-s}$ and $\beta(s) = \sum_n b_n n^{-s}$ that satisfy $\beta(s) = \alpha_1(s)\alpha_2(s)\alpha_3(s)$, one has

$$b_n = \sum_{n_1 n_2 n_3 = n} a_{1,n_1} a_{2,n_2} a_{3,n_3}.$$

Exercise 3. The OGFs of $d(k)$ and $v_2(k)$ are given by

$$D(z) = \sum_{k=1}^{\infty} d(k) z^k = \sum_{m=1}^{\infty} \frac{z^m}{1 - z^m}, \quad V_2(z) = \sum_{k=1}^{\infty} v_2(k) z^k = \sum_{m=1}^{\infty} \frac{z^{2^m}}{1 - z^{2^m}}.$$

Show that $D(z) \sim (1 - z)^{-1} \log(1 - z)^{-1}$ as $z \to 1^-$ and $V_2(z) \sim \log_2(1 - z)^{-1}$, but that singularity analysis cannot be applied as the functions cannot be extended beyond $|z| = 1$.

Exercise 4. Prove the translation of the unlabelled cycle construction stated in Chapter 1.

Exercise 5. Assuming that $g(x)$ is analytic at 0 and $g(0) = g'(0) = 0$, then

$$\sum_{n=1}^{\infty} f(nx) = g(x) \quad \Longrightarrow \quad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx).$$

$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} \qquad (\Re(s) > 1)$	Definition
$(1-2^{1-s})\zeta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s} \qquad (\Re(s) > 0)$	Alternating zeta function
$\zeta(s) = \prod_p (1-p^{-s})^{-1} \qquad (\Re(s) > 1)$	Euler product
$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s} \qquad (\Re(s) > 1)$	Moebius function
$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!}(-1)^{m-1}B_{2m}$	Bernoulli numbers and zetas
$\zeta(-2m) = 0, \quad \zeta(-2m+1) = -\frac{B_{2m}}{2m}$	Bernoulli numbers and zetas
$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = \log\sqrt{2\pi}$	
$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\zeta(s)\cos(\frac{\pi s}{2})$	Functional equation (1)
$\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s) = \Gamma(\frac{1-s}{2})\pi^{-(1-s)/2}\zeta(1-s)$	Functional equation (2)
$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \cdots \quad (s \to 1)$	Singular expansion at $s = 1$
$\zeta(s) = \mathcal{O}(t^{1/2-\Re(s)}) \qquad (\Im(s) \to \pm\infty, \Re(s) < 0)$	Growth at $\pm i\infty$

Figure 7.1: A summary of the main properties of the zeta function.

Analysis of Dirichlet series. A clear introduction to the analytic properties of Dirichlet series is given in Chapter IX of Titchmarsh's book [42] to which we globally refer for this section. It is a standard theorem of the theory of Dirichlet series that they converge in a half–plane $\Re(s) > \sigma_c$ and converge absolutely in a (possibly smaller) half–plane $\Re(s) > \sigma_a$, where by elementary analysis $0 \leq \sigma_a - \sigma_c \leq 1$, see [42, p. 290–292]. Thus, Dirichlet series exist and are analytic in half-planes. As we have seen, the pair (σ_c, σ_a) is $(1, 1)$ for $\zeta(s)$. For the alternating zeta function, it is $(0, 1)$, for the characteristic function of powers of 2, it is $(0, 0)$. The half-plane may be empty as in the DGF of $a_n = 2^n$ for which $\sigma_c = \sigma_a = +\infty$ or equal to the whole complex plane as for $a_n = 2^{-n}$ for which $\sigma_c = \sigma_a = -\infty$. In full generality, a Dirichlet series exists analytically in some region only if its coefficients are polynomially bounded.

It is a classical theorem [42, 44] that $\zeta(s)$ admits an analytic continuation: $\zeta(s)$ extends to a meromorphic function in the whole of C with only a simple pole at $s = 1$, near which

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots, \tag{7.2}$$

where γ is the usual Euler constant,

$$\gamma = \lim_{n\to+\infty} H_n - \log n,$$

and the γ_j are sometimes called the Stieltjes constants. First, the extension to $\Re(s) > 0$ can be deduced elementarily from the fact that the alternating zeta function is by design analytic in $\Re(s) > 0$. Next, the extension to the whole of C results from a much stronger property, the functional equation of Riemann given below that relates $\zeta(s)$ to $\zeta(1-s)$ and thus permits to continue $\zeta(s)$ to the half-plane $\Re(s) \le 0$.

The functional equation of the zeta function admits two standard forms: the asymmetrical form

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\zeta(s)\cos(\frac{\pi s}{2}), \tag{7.3}$$

and the symmetrical form

$$\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s) = \Gamma(\frac{1-s}{2})\pi^{-(1-s)/2}\zeta(1-s). \tag{7.4}$$

The basic properties of the zeta function including a proof of the functional equation form the subject of [44, Chap. XIII] to which the reader is referred for the exercises that follow.

Exercise 6. [Euler] Prove that the Euler product formula holds true analytically for any $s > 1$. Prove that $\zeta(s) \to +\infty$ as $s \to 1^+$. Show that the property $\zeta(s) \to \infty$ as $s \to 1$ implies that there exist infinitely many prime numbers.

Exercise 7. Use the alternating zeta function to justify directly that $\zeta(s)$ has only a simple pole at $s = 1$ in $\Re(s) > 0$.

EXERCISE 8. Let $B_n = n![z^n]z/(e^z - 1)$ denote the Bernoulli numbers. Then, for m a positive integer,

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!}(-1)^{m-1}B_{2m}.$$

[Hint. Determine the partial fraction decomposition of $\coth z$ and expand. See Chapter 4.]

Prove that for m a positive integer,

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \quad \zeta(0) = -\frac{1}{2}.$$

EXERCISE 9. Use the asymptotic expansion of the harmonic numbers H_n to deduce that the corresponding DGF is meromorphic in the whole of C with poles at a subset of $\{\ldots, -2, -1, 0, 1\}$.

EXERCISE 10. Give an alternative definition of the Stieltjes constants γ_j that generalizes the classical definition of the Euler constant.

Inversion formulae The Cauchy coefficient formula provides a way of relating a power series to its coefficients by means of an integral representation. There is a similar formula for Dirichlet series, although it gives partial sums of coefficients, rather than directly the coefficients themselves. This formula is known as Perron's formula and, as we shall see, it also relates to the collection of general methods based on Mellin transforms.

Theorem 7.1 (Mellin-Perron's coefficient formula) *Let $\alpha(s)$ be the Dirichlet generating function of the sequence $\{a_n\}$. Let $c > 0$ lie inside the half-plane of absolute convergence of $\alpha(s)$.*

(i) The partial sums of the a_n are given by

$$\sum_{1 \le k < n} a_k + \frac{1}{2}a_n = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \alpha(s) n^s \, \frac{ds}{s}. \tag{7.5}$$

(ii) Iterated sums of the a_n for $m \ge 1$ are given by

$$\frac{1}{m!}\sum_{1 \le k < n} a_k(1 - \frac{k}{n})^m = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \alpha(s) n^s \frac{ds}{s(s+1)\cdots(s+m)}. \tag{7.6}$$

It is interesting to observe that power series are "generated" by z^n while Dirichlet series are "generated" by n^{-z}; accordingly, the inversion formulae involve z^{-n} for the former (Cauchy's coefficient formula) and n^z for the latter (Mellin-Perron's coefficient formula).

Proof. We establish here the case $m = 1$ of part (ii). Detailed proofs may be found in almost any book on analytic number theory, see for instance [2], the basic version ($m = 0$) appearing also in Titchmarsh's book [42, p. 301]. Take $x > 0$ and consider the integral

$$J_1(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s(s+1)}. \tag{7.7}$$

Closing the line of integration by a large semi-circle to the left, when $x \geq 1$, and taking residues at $s = 0, -1$ into account, one finds that $J_1(x) = 1 - x^{-1}$ in that case (the assumption that $x \geq 1$ is necessary to ensure convergence of the integral taken along the semi-circle). When $x \leq 1$, one can close the contour by a large semi-circle to the right; since there are no poles, one finds that $J_1(x) = 0$ in this case. In summary, we have

$$J_1(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - x^{-1} & \text{if } x \geq 1. \end{cases} \tag{7.8}$$

Now, the left hand side of Equation (7.6) is equal to

$$\sum_{k=1}^{\infty} J_1(\frac{n}{k}) a_k = \frac{1}{n} \sum_{k=1}^{n-1} (n-k) a_k.$$

Thus, the right hand side of (7.6) develops by linearity as (7.7), the exchange of sum and integral being permitted as c lies, by assumption, in the domain where the DGF converges absolutely.

The other cases follow by a similar argument. For instance, for $m = 0$ corresponding to part 1, one operates with the function $J_0(x)$ defined like in (7.7), but with $s(s+1)$ replaced by s, and one finds that $J_0(x) = 0, \frac{1}{2}, 1$ when $x < 1$, $x = 1$ and $x > 1$. □

The connection of the sums in (7.6) with iterated sums (iterated Cesàro averages) should also be clear, as for instance when $m = 1$,

$$\sum_{k=1}^{n-1} (n-k) a_k = \sum_{k=1}^{n-1} \left(\sum_{\ell=1}^{k} a_\ell \right).$$

A favorable situation for applying Theorem 7.1 is to sequences whose *differences* (of some order ≥ 1) admit Dirichlet series that are expressible in terms of standard functions.

EXERCISE 11. State conditions that justify the Mellin-Perron formula of "order -1":

$$a_n = \lim_{T\to\infty} \frac{1}{2i\pi} \int_{c-iT}^{c+iT} \alpha(s) n^s \, ds.$$

EXERCISE 12. Evaluate

$$J_m(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^s \, \frac{ds}{s(s+1)\cdots(s+m-1)}.$$

EXERCISE 13. Evaluate

$$K_m(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} x^s \, \frac{ds}{s^{m+1}}.$$

Deduce a formula for "logarithmic averages" of coefficients of a Dirichlet series that are defined as

$$\sum_{k<n} a_k \left(\log(\frac{n}{k})\right)^m .$$

Growth conditions. The simplest case of application of Mellin-Perron formulae is when the DGFs involved admit a meromorphic continuation in some extended region of the complex plane. A natural approach then consists in pushing the line of integration past the leftmost poles. This situation is in a sense analogous to the analysis of coefficients for power series that are meromorphic. However, as the contour of integration is now infinite, it requires some supplementary condition in order for integrals to be convergent: *the* growth *of the DGF $\alpha(s)$ on vertical lines must be less than the the growth of the kernel's denominator polynomial $s(s+1)\cdots(s+m)$ when $|\Im(s)| \to \infty$.*

The analysis of the growth of Dirichlet series is often not easy, and it sometimes relates to deep conjectures of number theory. Though a DGF is clearly $\mathcal{O}(1)$ inside its half-plane of absolute convergence, it tends to oscillate heavily there [42, p. 293], and the behaviour of its analytic continuation can be rather wild (though constrained by a few general theorems, see [42, p. 299]). For instance, it is still an unproven conjecture, known as Riemann's hypothesis[1], that all the zeros of the $\zeta(s)$ in the strip $0 < \Re(s) < 1$ satisfy $\Re(s) = \frac{1}{2}$. There is however a well established body of knowledge on these questions, and we shall freely borrow from the classical literature. For instance, for $s = \sigma + it$ with $\sigma < 0$, the zeta function is known to satisfy the estimate

$$\zeta(\sigma + it) = \mathcal{O}(t^{\frac{1}{2}-\sigma}) \qquad \text{as} \quad t \to \pm\infty. \tag{7.9}$$

In summary, the Mellin Perron formula plays a rôle analogous to the Cauchy coefficient formula. There is however *a priori* a difficulty since integration is required along vertical lines rather than just circles or closed curves, a fact which may pose delicate convergence problems. Fortunately, in many discrete mathematics problems, only basic growth conditions like (7.9) are needed in order to apply the method.

Exercise 14. Prove that the two functions

$$h(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s}, \qquad \hat{h}(s) = \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^s}$$

are of "finite order": there exists a function $\alpha(s)$ such that $h(s), \hat{h}(s) = \mathcal{O}(|s|^{\alpha(\sigma)})$ in any half-plane $\Re(s) \geq \sigma$.

7.2 Asymptotics with the Mellin-Perron formula

In this section, we focus on the use of the Mellin-Perron formula for estimating asymptotically coefficients of a Dirichlet series. We first treat in detail the estimation of the number of 1-digits in the binary representations of integers, and then extract the general method underlying this special treatment. The next section provides further applications to divide–and conquer recurrences. This section and the next are largely based on [15, 17].

What is striking in all these problems is the occurrence of *fractal* periodic functions with explicit Fourier expansions that are easily captured by the Mellin-Perron formula.

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[1]Such properties directly influence our knowledge of the distribution of primes that is closely related to analytic properties of $\zeta'(s)/\zeta(s)$ or $\log \zeta(s)$.

The sum-of-digits function. The Trollope–Delange formula [10] expresses the quantity $S(n)$ representing the total number of 1-digits in the representation of integers $1, \ldots, n-1$ in terms a *fractal* function that is continuous but nowhere differentiable.

First, it is not hard to see that

$$S(n) = \frac{1}{2} n \log_2 n + o(n \log n),$$

since, asymptotically, the binary representations contains roughly as many 0's as 1's. A plot of the difference between $S(n)$ and its asymptotic equivalent $\frac{1}{2} n \log_2 n$ reveals a somewhat erratic behaviour that still seems to obeys three laws: the difference increases in proportion to n; a similar behaviour repeats itself between consecutive powers of 2; that behaviour is not smooth but rather strongly oscillating. In fact the local oscillations have a self-similar aspect that is characteristic of a *fractal* behaviour. Delange's theorem precisely quantifies what happens.

Proposition 7.1 (Delange's digital theorem) *The number of 1-digits in the binary representation of numbers* $1, \ldots, n-1$ *satisfies*

$$S(n) = \frac{1}{2} n \log_2 n + nP(\log_2 n),$$

where $P(u)$ *is a periodic function of period 1 that is representable by the Fourier series* $P(u) = \sum_{k \in Z} p_k e^{2ik\pi u}$, *and*

$$\begin{aligned} p_0 &= \log_2 \sqrt{\frac{\pi}{2}} - \frac{3}{4} \\ p_k &= -\frac{1}{\log 2} \frac{\zeta(\chi_k)}{\chi_k(\chi_k + 1)} \qquad \textit{with} \qquad \chi_k = \frac{2ik\pi}{\log 2} \quad (k \in Z \setminus \{0\}). \end{aligned}$$

The proof of this proposition reveals most of the features of the Mellin-Perron formula that are essential for the analysis of divide-and-conquer recurrences and algorithms.

Proof. We note that by definition,

$$S(n) = \sum_{k=1}^{n-1} \nu(k),$$

with $\nu(k)$ the number of ones in the binary representation of k. Next, the function $\nu(k)$ satisfies the obvious identity

$$\nu(k) - \nu(k-1) = 1 - v_2(k) \qquad \text{or} \qquad \nu(k) = k - \sum_{\ell=1}^{k} v_2(\ell),$$

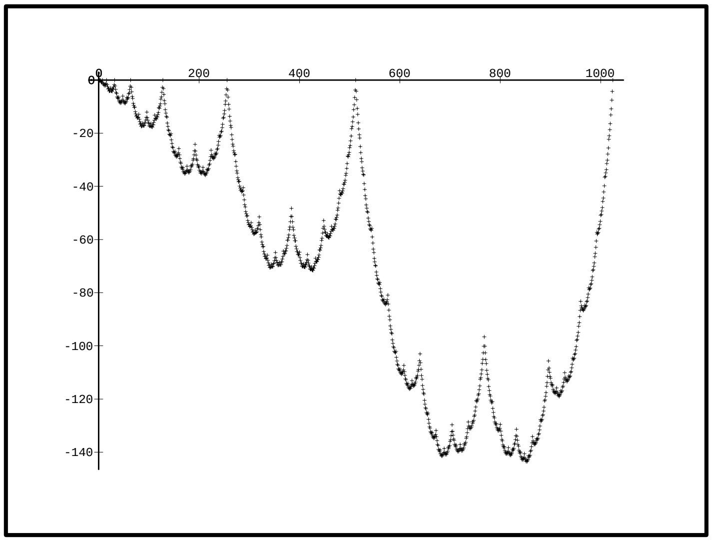

Figure 7.2: The sequence $S(n) - \frac{1}{2} n \log_2 n$, for $n \leq 1024$.

as follows from comparing the trailing zeros in k and $k-1$. Thus, up to a normalization polynomial, S is a double sum of v_2:

$$S(n) = \frac{n(n-1)}{2} - \sum_{k=1}^{n-1} \left(\sum_{\ell=1}^{k} v_2(\ell) \right).$$

This simple observation brings $S(n)$ to the realm of Dirichlet series: the DGF of $v_2(k)$ is

$$\sum_{k=1}^{\infty} \frac{v_2(k)}{k^s} = \frac{2^{-s}}{1-2^{-s}} \zeta(s) = \frac{\zeta(s)}{2^s - 1},$$

as directly results from the Dirichlet convolution formula.

Thus, the Mellin-Perron formula entails

$$S(n) = \frac{n(n-1)}{2} - \frac{n}{2i\pi} \int_{2-i\infty}^{2+i\infty} \frac{\zeta(s)}{2^s - 1} \frac{ds}{s(s+1)}. \tag{7.10}$$

This integral form constitutes the basis of an evaluation by residues. The idea is to push the line of integration to the *left* and take residues of the integrand into account by Cauchy's theorem. This very much resembles the analysis of coefficients of meromorphic OGFs or EGFs by pushing the contour of integration away from the original domain of analyticity.

To justify this, one considers the integral in(7.10) taken along a rectangle $\mathcal{R}_T$ with vertical sides $\Re(s) = 2$, and $\Re(s) = c$ for some c, and with vertical sides $\Im(s) = \pm T$. From the growth properties (7.9) of $\zeta(s)$, one may take $c = -\frac{1}{4}$ which ensures convergence of the integral along the western side. At the same time, one may let T tend to ∞, and, by selecting $T = (2m + 1)\pi/\log 2$ for integer values of m, poles are avoided in such a way that the corresponding horizontal integrals tend to 0. Thus, with $\omega(s)$ denoting the integrand in (7.10), one has

$$\frac{1}{2i\pi}\left[\int_{2-i\infty}^{2+i\infty} - \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty}\right]\omega(s)\,ds = \sum_{s\in\mathcal{P}} \operatorname{Res}\left(\omega(s)\right), \tag{7.11}$$

where the sum is extended to the set $\mathcal{P}$ of poles with $-\frac{1}{4} < \Re(s) < 2$. This set $\mathcal{P}$ consists precisely of the points

$$s = 0, \qquad \text{and} \qquad s = \frac{2ik\pi}{\log 2} \ \text{ for } k \in Z \setminus \{0\},$$

the poles being simple except for $s = 0$ that is a double pole.

The left hand side of (7.11) contains two integrals, the one taken along $\Re(s) = 2$ being related in a simple algebraic way to $S(n)$, the other one along the line of abscissa $s = -\frac{1}{4}$ being by basic bounds $\mathcal{O}(n^{-1/4})$. The residues are computable from known special values of the zeta function, like

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta'(0) = -\log\sqrt{2\pi},$$

that derive from the functional equation (7.3), (7.4). The pole at $s = 0$ yields a residue of

$$-\frac{1}{2}\log_2 n - \log_2\sqrt{\frac{\pi}{2}} - \frac{1}{4}.$$

The poles at the χ_k for $k \neq 0$ contribute the Fourier series $-P(\log_2 n) + p_0$, as is easliy checked by computing the residue of the integrand at each of the points $s = \chi_k$. Thus, all computations done, we find a summation formula with a remainder term:

$$S(n) = \frac{1}{2}n\log_2 n + nP(\log_2 n) + R(n) \qquad \text{where} \qquad R(n) = \mathcal{O}(n^{3/4}).$$

In most cases, things end there, and a formula with some $\mathcal{O}$-error term results from the Mellin-Perron approach. However, in this particular case, the formula turns out to be *exact*, that is to say, $R(n)$ is identically 0. This results from the two identities:

$$\int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \zeta(s)n^s \frac{ds}{s(s+1)} = 0, \qquad \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\zeta(s)}{2^s-1} n^s \frac{ds}{s(s+1)} = 0.$$

(See the following exercise for a proof of this simple fact.) □

The fluctuations directly reflect the fractal nature of binary representations of numbers.

EXERCISE 15. Use the Mellin-Perron formula to establish, for n an integer,

$$\frac{n-1}{2} = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} \zeta(s)n^s \frac{ds}{s(s+1)},$$

and deduce by shifting the contour

$$0 = \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \zeta(s)n^s \frac{ds}{s(s+1)}.$$

By further expanding $(2^s-1)^{-1} = -1 - 2^s - 2^{2s} - \cdots$ inside the integrand, deduce

$$0 = \int_{-\frac{1}{4}-i\infty}^{-\frac{1}{4}+i\infty} \frac{\zeta(s)}{2^s-1} n^s \frac{ds}{s(s+1)}.$$

EXERCISE 16. [17] Analyse the number of blocks 01 in the binary representations of integers of $\{1, \ldots, n-1\}$.

Analyse the number of 1-digits in the Gray code representation of integers of $\{1, \ldots, n-1\}$.

The general method. It is not difficult to encapsulate the technique used for the sum-of-digits function into a general theorem. The following result is but one of a large number of possible statements that formalize the analysis of coefficients of meromorphic Dirichlet series.

Theorem 7.2 (Mellin-Perron asymptotics) *Let $\alpha(s) = \sum_n a_n n^{-s}$ be a Dirichlet series that converges absolutely in $\Re(s) \geq c > 0$, is meromorphic in $\Re(s) \geq d$ for some $d < c$, and is analytic on the line $\Re(s) = d$. Assume further that for some $r < 2$ and some $\{\tau_i\}$ with $\tau_i \to +\infty$, one has*

$$\alpha(s) = \mathcal{O}(|s|^r), \qquad \text{uniformly for } s \text{ such that } d \leq \Re(s) \leq c,\ |\Im(s)| = \tau_i.$$

Dirichlet series $\alpha(s)$	*Asymptotic expansion of coefficient sum* S_n
Pole at $s_0 = \sigma_0 + i\tau_0$	Term $\approx n^{s_0} = n^{\sigma_0} e^{i\tau_0 \log n}$
• — with smaller real value	• smaller term in expansion (n^{σ_0})
• — with imaginary part $\neq 0$	• fluctuating factor $(e^{i\tau_0 \log n})$
• — with multiplicity r	• extra factor of $(\log n)^{r-1}$
• regularly spaced poles	• Fourier series in $\log n$
Simple pole $\dfrac{1}{(s-s_0)}$	$\dfrac{n^{s_0}}{s_0(s_0+1)}$
Double pole $\dfrac{1}{(s-s_0)^2}$	$\dfrac{n^{s_0}\log n}{s_0(s_0+1)} - \dfrac{n^{s_0}(2s_0+1)}{s_0^2(s_0+1)^2}$
Triple pole $\dfrac{1}{(s-s_0)^3}$	$\dfrac{n^{s_0}\log^2 n}{2s_0(s_0+1)} - \dfrac{n^{s_0}\log n(2s_0+1)}{s_0^2(s_0+1)^2} + 3\dfrac{n^{s_0}}{s_0^2(s_0+1)^2}$

Figure 7.3: The correspondence between singularities of a Dirichlet series at s_0 ($s_0 \neq 0, -1$) and induced terms in the asymptotic expansion of coefficient sums.

Then:

$$\frac{1}{n}\sum_{k=1}^{n-1} a_k(n-k) = \sum_{d \leq \Re(s) \leq c} Res\left(\alpha(s)\frac{n^s}{s(s+1)}\right) + \mathcal{O}(n^d), \tag{7.12}$$

where the sum is extended to all poles in the strip $d \leq \Re(s) \leq c$.

Proof. Integrate along a rectangle whose horizontal sides are $\pm\tau_j$ and vertical sides are $\Re(s) = c$ and $\Re(s) = d$ and apply the residue theorem. □

It is crucial to observe that a pole of $\alpha(s)$ at some point $s_0 = \sigma_0 + i\tau_0$ leads to a residue involving the quantity

$$n^{s_0} = n^{\sigma_0} e^{i\tau_0 \log n},$$

so that poles with larger real parts bring dominant contributions while their imaginary parts induce a periodic function of $\log n$. The computation is easily carried out in full generality and is summarized in Fig. 7.3), in the case of the formula for the sum of coefficients, $S_n = \sum_{k=1}^{n-1} a_k(n-k)$, corresponding to the Mellin-Perron formula of order $m = 1$. We have:

— *Poles of a Dirichlet series farther to the left contribute smaller terms in an asymptotic expansion of coefficients. The growth is dictated by the real parts of the singularities; the imaginary parts induce periodic fluctuations.*

— *Regularly spaced poles on a vertical line correspond to a Fourier series in* $\log n$.

— *A pole of multiplicity* r *introduces a factor of* $(\log n)^{r-1}$.

Figure 7.3 gives the residues induced by simple singular elements of the form $(s - s_0)^{-j}$, for $j = 1, 2, 3$. It is asumed there that s_0 is distinct from the values $0, -1$ that render the kernel $(s(s+1))^{-1}$ singular (the calculation otherwise obeys the same principles but there is an extra factor of $\log n$).

7.3 Divide-and-conquer recurrences

Many algorithms are based on a recursive divide-and-conquer strategy. Typically a problem of size n is split into two subproblems of size $n/2$ or about (balancing usually pays!), the subproblems are solved independently, and the two partial solutions are woven back together. Since $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$, the cost of such an algorithm obeys the classical *divide-and-conquer recurrence*

$$f_n = f_{\lceil n/2 \rceil} + f_{\lfloor n/2 \rfloor} + e_n, \tag{7.13}$$

where e_n, often called the *"toll function"*, is the cost of splitting the original problem into two subproblems and of recombining the two partial solutions. Examples are provided by mergesort, binomial queues, sorting networks, and many computational geometry algorithms [7, 40, 41]. The recurrence (7.13) is most frequently used for describing worst–case performance but it may also be used for average–case analysis of algorithms, provided randomness is inherited by the decomposition into smaller subproblems.

Although it was not stressed earlier, the sum-of-digit function falls into this category since $f_n = S(n+1)$ satisfies the recurrence

$$f_n = f_{\lceil n/2 \rceil} + f_{\lfloor n/2 \rfloor} + \lfloor \frac{n}{2} \rfloor,$$

an equality that may be checked by separating the odd and even numbers in the table of binary representations and that is equivalent to the divide-and-conquer recurrence for $\nu(n)$,

$$\nu(1) = 1, \quad \nu(2n) = \nu(n), \quad \nu(2n+1) = 1 + \nu(n).$$

Exercise 17. The OGF of f_n is determined by the OGF of e_n:

$$f(z) = \sum_{k=0}^{\infty} \frac{1 - z^{2^k}}{1 - z} e(z^{2^k}).$$

Dirichlet series for divide-and-conquer recurrences. In order to analyze (7.13), one should look for a simple form of some Dirichlet series associated with f_n. This is easily found by differencing twice. Let $\{u_n\}$ be a sequence of numbers; the second (centered) difference of the sequence is defined as

$$\Delta\nabla u_n = u_{n+1} - 2u_n + u_{n-1}.$$

By separating the odd and even cases, we find

$$\begin{aligned} \Delta\nabla f_{2m} &= \Delta\nabla f_m + \Delta\nabla e_{2m} \\ \Delta\nabla f_{2m+1} &= \qquad\quad + \Delta\nabla e_{2m+1} \end{aligned} \qquad (7.14)$$

If the relation of (7.13) is to hold true for any n, we must have for consistency at $n = 0, 1$ the conditions $e_0 = e_1 = 0$, which we now assume. This is not a severe restriction as the choice $e_n = \delta_{n,1}$ in conjunction with (7.13) for $n \geq 2$ yields $f_n = n$. Then forming DGFs in the usual way from (7.13), through multiplication by n^{-s} and summing yields

$$\sum_{n=1}^{\infty} \frac{\Delta\nabla f_n}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{\Delta\nabla e_n}{n^s}. \qquad (7.15)$$

We restrict attention to the case where the e_n are of at most polynomial growth: $e_n = \mathcal{O}(n^d)$ for some d. This guarantees that the DGFs of $\Delta\nabla e_n$ and $\Delta\nabla f_n$ have nonempty half-planes of convergence. The Mellin-Perron formula with $m = 1$ applies, and with $c > 0$ taken in the interior of the half-plane of absolute convergence of the DGF of $\Delta\nabla e_n$, one has

$$f_n = \frac{n}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Xi(s)}{1 - 2^{-s}} n^s \frac{ds}{s(s+1)} \qquad \text{with} \qquad \Xi(s) = \sum_{n=1}^{\infty} \frac{\Delta\nabla e_n}{n^s}. \quad (7.16)$$

This relation holds under the initial condition $f_1 = 0$; in other cases, a term of nf_1 must be added to the integral.

In general the e_n that are "known" are simple enough so that the DGF may be related to standard Dirichlet series with easily located singularities.

The DGF of the $\Delta\nabla f_n$ that are "unknown" then has its singularities fully determined. In particular, the factor of $(1-2^{-s})^{-1}$ in (7.15) leads one to expect poles at the points

$$\chi_k = \frac{2ik\pi}{\log 2},$$

which induces periodic fluctuations in the form of a Fourier series in $\log_2 n$. Theorem 7.2 is applicable each time the DGF $\Xi(s)$ admits an analytic continuation and is of a sufficiently small growth outside its half-plane of absolute convergence.

The divide-and-conquer recurrences of the type (7.12) fall into three broad categories depending on the growth rate of the e_n. From their elementary theory [7, 41], it is known that

$$\begin{aligned} e_n &= \mathcal{O}(n^\alpha), \quad \alpha < 1 &&\Longrightarrow\quad f_n = \mathcal{O}(n) \\ e_n &= \mathcal{O}(n) &&\Longrightarrow\quad f_n = \mathcal{O}(n\log n) \\ e_n &= \mathcal{O}(n^\beta), \quad \beta > 1 &&\Longrightarrow\quad f_n = \mathcal{O}(n^\beta) \end{aligned} \tag{7.17}$$

All three cases lead to some fluctuating behaviour. In the linear case and superlinear case of (7.17) and for "smooth" e_n, fluctuations are present but they are restricted to subdominant asymptotic terms (as was the case for the sum of digit function). However, in the first case of (7.17) where e_n is sublinear so that f_n is of linear growth, the fluctuations appear in the main term of the asymptotics of f_n. In particular f_n/n does *not* generally tend to a limit but oscillates boundedly. The following result of Flajolet and Golin precisely quantifies what happens, assuming a minor technical condition. It is a direct consequence of Theorem 7.2.

Theorem 7.3 (Divide-and-conquer recurrence) *Assume that the series $\sum_n |\Delta\nabla e_n| n^\epsilon$ is convergent for some $\epsilon > 0$. Then the solution to the divide-and-conquer recurrence (7.13) satisfies*

$$f_n = nQ(\log_2 n) + \mathcal{O}(n^{1-\epsilon}),$$

where $Q(u)$ is a periodic function with mean value

$$q_0 = \frac{1}{\log 2}\sum_{m=2}^{\infty} e_m \log\frac{m^2}{m^2-1}.$$

The Fourier coefficients are given by (7.18) below.

Proof. Theorem 7.2 applies here as the DGF of $\Delta\nabla e_n$ has, by assumption a half-plane of convergence that contains $\Re(s) \geq -\epsilon$. Thus, it suffices to take $c = 1$, choose $d = -\epsilon$, and $\tau_k = (2k+1)i\pi/\log 2$ so as to avoid poles. There is only a simple pole of the integrand at $s = 0$ since $\Xi(0) = 0$ as it is a sum of (second) differences. Accordingly the residue of the Mellin-Perron integral at $s = 0$ is $\Xi'(0)/\log(2)$, which gives the form of q_0. To the Fourier series,

$$Q(u) = q_0 + \sum_{k \neq 0} q_k e^{2ik\pi u},$$

the imaginary poles contribute the Fourier coefficient ($k \neq 0$),

$$q_k = -\frac{1}{\log 2} \frac{1}{\chi_k(\chi_k+1)} \sum_{n \geq 1} \frac{\Delta\nabla e_n}{n^{\chi_k}}. \tag{7.18}$$

□

Maxima finding. One of the fundamental problems of computational geometry is to determine in a collection of points which ones are visible from a certain direction. There is a very simple and surprisingly efficient algorithm for the closely related problem of maxima-finding that works in any dimension. To keep notations concise, we base our discussion on the already nontrivial case of dimension $d = 2$.

A point $P = (x, y)$ dominates a point $P' = (x', y')$ which we write as $P \succ Q$ iff $P \neq P'$ and the two simultaneous conditions $x \geq x'$ and $y \geq y'$ are met. A *maximal* element of a finite set of points $\mathcal{F} = (P_1, \ldots, P_n)$ is a point of $\mathcal{F}$ that is *not* dominated by any other point in the set. In the case of 2-dimensional space and with the usual orientation of the axes, a point dominates all points that lie to its South-West. Thus, in this perspective, maximal elements are the ones that do not lie indolently in the "shade" of any other point to their North-East.

The maxima finding algorithm studied here determines the set $\mu(\mathcal{F})$ of maxima in $\mathcal{F} = (P_1, \ldots, P_n)$ in the following sequence of steps:

- find recursively $M_1 = \mu(P_1, \ldots, P_{\lceil n/2 \rceil})$;
- find recursively $M_2 = \mu(P_{\lceil n/2 \rceil + 1}, \ldots, n)$;
- compute the "merge" of M_1 and M_2 by pairwise comparisons of their elements:

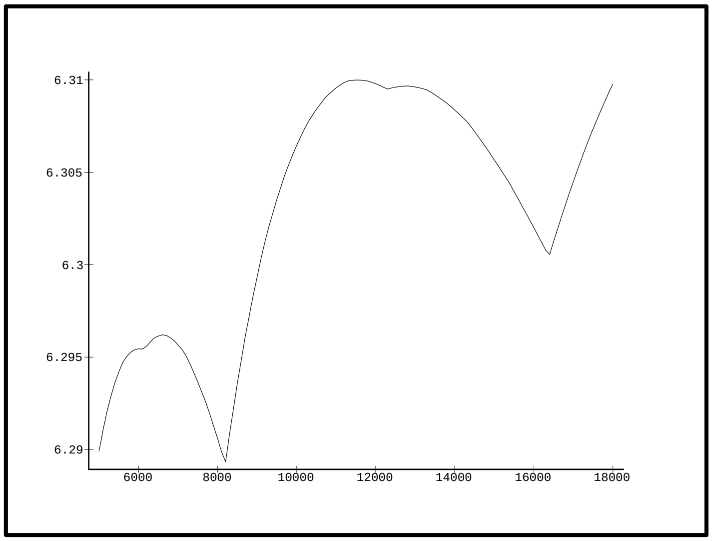

Figure 7.4: Maxima finding —a plot of f_n/n for $n = 5000, \ldots, 18000$.

- filter the elements of M_1 and retain only the set M_1' of those that are not dominated by any element of M_2:

$$M_1' = \{P \in M_1 \mid \forall Q \in M_2,\ Q \not\succ P\};$$

- filter the elements of M_2 and retain only the set M_2' of those that are not dominated by any element of M_1:

$$M_2' = \{P \in M_2 \mid \forall Q \in M_1,\ Q \not\succ P\};$$

- return $\mu(\mathcal{F}) = M_1' \cup M_2'$.

Now comes the average–case analysis. Assume that the points to be operated on are all independently drawn according to the uniform distribution in the unit square $[0,1]^2$. Given n such points, the random variable representing the number of maxima has the same distribution as the number of records (or left-to-right maxima) in a random permutation of $1\,..\,n$. (The distribution involves, as we know already from Chapters 2, 3, the Stirling cycle numbers.) In particular, the mean number of maxima for n points is given by the harmonic number, H_n.

Consider a naïve implementation where the sets M_1' and M_2' are each computed by performing all pairwise comparisons (a clear optimization would result by using a while loop rather than a for–loop as suggested here, but the naïve implementation is already good enough to guarantee linear time behaviour). Then, the mean number of comparisons required for finding maxima in a set of n random points satisfies the recurrence:

$$f_n = f_{\lceil n/2\rceil} + f_{\lfloor n/2\rfloor} + 2H_{\lceil n/2\rceil}H_{\lfloor n/2\rfloor}. \tag{7.19}$$

The initial conditions are $f_0 = f_1 = 0$.

Fig. 7.4 displays the value of f_n/n computed from this recurrence. There is a clear periodicity phenomenon, and the graph presents marked cusps at powers of 2, with a very few secondary cusps being apparent. The aspect is only superficially different from that of Fig. 7.2, as it can be proved that similar fractal phenomena take place.

A simple computation shows that

$$\Delta\nabla e_{2m} = -\frac{H_m}{m(m+1)}, \qquad \Delta\nabla e_{2m+1} = -\frac{1}{(m+1)^2},$$

so that the conditions of Theorem 7.3 relative to the DGF $\Xi(s)$ of the differences $\Delta\nabla e_n$ are satisfied.

Proposition 7.2 (Maxima finding) *Maxima-finding has an expected cost f_n that satisfies*

$$f_n \sim nQ(\log_2 n),$$

where $Q(u)$ is a periodic function with mean value

$$q_0 = 2\sum_{m=1}^{\infty} H_m^2 \log\frac{1}{1-(2m)^2} + 2\sum_{m=1}^{\infty} H_m H_{m+1} \log\frac{1}{1-(2m+1)^2}.$$

Numerically, one finds $q_0 \doteq 6.3257$.

Linear time performance is not *a priori* obvious on this maxima-finding problem. A direct algorithm consists in a sort along the x-coordinate and followed by a left-to-right scan to eliminate nonmaximal elements. However, this solution has $\mathcal{O}(n\log n)$ cost when comparison-based sorting is used. Linearity of the divide-and-conquer solution even persists in all dimensions when the divide-and-conquer paradigm is used, and the methods discussed here provide precise analyses, see [14].

Exercise 18. Use a computer algebra system to determine q_0 to 50 significant digits. (Hint: reorganize the sum as a sum of zeta functions.) Determine numerically the first 5 Fourier coefficients of $Q(u)$.

EXERCISE 19. [Buchta [5]] Determine the expected number of maxima of n random points in the unit hypercube of dimension d. (The answer involves generalized harmonic numbers.)

EXERCISE 20. [14] Analyze the maxima finding algorithm in dimension $d > 2$, and estimate the corresponding mean-value constants for $d = 3, 4, 5$.

EXERCISE 21. Analyze the improved maxima-finding algorithm where a scan is stopped as soon as a dominating element is found (a for-loop is replaced by a while loop).

Mergesort. Top-down recursive mergesort is a popular sorting algorithm that sorts an array $t[1\,..\,n]$ of n numbers according to the following principle [40, p. 165]

- sort recursively the "first half": $t[1\,..\,\lceil\frac{n}{2}\rceil]$;
- sort recursively the "second half": $t[1+\lceil\frac{n}{2}\rceil\,..\,n]$;
- merge the two halves.

The divide and conquer recurrence applies equally well to the worst-case cost $T(n)$ and the average–case cost $U(n)$, with the cost being measured in the number of comparisons:

$$\begin{aligned} T(n) &= T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor) + n - 1 \\ U(n) &= U(\lceil n/2\rceil) + U(\lfloor n/2\rfloor) + n - \gamma_n \\ &\quad \text{where } \gamma_n = \frac{\lfloor n/2\rfloor}{\lceil n/2\rceil + 1} + \frac{\lceil n/2\rceil}{\lfloor n/2\rfloor + 1} \end{aligned} \tag{7.20}$$

These equalities result from the fact that the cost of merging two files of sizes a and b is

$$a + b - 1 \qquad \text{and} \qquad a + b - \frac{a}{b+1} - \frac{b}{a+1},$$

in the worst case and in the average case respectively [31, ex. 5.2.4-2].

The worst case is easy to analyze, and one may check directly [31, 41] from the first recurrence of (7.20) that

$$T(n) = \sum_{k=1}^{n} \lceil \lg n \rceil = n\lceil \lg n \rceil - 2^{\lceil \lg n \rceil} + 1,$$

so that periodicities are apparent without a need to appeal to Theorem 7.3: with $\{x\}$ representing the fractional part of x, one has the exact expression

$$T(n) = n \lg n + nA(\lg n) + 1 \qquad \text{where} \qquad A(u) = 1 - \{u\} - 2^{1-\{u\}}.$$

Although $U(n)$ grows like $n \log n$, it is still possible to determine $U(n)$ by Theorem 7.3: the recurrences being linear, the quantity $T(n) - U(n)$ also satisfies a divide-and-conquer recurrence but with $e_n = \gamma_n - 1 = \mathcal{O}(1)$. This is a case covered by Theorem 7.3, and all computations done, one finds (see [15] for details):

Proposition 7.3 (Mergesort) *The average cost of mergesort satisfies*

$$U(n) = n \lg n + nB(\lg n) + \mathcal{O}(\sqrt{n}),$$

where $B(u)$ is a continuous 1-periodic function with mean value

$$b_0 = \frac{1}{2} - \frac{1}{\log 2} - \frac{1}{\log 2} \sum_{m=1}^{\infty} \frac{2}{(m+1)(m+2)} \log \frac{2m+1}{2m},$$

$b_0 \doteq -1.24815\,20420$, *and with amplitude less than* 10^{-2}.

By Prop. 7.3, mergesort has an average complexity that is about

$$n \log_2 n - (1.25 \pm 0.01)n + o(n).$$

This appears to be not far from the information-theoretic lower bound which is

$$\log_2 n! \approx n \log_2 n - 1.44n + o(n).$$

Exercise 22. Use DGFs to prove that $A(u)$ has mean value $a_0 = 1/2 - 1/\log 2$, and Fourier coefficients ($k \in Z \setminus \{0\}$)

$$a_k = \frac{1}{\log 2} \frac{1}{\chi_k(\chi_k + 1)}.$$

Verify these results by a direct calculation.

EXERCISE 23. [15] Show that the function $B(u)$ is not differentiable at a dense set of points of $[0,1]$. [Hint. Express $B(u)$ in terms of $A(u)$ by suitably reorganizing the Fourier expansion of B.]

Other recurrences. Divide-and-conquer recurrences of a type other than (7.12) can also be treated by these methods. However, each recurrence (or better perhaps each solution sequence) carries with it a certain degree of "smoothness" that is also related to the growth of the intervening DGFs. If the fluctuations of the solution sequence are too wild, then Mellin-Perron formulae of a higher order (a larger value of m in Theorem 7.1) will need to be applied resulting in estimates in the sense of Cesàro averages. This may be seen as the current limitation of the method for divide-and-conquer recurrences of practical interest.

7.4 Mellin transforms

Hjalmar Mellin (1854-1933) gave his name to the *Mellin transform* that associates to a function $f(t)$ defined over the positive reals the complex function $f^*(s)$ where

$$f^*(s) = \int_0^\infty f(t) t^{s-1}\, dt.$$

The Mellin transforms generalizes in many ways Dirichlet series that have been studied earlier.

The primary reason why Mellin transforms are useful in *asymptotic analysis* is the following:

> ***Mapping property.*** *Mellin transforms establish a correspondence between the asymptotic expansions at* 0 *and* $+\infty$ *of the original function* $f(x)$ *and the set of singularities of the transformed function* $f^*(s)$.

One of the major uses of Mellin transforms is for the asymptotic analysis of sums obeying the general pattern

$$F(x) = \sum_k \lambda_k f(\mu_k x),$$

either as $x \to 0$ or as $x \to +\infty$. Such a sum is called a *harmonic sum*. Many sums of this type present themselves in combinatorial enumerations and the analysis of algorithms, especially in expressions of average values of parameters of combinatorial objects. Typical examples include the height of plane planted trees studied by De Bruijn, Knuth, and Rice [9] or the basic parameters of randomly grown tries [31, 35]

Mellin transforms nicely "separate" the components of a harmonic sum:

> ***Separation property.*** *The Mellin transform of a harmonic sum factorizes as the* product *of the transform $f^*(s)$ of the base function $f(x)$ with a generalized Dirichlet series that only depends on the coefficients λ_k, μ_k, namely $\sum_k \lambda_k \mu_k^{-s}$.*

It is the combination of the mapping property and the separation property that gives its full power to Mellin transform asymptotics:

> ***Mellin asymptotic summation.*** *To analyse asymptotically a harmonic sum, determine its Mellin transform that factorises by the separation property. Locate the singularities of its components. Find the asymptotic behaviour of the original harmonic sum by translating these singularities into asymptotic expansions by means of the mapping property.*

7.5 Mellin transforms: basic properties

In this section we develop the basic definitions and functional properties of Mellin transforms.

Definition 7.2 *The Mellin transform of a complex-valued function $f(x)$ that exists over $(0, +\infty)$ and is locally integrable is defined by*

$$\mathcal{M}[f(x); s] = f^*(s) = \int_0^\infty f(x) x^{s-1}\, dx. \tag{7.21}$$

Existence. We recall that a function is locally integrable if it is integrable over any finite closed subinterval of the open set $(0, +\infty)$. Technically, the definition above (7.21) is taken in the sense of Lebesgue integrals[2] as this

[2]We refer to [42] for a good introduction to measure and Lebesgue integrals and we shall develop the general theory Mellin transforms along these lines though the examples that we treat are also Riemann integrable functions.

allows for a presentation unencumbered by special conditions. It should also be remembered that Lebesgue integrability is a notion of absolute integrability.

Theorem 7.4 (Fundamental strip) *There is a maximal open strip* $\Omega = \{\alpha < \Re(s) < \beta\}$ *such that the integral giving the Mellin transform of* $f(x)$ *is defined. This strip is called the* fundamental strip. *The transformed function* $f^*(s)$ *is analytic in the fundamental strip.*

Proof. As $f(x)$ and x^{s-1} are both locally integrable, what can restrict the existence of a Mellin transform is only the behaviour of f at the boundary points 0 and $+\infty$. The basic decomposition

$$f^*(s) = \int_0^1 f(x)x^{s-1}\,dx + \int_1^\infty f(x)x^{s-1}\,dx,$$

is such that the first integral exists in some right half-plane $\Re(s) > \alpha$, and the second one in some left half-plane $\Re(s) < \beta$: the quantity α is the supremum of all real a's such that $f(x)x^{a-1}$ is integrable over $(0,1]$, and β is the infimum of all real b's such that $f(x)x^{b-1}$ is integrable over $[1,+\infty)$. Such quantities are well defined as follows from the dominated convergence theorem of Lebesgue integration [42, §10.8]. □

Thus Mellin transforms exist in strips. In particular, if

$$f(x) = \mathcal{O}(x^a) \text{ as } x \to 0^+, \qquad f(x) = \mathcal{O}(x^b) \text{ as } x \to +\infty,$$

then the transform $f^*(s)$ exists and is analytic in

$$-a < \Re(s) < -b$$

(again by virtue of the dominated convergence theorem) which is thus a substrip of the fundamental strip. The strip above is nonempty provided $b < a$. Thus, for functions with definite orders at 0 and ∞, *the Mellin transform exists provided the exponent* a *at 0 is larger than the exponent* b *at* ∞. Surprisingly perhaps, constants and polynomials have no Mellin transforms in the sense of Definition 7.2.

At this stage, it is convenient to introduce a concise notation for open strips of the complex plane: we define

$$\langle \alpha, \beta \rangle = \{s \in C \mid \alpha < \Re(s) < \beta\}.$$

Function		*Transform*	*Fund. strip*
Exponential	e^{-x}	$\Gamma(s)$	$\langle 0, +\infty \rangle$
—	$e^{-x} - 1$	$\Gamma(s)$	$\langle -1, 0 \rangle$
—	$e^{-x} - 1 - x$	$\Gamma(s)$	$\langle -2, -1 \rangle$
Gaussian	e^{-x^2}	$\frac{1}{2}\Gamma(\frac{s}{2})$	$\langle 0, +\infty \rangle$
Heaviside step	$H(x)$	$\frac{1}{s}$	$\langle 0, +\infty \rangle$
— (compl.)	$1 - H(x)$	$-\frac{1}{s}$	$\langle -\infty, 0 \rangle$
	$\frac{1}{1+x}$	$\frac{\pi}{\sin \pi s}$	$\langle 0, 1 \rangle$
Logarithm	$\log(1+x)$	$\frac{\pi}{s \sin \pi s}$	$\langle -1, 0 \rangle$
	$\frac{e^{-x}}{1 - e^{-x}}$	$\Gamma(s)\zeta(s)$	$\langle 1, +\infty \rangle$

Figure 7.5: Mellin transforms of some common functions.

The fundamental strips of the functions

$$f_1(x) = \frac{1}{1+x}, \quad f_2(x) = e^{-x}, \quad f_3(x) = e^{-x} - 1, \quad f_4(x) = e^{-(x+1/x)},$$

are thus

$$\Omega_1 = \langle 0, 1 \rangle, \quad \Omega_2 = \langle 0, +\infty \rangle, \quad \Omega_3 = \langle -1, 0 \rangle, \quad \Omega_4 = \langle -\infty, +\infty \rangle.$$

Special transforms. A table of some of the most commonly used transforms appears in Fig. 7.5, see also [34, 46] for more. The corresponding formulae will be established in the course of the next few pages.

The prototypical Mellin transform is the Gamma function,

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, ds,$$

which is by this definition the transform of the exponential function; the fundamental strip is $\langle 0, +\infty \rangle$.

Let $H(x)$ be the Heaviside step function that takes the value 1 for $x \in [0,1]$ and the value 0 for $x > 1$. Then, by straight integration,

$$\mathcal{M}[H(x);s] = \frac{1}{s} \text{ in } \langle 0, +\infty\rangle, \quad \mathcal{M}[1-H(x);s] = -\frac{1}{s} \text{ in } \langle -\infty, 0\rangle.$$

We shall determine the other transforms of Fig. 7.5 in the next few pages.

Hankel contours. In all generality, Mellin transform are defined for functions on the real line. Stronger properties are available when the function to be transformed is analytic in a wider region.

We have seen in an earlier chapter that a Hankel contour provides a way to continue analytically the Gamma function. This method extends, by the same reasoning, to the transforms of all functions that are analytic at 0 and on the real half-line, while decaying fast enough towards infinity.

Proposition 7.4 (Transforms of analytic functions) *Let $f(w)$ be a function analytic in some neighbourhood Ω of the nonnegative real axis. Assume that $f(w) = \mathcal{O}(w^{-\beta})$ for w tending to $+\infty$ while being restricted to Ω. Then, for s in $\langle -\infty, \beta\rangle$, one has*

$$\mathcal{M}[f(x);s] = -\frac{1}{2i\sin\pi s}\int_{\mathcal{H}} f(w)(-w)^{s-1}\,dw,$$

where $\mathcal{H}$ is a contour inside Ω that starts at $+\infty$ above the real axis, encircles the origin counterclockwise, and returns to $+\infty$ below the real axis. The determination of $(-w)^s$ extends the principal determination for $w < 0$.

The situation where $f(w)$ has a pole of order k at 0 can also be accommodated since then $w^k f(w)$ is then regular at 0 and the proposition applies.

Via a residue computation, Prop. 7.4 permits to determine explicitly the Mellin transforms of all rational functions: it suffices to close the contour to the left by a large circle and take residues of poles into account. For instance, if $R(w)$ has only simple poles at a set $\mathcal{P}$ that does include 1, one has

$$\mathcal{M}[R(w);s] = \frac{\pi}{\sin\pi s}\sum_{\omega\in\mathcal{P}} c_\omega(-\omega)^{s-1} \qquad \text{with} \qquad c_\omega = \lim_{w\to\omega} R(w)(w-\omega).$$

This justifies for instance the fact that the transform of $(1+x)^{-1}$ is $\pi/\sin\pi s$. (Principal determinations must be taken for assigning a value to $(-\omega)^{s-1}$.)

Exercise 24. Determine the Mellin transforms of

$$\frac{1}{(1+x)^m}, \quad \frac{1}{1+x+x^2}, \quad \frac{1}{1+x^m},$$

with m an integer.

Function	*Transform*	*Fund. strip*	
$f(x)$	$f^*(s)$	$\langle \alpha, \beta \rangle$	
$f(\mu x)$	$\mu^{-s} f^*(s)$	$\langle \alpha, \beta \rangle$	$(\mu > 0)$
$f(x^\rho)$	$\frac{1}{\rho} f^*(\frac{s}{\rho})$	$\langle \rho\alpha, \rho\beta \rangle$	$(\rho > 0)$
		$\langle \rho\beta, \rho\alpha \rangle$	$(\rho < 0)$
$f(x) \log x$	$\frac{d}{ds} f^*(s)$	$\langle \alpha, \beta \rangle$	
$\Theta f(x)$	$-s f^*(s)$	$\langle \alpha', \beta' \rangle$	$(\Theta = x \frac{d}{dx})$
$\frac{d}{dx} f(x)$	$-(s-1) f^*(s-1)$	$\langle \alpha' - 1, \beta' - 1 \rangle$	
$\int_0^x f(t)\, dt$	$-\frac{1}{s} f^*(s+1)$	—	
$\sum_k \lambda_k f(\mu_k x)$	$\left(\sum_k \lambda_k \mu_k^{-s} \right) \cdot f^*(s)$	$\langle \alpha', \beta' \rangle$	*Harmonic sum rule*

Figure 7.6: Basic functional properties of Mellin transforms.

The method extends to functions that are meromorphic in C provided their growth remains moderate on a collection of large contours (*e.g.*, circles). When applied to the function $e^{-x}/(1-e^{-x})$ that has poles at $s = 2ik\pi$, this technique yields representation of the Mellin transform $F^*(s)$ of $F(x)$ that involves $\sum_k (2ik\pi)^{s-1}$ and thus does not immediately reduces to the form $\Gamma(s)\zeta(s)$ of (7.23). Comparing both forms establishes the *functional equation* of the Riemann zeta function as given in (7.3-7.4). (Details of a proof along these lines may be found in [42, p. 148-153].)

Exercise 25. Complete the proof of the functional equation of the Riemann zeta function.

Functional properties. These are summarized in Fig. 7.6. The most important property for us is the *rescaling rule*,

$$f(\mu x) \quad \hookrightarrow \quad \mathcal{M}[f(\mu x); s] = \mu^{-s} f^*(s), \qquad \mu > 0,$$

that derives from the change of variables in the Mellin integral $x \mapsto \mu x$. Similarly,

$$f(x^\rho) \quad \hookrightarrow \quad \frac{1}{\rho} f^*(\frac{s}{\rho})$$

results from $x \mapsto x^\rho$. The two most important cases are

$$f(\frac{1}{x}) \quad \hookrightarrow \quad -f^*(-s), \qquad f(x^2) \quad \hookrightarrow \quad \frac{1}{2}f^*(\frac{s}{2}),$$

corresponding to $\rho = -1, 2$. In particular, this justifies the Gaussian entry of Fig. 7.5.

The rule for $f(x)\log x$ follows from differentiation under the integral sign. The rule for $\Theta f(x) = x\frac{d}{dx}f(x)$ results from integration by parts,

$$\mathcal{M}[x\frac{d}{dx}f(x); s] = [f(x)x^s]_0^\infty - s\mathcal{M}[f(x); s],$$

with the validity region $\langle \alpha', \beta' \rangle \subseteq \langle \alpha, \beta \rangle$ dictated by the growth properties of the function. The application of the related antiderivative rule permits us to deduce the transform of $\log(1+x)$ from that of $(1+x)^{-1}$, see Fig. 7.5.

An important consequence of the rescaling rule together with the linearity of the transform is to the *harmonic sums* defined by

$$F(x) = \sum_k \lambda_k f(\mu_k x) \tag{7.22}$$

In view of the importance of this rule, we state it as a theorem.

Theorem 7.5 (Transforms of harmonic sums) *The Mellin transform of the harmonic sum*

$$F(x) = \sum_k \lambda_k f(\mu_k x), \qquad \mu_k > 0,$$

is defined in the intersection of the fundamental strip of $f(x)$ and of the domain of absolute convergence of the generalized Dirichlet series $\sum_k \lambda_k \mu_k^{-s}$. In that intersection domain, its value is

$$F^*(s) = \left(\sum_k \lambda_k \mu_k^{-s}\right) \cdot f^*(s).$$

Without substantial loss of generality, the μ_k can be taken to be either strictly increasing (and tending to $+\infty$) or strictly decreasing (and tending to 0). In that case, the Dirichlet series associated with the harmonic sum is known to have a half plane of absolute convergence.

Proof. The conditions of the theorem legitimate the interchange of summation and integration in the integral defining $F^*(s)$. □

For instance, the Mellin transform of $e^{-x}/(1-e^{-x})$ (see Fig. 7.5) results from the expansion

$$F(x) = \frac{e^{-x}}{1-e^{-x}} = e^{-x} + e^{-2x} + e^{-3x} + \cdots \quad \Longrightarrow \quad F^*(s) = \Gamma(s)\zeta(s), \quad (7.23)$$

with a validity strip that is $\langle 1, +\infty \rangle$.

Inversion. Mellin transforms have an inversion formula very much in line with what exists for Laplace or Fourier transforms[3].

Theorem 7.6 (Mellin inversion) *Let $f(x)$ be locally integrable with fundamental strip $\langle \alpha, \beta \rangle$. Then, provided $f(x)$ is of bounded variation in a neighbourhood of x_0, one has for any c in the interval (α, β):*

$$\frac{f(x_0^-) + f(x_0^+)}{2} = \lim_{T\to\infty} \frac{1}{2i\pi} \int_{c-iT}^{c+iT} f^*(s) x_0^{-s}\, ds. \quad (7.24)$$

If in addition $f(x)$ is continuous at x_0, then

$$f(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} f^*(s) x_0^{-s}\, ds. \quad (7.25)$$

We refer the reader to the literature for a proof of this classical theorem, see [45, p. 246].

Exercise 26. Verify directly by a residue computation that, for $c = \frac{1}{2}$,

$$e^{-x} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s}\, ds.$$

Show that taking $c = -\frac{1}{2}$ gives $e^{-x} - 1$ and generalize. Proceed similarly for $(1+x)^{-1}$.

[3] For instance, the change of variables $x = e^t$ reduces a Mellin transform to a two-sided Laplace transform,

$$\int_0^\infty f(x) x^{s-1}\, dx = \int_{-\infty}^{+\infty} f(e^t) e^{st}\, dt,$$

and a further change of variables $t = i\omega$ yields a Fourier transform.

7.6 Mellin transforms: asymptotic properties

There is a very precise correspondence between *the asymptotic expansion of a function* $f(x)$ at 0 (resp. $+\infty$) and the *singularities of the Mellin transform* $f^*(s)$ in a left half-plane (resp. right half-plane). Each term of the form

$$x^\rho (\log x)^k$$

in any of the asymptotic expansions of $f(x)$ at 0 or $+\infty$ induces for $f^*(s)$ a pole of order $k+1$ at $s = -\rho$, so that

$$f^*(s) = \mathcal{O}\left(\frac{1}{(s+\rho)^{k+1}}\right) \qquad (s \to -\rho).$$

The converse property that poles of f^* induce asymptotic terms of f is also true under some mild conditions; it can be proved by means of the inversion theorem and a residue calculation, very much along the lines of Theorem 7.2.

Singular expansions. The mapping properties are conveniently expressed in terms of "singular expansions" that we first introduce.

Definition 7.3 *Let $\phi(s)$ be meromorphic in a domain Ω with $\mathcal{S}$ the set of its poles in Ω. A* singular expansion *of $\phi(s)$ in Ω is defined as a formal sum*

$$\sum_{s_0 \in \mathcal{S}} \Delta_{s_0}(s),$$

where each $\Delta_{s_0}(s)$ is a truncation of the Laurent expansion of $\phi(s)$ at s_0 till terms of order $\mathcal{O}(1)$ at least. One writes

$$\phi(s) \asymp \sum_{s_0 \in \mathcal{S}} \Delta_{s_0}(s) \qquad (s \in \Omega).$$

For instance, one has

$$\frac{1}{s^2(s+1)} \asymp \left[\frac{1}{s+1} + 2\right]_{s=-1} + \left[\frac{1}{s^2} - \frac{1}{s}\right]_{s=0} + \left[\frac{1}{2}\right]_{s=1} \qquad (s \in \langle -2, +2 \rangle), \tag{7.26}$$

where the point of expansion may be indicated whenever needed as a subscript to the corresponding singular element. The expansion (7.26) is a concise way of combining information contained in the Laurent expansions of the function $\phi(s) \equiv s^{-2}(s+1)^{-1}$ at the three points of $\mathcal{S} = \{-1, 0, 1\}$:

$$\phi(s) \underset{s \to -1}{=} (s+1)^{-1} + 2 + 3(s+1) + \cdots, \quad \phi(s) \underset{s \to 0}{=} s^{-2} - s^{-1} + 1 + \cdots,$$

$$\phi(s) \underset{s\to 1}{=} \frac{1}{2} - \frac{5}{4}(s-1) + \frac{17}{8}(s-1)^2 + \cdots .$$

The fact that $\Gamma(s)$ has poles at the integers ≤ 0 with residue $(-1)^k/k!$ at $s = -k$ is expressed concisely as

$$\Gamma(s) \asymp \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(s+k)} \qquad (s \in \langle -\infty, +\infty \rangle).$$

As we shall see shortly, this singular expansion directly reflects the asymptotic behaviour of e^{-x} (of which $\Gamma(s)$ is the Mellin transform) at 0:

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \qquad (x \to 0).$$

Exercise 27. Let $\omega(s)$ be analytic in Ω and let $\phi(s)$ have there only simple poles at the points s_k. Show that

$$\phi(s) \asymp \sum_k \frac{c_k}{(s-s_k)} \qquad \Longrightarrow \qquad \phi(s)\omega(s) \asymp \sum_k \frac{c_k \omega(s_k)}{(s-s_k)} \qquad (s \in \Omega).$$

Treat similarly the case when $\omega(s)$ has simple poles and/or $\phi(s)$ has double poles.

Exercise 28. Establish the singular expansion

$$(\Gamma(s))^2 \asymp \sum_{k=0}^{\infty} \left[\frac{c_k}{(s+k)^2} + \frac{d_k}{(s+k)} \right],$$

prove that $c_k = (k!)^{-2}$ and that d_k is expressible in terms of Euler's constant and harmonic numbers.

Direct mapping. We show now that the empirical observation made for the transforms of e^{-x} corresponds to a general schema.

Theorem 7.7 (Direct mapping) *Let $f(x)$ have a transform $f^*(s)$ with non-empty fundamental strip $\langle \alpha, \beta \rangle$.*

(i) Assume that $f(x)$ admits as $x \to 0^+$ a finite asymptotic expansion of the form

$$f(x) = \sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k + \mathcal{O}(x^{\gamma}) \qquad (x \to 0^+), \tag{7.27}$$

for some finite set A of pairs (ξ, k), where the ξ satisfy $-\gamma < -\xi \le \alpha$ and the k are nonnegative. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle -\gamma, \beta \rangle$ where it admits the singular expansion

$$f^*(s) \asymp \sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \qquad (s \in \langle -\gamma, \beta \rangle).$$

(ii) Similarly, assume that $f(x)$ admits as $x \to +\infty$ a finite asymptotic expansion

$$f(x) = \sum_{(\xi,k)\in B} c_{\xi,k} x^{\xi} (\log x)^k \qquad (x \to +\infty) + \mathcal{O}(x^{\delta}), \tag{7.28}$$

for a finite set B of pairs (ξ, k) where now $\beta \le -\xi < -\delta$. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle \alpha, -\delta \rangle$ where

$$f^*(s) \asymp - \sum_{(\xi,k)\in B} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \qquad (s \in \langle \alpha, -\delta \rangle).$$

Proof. Since $\mathcal{M}(f(1/x), s) = -\mathcal{M}(f(x), -s)$, it suffices to treat the case (i) corresponding to $x \to 0^+$. By assumption, the function $g(x)$

$$g(x) = f(x) - \sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k$$

satisfies $g(x) = \mathcal{O}(x^{\gamma})$.

For s in the fundamental strip, a split of the definition domains yields

$$\begin{aligned} f^*(s) &= \int_0^1 f(x)\, x^{s-1}\, dx + \int_1^{\infty} f(x)\, x^{s-1}\, dx \\ &= \int_0^1 g(x) x^{s-1}\, dx + \int_0^1 \Big(\sum_{(\xi,k)\in A} c_{\xi,k} x^{\xi} (\log x)^k \Big) x^{s-1}\, dx + \int_1^{+\infty} f(x) x^{s-1}\, dx. \end{aligned} \tag{7.29}$$

In the last line of (7.29), the first integral defines an analytic function of s in the strip $\langle -\gamma, +\infty \rangle$ since $g(x) = O(x^{\gamma})$ as $x \to 0$; the third integral is analytic in $\langle -\infty, \beta \rangle$, so that the sum of these two is analytic in $\langle -\gamma, \beta \rangle$. Finally, straight integration expresses the middle integral as

$$\sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}}$$

Original $f(x)$	*Transform* $f^*(s)$
$x^\xi(\log x)^k \qquad (x \to 0)$	$\dfrac{(-1)^k k!}{(s+\xi)^{k+1}}$
$\mathcal{O}(x^\gamma)$	Meromorphicity to the left, till $\Re(s) = -\gamma$
$x^\xi(\log x)^k \qquad (x \to \infty)$	$-\dfrac{(-1)^k k!}{(s+\xi)^{k+1}}$
$\mathcal{O}(x^\delta)$	Meromorphicity to the right, till $\Re(s) = -\delta$

Figure 7.7: The correspondence between asymptotics of $f(x)$ and poles of $f^*(s)$.

which is meromorphic in all C and provides the singular expansion of $f^*(s)$ in the extended strip. (See also [11].) □

The notation of singular expansions gives a transparent form to the correspondence between asymptotic expansions of original functions and poles of transforms. For instance, in the simpler case of an asymptotic expansion in increasing powers of x that will frequently arise from a Taylor expansion at 0, we have

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^n \quad (x \to 0) \quad \Longrightarrow \quad f^*(s) \asymp \sum_{n=0}^{\infty} \frac{c_n}{(s+n)} \quad (s \in \langle -\infty, \beta \rangle).$$

Similarly, $\log(1+x)$ has for Mellin transform the function $\pi/(s \sin \pi s)$, as results from the transform of its derivative $(1+x)^{-1}$ that is rational. This function admits the two singular expansions

$$\left\{ \begin{array}{lll} \dfrac{\pi}{s \sin \pi s} & \asymp & \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{s+k} \qquad\qquad (s \in \langle -\infty, -\tfrac{1}{2} \rangle) \\ & \asymp & \displaystyle\frac{1}{s^2} + \frac{0}{s} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{s-k} \quad (s \in \langle -\tfrac{1}{2}, +\infty \rangle) \end{array} \right.$$

that correspond term by term to the asymptotic expansions of the original function at 0 and ∞:

$$\begin{cases} \log(1+x) & = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \qquad (x \to 0) \\ & = \log x + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^{-k} \quad (x \to \infty) \end{cases}$$

EXERCISE 29. Find the singular expansion in $\langle -10, 10 \rangle$ of the Mellin transform of $(\cosh x)^{-1/2}$.

As an illustration of the theorem, consider the Mellin pair

$$f(x) = \frac{x}{e^x - 1} \qquad f^*(s) = \Gamma(s+1)\zeta(s+1).$$

Theorem 7.7 entails that $\zeta(s)$ is meromorphic in the whole of C (without appeal to the functional equation!), and in addition

$$f(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \qquad \Longrightarrow \qquad \Gamma(s+1)\zeta(s+1) \asymp \sum_{n=0}^{\infty} \frac{B_n}{n!} \frac{1}{(s+n)}.$$

By comparing this to the known singular expansion of $\Gamma(s)$, we find that

$$\zeta(0) = -\frac{1}{2},\ \zeta(-1) = -\frac{1}{12},\ \zeta(-2) = 0,\ \zeta(-3) = \frac{1}{120}, \ldots$$

and more generally,

$$\zeta(-2m) = 0, \quad \zeta(1-2m) = -\frac{B_{2m}}{2m} = -(2m-1)!\,[z^{2m}]\frac{z}{e^z - 1},$$

this without any recourse to the functional equation.

EXERCISE 30. What can be deduced from consideration of $f(x) = (1+e^x)^{-1}$ regarding special values of $L(s) = 1^{-s} - 3^{-s} + \cdots \Gamma$

Note on analytic continuation of transforms. A general principle also derives from the proof of Theorem 7.7: *Subtracting from a function a truncated form of its asymptotic expansion at either 0 or ∞ does not alter its Mellin transform but only shifts the fundamental strip.* An instance is provided by the equalities

$$\mathcal{M}(e^{-x}, s) = \Gamma(s) \quad s \in \langle 0, +\infty\rangle, \qquad \mathcal{M}(e^{-x} - 1, s) = \Gamma(s) \quad s \in \langle -1, 0\rangle, \quad (7.30)$$

previously established using integration by parts and specific properties of the exponential. The following proof of (7.30) demonstrates the general process on this particular example. Take the function

$$F^*(s) = \frac{1}{s} + \int_0^1 (e^{-x} - 1)x^{s-1}\, dx + \int_1^\infty e^{-x} x^{s-1}\, dx.$$

Consideration of both integrals shows that the function is meromorphic in $\langle -1, +\infty\rangle$. Its restriction to $\langle 0, +\infty\rangle$ is

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx,$$

and its restriction to $\langle -1, 0\rangle$ is

$$\int_0^\infty (e^{-x} - 1)x^{s-1}\, dx.$$

This argument shows that the transforms of e^{-x} and of $e^{-x} - 1$ are elements of the same meromorphic function in different strips.

Reverse mapping. There is a converse to Theorem 7.7: under suitable growth conditions, the existence of a meromorphic continuation of $f^*(s)$ entails asymptotic properties of the function itself.

Theorem 7.8 (Reverse mapping) *Let $f(x)$ be continuous in $]0, +\infty[$ with Mellin transform $f^*(s)$ having a non-empty fundamental strip $\langle \alpha, \beta\rangle$.*

(i) Assume that $f^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta\rangle$ for some $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\Re(s) = \gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that

$$f^*(s) = \mathcal{O}(|s|^{-r}) \quad \textit{with} \quad r > 1, \quad (7.31)$$

when $|s| \to \infty$ in $\gamma \le \Re(s) \le \eta$. If $f^*(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha\rangle$

$$f^*(s) \asymp \sum_{(\xi,k)\in A} d_{\xi,k} \frac{1}{(s-\xi)^k}, \quad (7.32)$$

then an asymptotic expansion of $f(x)$ at 0 is

$$f(x) = \sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^{k-1} \right) + \mathcal{O}(x^{-\gamma}).$$

(ii) Similarly assume that $f^*(s)$ admits a meromorphic continuation to $\langle \alpha, \delta \rangle$ for some $\delta > \beta$ and is analytic on $\Re(s) = \delta$. Assume also that the growth condition (7.31) holds for $\eta \le \Re(s) \le \delta$, for some $\eta \in (\alpha, \beta)$. If $f^*(s)$ admits the singular expansion

$$f^*(s) \asymp \sum_{(\xi,k)\in B} d_{\xi,k} \frac{1}{(s-\xi)^k}, \tag{7.33}$$

for $s \in \langle \eta, \delta \rangle$, then an asymptotic expansion of $f(x)$ at ∞ is

$$f(x) = -\sum_{(\xi,k)\in B} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^{k-1} \right) + \mathcal{O}(x^{-\delta}).$$

Proof. The proof makes use of the inversion theorem and of a residue computation using large rectangular contours in the extended strip of $f^*(s)$. As before, it suffices to consider case (i) corresponding to continuation to the left.

Let $\mathcal{S}$ be the set of poles in $\langle \gamma, \beta \rangle$. Consider the integral

$$J(T) = \frac{1}{2i\pi} \int_{\mathcal{C}} f^*(s) x^{-s} \, ds,$$

where $\mathcal{C} \equiv \mathcal{C}(T)$ denotes the rectangular contour defined by the segments

$$[\eta - iT, \eta + iT], \quad [\eta + iT, \gamma + iT], \quad [\gamma + iT, \gamma - iT], \quad [\gamma - iT, \eta - iT].$$

Assume that T is larger than $|\Im(s_0)|$ for all poles $s_0 \in \mathcal{S}$. By Cauchy's theorem, $J(T)$ is equal to the sum of residues, which is by a direct computation

$$R = \sum_{(\xi,k)\in A} d_{\xi,k} \operatorname{Res} \left(\frac{x^{-s}}{(s-\xi)^k} \right)_{s=\xi} = \sum_{(\xi,k)\in A} d_{\xi,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-\xi} (\log x)^k \right).$$

Let now T tend to $+\infty$. The integral along the two horizontal segments is $\mathcal{O}(T^{-r})$ and thus tends to 0 as $T \to \infty$. The integral along the vertical line $\Re(s) = \eta$ that lies inside the fundamental strip tends to the inverse Mellin integral which converges given the growth assumption on f^* and equals $f(x)$

$f^*(s)$	$f(x)$
Pole at ξ	Term in asymptotic exp. $\approx x^{-\xi}$
left of fund. strip	expansion at 0
right of fund. strip	expansion at $+\infty$
Multiple pole	Logarithmic factor
left: $\dfrac{1}{(s-\xi)^{k+1}}$	$\dfrac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at 0
right: $\dfrac{1}{(s-\xi)^{k+1}}$	$-\dfrac{(-1)^k}{k!}x^{-\xi}(\log x)^k$ at ∞
Pole with imaginary part: $\xi = \sigma + it$	Fluctuations: $x^{-\xi} = x^{-\sigma}e^{it\log x}$
Regularly spaced poles	Fourier series in $\log x$

Figure 7.8: The fundamental correspondence: aspects of the reverse mapping.

by the inversion theorem (since $f(x)$ is continuous). The integral along the vertical line $\Re(s) = \gamma$ is bounded by a quantity of the form

$$\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} |f^*(s)|\,|x^{-s}|\,|ds| = \mathcal{O}(1)\int_0^\infty \frac{x^{-\gamma}dt}{(1+t)^r} = \mathcal{O}(x^{-\gamma}),$$

given the growth assumption on f^*.

Thus, in the limit, $J(\infty)$ equals $f(x)$ plus a remainder term that is $\mathcal{O}(x^{-\gamma})$ plus the sum of residues that is of the stated form in x and $\log x$. □

Theorems 7.7 and 7.8 are clearly dual. Fig. 7.8 illustrates the information that can be extracted from knowledge of the singularities of a Mellin transform and should be compared to Fig. 7.3 relative to Dirichlet series and the Mellin-Perron formula.

Asymptotics of harmonic numbers. The asymptotic expansion of the harmonic numbers provides a clear illustration of the reverse mapping correspondence in the context of the analysis of sums.

We consider the real function

$$h(x) = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+x}\right]$$

that "interpolates" the harmonic numbers in the sense that $h(n) = H_n$ for any integer n. It is also a harmonic sum, as results from the equality

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{\frac{x}{n}}{1 + \frac{x}{n}}.$$

From the harmonic sum theorem, we find the transform

$$h^*(s) = \zeta(1-s) \cdot \left(-\frac{\pi}{\sin \pi s} \right),$$

with fundamental strip $\langle -1, 0 \rangle$.

Singular expansions of $\zeta(s)$ and $\pi / \sin \pi s$ are already known,

$$\zeta(s) \asymp \left[\frac{1}{s-1} + \gamma \right]_{s=1}, \qquad -\frac{\pi}{\sin \pi s} \asymp \frac{-1}{s} + \frac{1}{s-1} - \frac{1}{s-2} + \cdots,$$

with the latter holding in $\langle -1/2, +\infty \rangle$, so that, in $\langle -\frac{1}{2}, +\infty \rangle$,

$$h^*(s) \asymp \left[\frac{1}{s^2} - \frac{\gamma}{s} \right] - \frac{1}{2} \frac{1}{s-1} + \frac{1}{12} \frac{1}{s-2} - \frac{1}{120} \frac{1}{s-4} + \cdots.$$

The transform $h^*(s)$ is small towards $\pm i\infty$ since $\zeta(s)$ is only of polynomial growth in finite strips, while $\frac{\pi}{\sin \pi s}$ decreases exponentially. Thus, theorem 7.8 applies and by a direct term-by-term translation, we get

$$h(x) \sim \log x + \gamma + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} + \cdots,$$

and a full expansion results for $h(x)$, hence also for the harmonic numbers:

$$H_n \sim \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}.$$

The derivation makes use of the values of the zeta function at negative integers. The function $h(x)$ is closely related to the logarithmic derivative of the Gamma function, since $h(z) = z^{-1} + \gamma + \psi(z)$ where $\psi(z) = \frac{d}{dz} \log \Gamma(z)$. (The common elementary derivation of this results uses Euler-Maclaurin summation, see [41, §4.5].)

Exercise 31. Develop a complete asymptotic expansion for the generalized harmonic numbers of some integer order r,

$$H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

In particular,

$$H_n^{(2)} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2\,n^2} - \frac{1}{6\,n^3} + \frac{1}{30\,n^5} - \frac{1}{42\,n^7} + \frac{1}{30\,n^9} + O(n^{-11})$$

$$H_n^{(3)} = \zeta(3) - \frac{1}{2\,n^2} + \frac{1}{2\,n^3} - \frac{1}{4\,n^4} + \frac{1}{12\,n^6} - \frac{1}{12\,n^8} + O(n^{-10})$$

Exercise 32. [Ramanujan] Analyse asymptotically

$$H_n^{(1/2)} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

Hint: one may consider

$$h(x) = \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+x}}\right] = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[1 - \frac{1}{\sqrt{1+x/n}}\right].$$

7.7 Harmonic sums

In this section we develop applications of the previous theorems specialized to the analysis of sums. First, we state a complete set of definitions relative to harmonic sums.

Definition 7.4 *A* harmonic sum *is a sum of the form*

$$F(x) = \sum_k \lambda_k f(\mu_k x).$$

The function $f(x)$ is called the base function. *The λ_k are the* amplitudes *and the μ_k are the* frequencies. *The (generalized)* Dirichlet series *of the harmonic sum is the series*

$$\Xi(s) := \sum_k \frac{\lambda_k}{\mu_k^s}.$$

The denomination is motivated by the fact that harmonic sums vastly generalize Fourier series that correspond to frequencies $\mu_k = k$ and to a base function f that is a complex exponential, $f(x) = e^{\pm ix}$. To avoid degeneracies, we further assume that the frequencies μ_k are nonzero reals and that either $\mu_k \to 0$ or $\mu_k \to +\infty$.

As expressed by Theorem 7.5, the Mellin transform of a harmonic sum factorizes as

$$F^*(s) = \Xi(s) \cdot f^*(s) \qquad \text{where} \qquad \Xi(s) = \left(\sum_k \lambda_k \mu_k^{-s} \right)$$

which is the product of its Dirichlet series (determined by the amplitude-frequency pair) and of the Mellin transform of the base function.

Mellin transform analysis of harmonic sums requires *a priori* two conditions so that the mapping theorem (Thm. 7.8) be applicable:

- *Analytic continuation.* Both $f^*(s)$ and $\Xi(s)$ must be meromorphically continuable beyond their original domains of existence. This is guaranteed by the direct mapping theorem 7.7 for $f^*(s)$ as soon as $f(x)$ has asymptotic expansions at 0 and ∞. For $\Xi(s)$ many number–theoretic functions related to divisors, or binary representations also share this property.

- *Smallness at imaginary infinity.* The most commonly encountered situation is where $f^*(s)$ decays exponentially along vertical lines (this is the case for the transforms of e^{-x}, $(1+x)^{-1}$ and many more functions) while $\Xi(s)$ only grows at most polynomially along vertical lines (the function $\zeta(s)$ is itself an instance of this situation). In this case the "balance" is in favour of fast decay and the reverse mapping theorem is guaranteed to apply.

To formalize this process, we make precise the notions of fast decay and moderate growth, then state a theorem that summarizes the whole chain.

Definition 7.5 *A function $\phi(s)$ that is meromorphic in C is said to be of* fast decrease *if in any finite strip of the complex plane, it satisfies for any $r > 0$,*

$$\phi(s) = \mathcal{O}(|s|^{-r}),$$

as $|s| \to +\infty$ in the strip.

A function $\alpha(s)$ that is meromorphic in C is said to be of moderate growth *if in any finite strip of the complex plane, it satisfies for some $r > 0$,*

$$\alpha(s) = \mathcal{O}(|s|^{r}),$$

as $|s| \to +\infty$ in the strip.

We shall also say that a meromorphic $\alpha(s)$ is of *moderate growth in the weak sense* if the condition $\alpha(s) = \mathcal{O}(|s|^r)$ is only required to hold alongs two sets of horizontal lines $\Im(s) = -T_j$ and $\Im(s) = U_j$, where $T_j, U_j \to +\infty$.

The following theorem encapsulates most of the (currently known) Mellin technology in analytic combinatorics.

Theorem 7.9 (Mellin asymptotic summation) *Consider a harmonic sum $F(x) = \sum_k \lambda_k f(\mu_k x)$ that is a continuous function of x, for $x \in]0, +\infty[$. Assume the following conditions:*

(M0) There is a nonempty intersection of the open fundamental strip of $f^*(s)$ and of the open half-plane of absolute convergence of the Dirichlet series of the harmonic sum.

(M1) The transform $f^*(s)$ of the base function is meromorphic in C and of fast decrease.

(M2) The generalized Dirichlet series $\Xi(s)$ of the sum is meromorphic in C and of moderate growth (possibly in the weak sense).

Then, $F(x)$ admits asymptotic expansions at 0 and $+\infty$ determined by

$$\begin{aligned}\sum_k \lambda_k f(\mu_k x) \quad & \sim_{x \to 0} \quad \sum_{\Re(s) < c} Res\left[f^*(s) \cdot \Xi(s) x^{-s}\right] \\ & \sim_{x \to +\infty} \quad - \sum_{\Re(s) > c} Res\left[f^*(s) \cdot \Xi(s) x^{-s}\right],\end{aligned}$$

where c lies in the intersection of the fundamenatl strip of $f^*(s)$ and the half-plane of abolute convergence of $\Xi(s)$.

The treatment of harmonic numbers in the last section provides a clear instance of Mellin asymptotic summation: the Dirichlet series $\zeta(1-s)$ is of moderate growth while the transform of the base function $\pi / \sin(\pi s)$ is of fast decrease.

Naturally, arbitrarily many variants of this theorem could be generated, for instance by assuming only partial meromorphic extension of $F^*(s)$ and one-sided asymptotic expansions of $F(x)$ or by modifying the growth conditions in various ways (see [16]).

Exercise 33. By considering sums where the base function is the Heaviside function $H(x)$ or some of its antiderivatives, relate the Mellin-Perron formula and the Mellin summation formula.

Stirling's formula. From the product decomposition of the Gamma function, one has

$$\ell(x) := \log \Gamma(x+1) + \gamma x = \sum_{n=1}^{\infty} \left[\frac{x}{n} - \log(1 + \frac{x}{n}) \right], \quad (s \in \langle -2, +\infty \rangle).$$

The Mellin transform is

$$\ell^*(s) = -\zeta(-s) \frac{\pi}{s \sin \pi s},$$

with fundamental strip $\langle -2, -1 \rangle$, since the Dirichlet series $\zeta(-s)$ converges in $\Re(s) < -1$ while the strip of the base function is $\langle -2, -1 \rangle$.

There are double poles at $s = -1$, $s = 0$ and simple poles at the positive integers,

$$\ell^*(s) \asymp \left[\frac{1}{(s+1)^2} + \frac{1-\gamma}{(s+1)} \right] + \left[\frac{1}{2s^2} - \frac{\log \sqrt{2\pi}}{s} \right] + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \zeta(-n)}{n(s-n)}.$$

Hence Stirling's formula

$$\log(x\,!) \sim \log \left(x^x e^{-x} \sqrt{2\pi x} \right) + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}}.$$

Euler-Maclaurin summations. Given a continuous function $f(x)$, we consider the sum

$$F(x) = \sum_{n=1}^{\infty} f(nx),$$

where for the sake of convergence it is assumed that $f(x) = \mathcal{O}(x^{-1-\epsilon})$ as $x \to +\infty$, for some $\epsilon > 0$. Clearly as $x \to 0$, the behaviour of the sum should depend on properties of $f(x)$ near 0.

The Mellin transform of $F(x)$ is simply

$$\zeta(s) \cdot f^*(s),$$

and $\zeta(s)$ is of moderate growth. Assume first, like in the classical form of Euler-Maclaurin summation, that $f(x)$ has a standard asymptotic expansion,

$$f(x) \underset{x \to 0}{\sim} \sum_{k=0}^{\infty} c_k x^k, \quad \text{so that} \quad f^*(s) \asymp \sum_{k=0}^{\infty} \frac{c_k}{s+k},$$

in $\langle -\infty, 1+\epsilon\rangle$. Thus, taking into account the pole of $\zeta(s)$ at $s=1$, we find

$$F^*(s) \asymp \frac{f^*(1)}{s-1} + \sum_{k=0}^{\infty} \frac{c_k \zeta(-k)}{s+k}.$$

Under smallness of the transform $f^*(s)$, the conditions of Mellin asymptotic summations are satisfied. Given that $f^*(1)$ is the integral of $f(x)$ and given the known values of the zeta function, the asymptotic expansion of $F(x)$ results:

$$F(x) \sim \frac{1}{x}\int_0^\infty f(x)\,dx - \frac{1}{2}f(0) - \sum_{k\geq 1} \frac{B_{2k}}{2k} c_{2k-1} x^{2k-1}.$$

This is nothing but a form of Euler-Maclaurin summation specialized to the interval $(0, +\infty)$.

The approach developed here has the great advantage of generalizing to arbitrary asymptotic expansions with "fractional" exponents.

Proposition 7.5 (General Euler-Maclaurin summation) *Assume that $f(x)$ satisfies an asymptotic expansion*

$$f(x) \underset{x\to 0}{\sim} \sum_{k=1}^{\infty} c_k x^{\alpha_k},$$

where $-1 < \alpha_1 < \alpha_2 < \cdots$ and that $f^*(s)$ is of fast decrease. Then:

$$F(x) = \sum_{n=1}^{\infty} f(nx) \sim \frac{1}{x}\int_0^\infty f(x)\,dx + \sum_{k\geq 1} c_k \zeta(-\alpha_k) x^{\alpha_k}.$$

The method can also accommodate logarithmic terms in the expansion of $f(x)$, in which case derivatives of the zeta function will be involved. Such generalizations of Euler-Maclaurin summation have been considered by Barnes and Gonnet. Typical instances are the summatory formulæ [16],

$$\begin{aligned}
\sum_{k=1}^{\infty}(-1)^{k-1} f(kx) &\sim \sum_{k=0}^{\infty} c_k (1-2^{1+\alpha_k})\zeta(-\alpha_k) x^{\alpha_k} \\
\sum_{k=1}^{\infty}(\log k)\, f(kx) &\sim \frac{1}{x}\log\frac{1}{x}\int_0^\infty f(x)\,dx + \frac{1}{x}\int_0^\infty f(x)(\log x)\,dx \\
&\quad + \sum_{k=0}^{\infty} c_k \zeta'(-\alpha_k) x^{\alpha_k},
\end{aligned}$$

that are established by Estrada and Kanwal [12, Ch. 3] by means of the theory of distributions.

EXERCISE 34. Analyze asymptotically

$$\sum_{n\geq 1}\frac{\sqrt{nx}}{1+n^2x^2},\quad \sum_{n\geq 1}\frac{\log nx}{1+n^2x^2}.$$

EXERCISE 35. [16] Analyze asymptotically

$$\sum_{n\geq 1} e^{-\sqrt{nx}}.$$

EXERCISE 36. [Ramanujan] Show that

$$\sum_{k\geq 1} e^{-kx}\log k = \frac{1}{x}(\log\frac{1}{x}-\gamma)+\log\sqrt{2\pi}+\mathcal{O}(x).$$

EXERCISE 37. Develop asymptotic summatory formulæ for

$$\sum_k (-1)^k \log k f(kx),\quad \sum_k (-1)^k H_k f(kx),\quad \sum_k H_k f(kx).$$

A divisors sum. The problem here is to analyze the harmonic sum

$$D(x)=\sum_{k=1}^{\infty} d(k)e^{-kx}, \tag{7.34}$$

with $d(k)$ the number of divisors of k. Consideration of this sum is suggested by the analysis of the expected height of Catalan trees discussed in the next section. The function $d(k)$ fluctuates rather heavily, and for instance, the value of $d(k)$ equals 2 iff k is a prime number; for a highly composite number like $N=30^n$, one has $d(N)=(n+1)^3$ which is about $(\log N)^3$, etc.

With the trivial inequality $d(k)\leq k$, it is apparent that the sum in (7.34) is convergent and continuous for any $x>0$. Also, one has $D(x)\sim e^{-x}$ as $x\to+\infty$. As $x\to 0$, a direct estimate $D(x)=\mathcal{O}(x^{-2})$ results from the crude bound $d(k)<k$. Therefore, the Mellin transform of $D(x)$ exists at least in the strip $\langle 2,+\infty\rangle$.

The harmonic sum property implies that the Mellin transform of $D(x)$ is

$$F^*(s)=(\zeta(s))^2\cdot\Gamma(s)$$

since ζ^2 is the DGF of the divisor function. That transform is meromorphic in the whole of C; it is also exponentially small towards $\pm i\infty$ in any finite strip of C since $\zeta^2(s)$ is of polynomially bounded growth while $\Gamma(s)$ decreases exponentially.

There are singularities at $s = 1$, $s = 0$, then at all the odd negative integers, so that

$$D^*(s) \asymp \left[\frac{1}{(s-1)^2} + \frac{\gamma}{s-1}\right] + \left[\frac{1}{4s}\right]_{s=0} - \sum_{k=0}^{\infty} \frac{(\zeta(-2k-1))^2}{(2k+1)!} \frac{1}{s+2k+1}.$$

This translates into the asymptotic expansion of $D(x)$ as $x \to 0$:

$$D(x) \sim \frac{1}{x}(-\log x + \gamma) + \frac{1}{4} - \sum_{k=0}^{\infty} \frac{(\zeta(-2k-1))^2}{(2k+1)!} x^{2k+1}.$$

Clearly sums of divisors can be treated in a similar way as their Dirichlet series are expressible in terms of the Riemann zeta function. Ramanujan discovered a number of related formulæ later established systematically by Berndt and Evans using Mellin transforms [4].

A doubly exponential sum. This sum is motivated by the analysis of the Bernoulli splitting process treated in the next section. It illustrates the occurrence of tiny periodic fluctuations under the form of Fourier series related to regularly spaced poles of a Mellin transform that arise from a component of the form $(1-2^s)^{-1}$. The sum

$$F(x) = \sum_{k=0}^{\infty} [1 - e^{-x/2^k}],$$

is to be analyzed as $x \to +\infty$. The Mellin transform is

$$F^*(s) = -\frac{\Gamma(s)}{1-2^s},$$

with fundamental strip $\langle -1, 0\rangle$. (It actually suffices to prove existence of of F^* in any smaller nonemepty substrip, by elementary arguments of sorts.)

There is a double pole of $F^*(s)$ at $s = 0$, but also imaginary poles at

$$s = \chi_k \equiv \frac{2ik\pi}{\log 2},$$

which are expected to introduce periodic fluctuations. The singular expansion of $F^*(s)$ in $\langle -\frac{1}{2}, +2\rangle$ is

$$F^*(s) \asymp \left[\frac{1}{\log 2}\frac{1}{s^2} - \frac{\gamma + \frac{1}{2}\log 2}{s\log 2}\right] + \frac{1}{\log 2}\sum_{k\in Z\setminus\{0\}} \frac{\Gamma(\chi_k)}{s-\chi_k}.$$

Accordingly, one finds

$$F(x) = \frac{\log x}{\log 2} + \frac{1}{2} + \frac{\gamma}{\log 2} + Q(\log_2(x)) + \mathcal{O}(x^{-2}),$$

where

$$Q(u) = \frac{-1}{\log 2}\sum_{k\in Z\setminus\{0\}} \Gamma(\chi_k)e^{-2ik\pi u}.$$

Like before, the error term can be taken to be $\mathcal{O}(x^{-M})$ for any $M > 0$.

The Fourier coefficients decrease very fast, and one has for instance

$$\Gamma(\chi_1) = \overline{\Gamma(\chi_{-1})} = -0.41767\, 10^{-6} - 0.35043\, 10^{-6}\, i,$$

$$\Gamma(\chi_2) = \overline{\Gamma(\chi_{-2})} = -0.14763\, 10^{-12} + 0.20480\, 10^{-12}\, i,$$

and so on, so that *the amplitude of the Fourier series does not exceed* 10^{-6}.

Though the fluctuations are tiny (and may be safely ignored for most practical purposes), their presence makes the asymptotic analysis of such sums intrinsically nonelementary.

Thus, Mellin transform —like Dirichlet series— can capture periodic fluctuations. The results bear some resemblance to the analysis of the sum-of-digits function and of divide-and-conquer recurrences discussed earlier in this chapter. However, the presence of the exponential in the original function of the example just discussed entails fast convergence of the Fourier series (via smallness of the Gamma function), so that no fractal phenomenon appears here.

Dyadic sums. Mellin transforms make it possible to discuss in general terms harmonic sums involving powers of two as frequencies. This is close in spirit to our earlier discussion of generalized Euler-Maclaurin summations

Sums involving powers of 2, of the type

$$G_w(x) = \sum_{k\geq 0} 2^{-kw} g(\frac{x}{2^k}),$$

are particularly frequent in the analysis of algorithms and we call them *dyadic* sums. In applications x usually represents a large parameter.

Let $g(x)$ be such that

$$g(x) \underset{x\to\infty}{\sim} \sum_{k=0}^{\infty} d_k x^{-\beta_k},$$

for some increasing sequence $\{\beta_k\}$. We assume naturally that $g(x)$ is $\mathcal{O}(x^{-\alpha})$ at 0, for some $\alpha < \beta_0$, and that $g^*(s)$ is of fast decrease towards $\pm i\infty$. We examine the case of $G_0(x)$ whose transform is (formally)

$$G_0^*(s) = \frac{g^*(s)}{1-2^s}.$$

Assume first that $\beta_0 > 0$, so that $g(\infty) = 0$. Then the transform has a nonempty fundamental strip. Theorem 7.9 applies and gives

$$G_0(x) \equiv \sum_{k\geq 0} g(\frac{x}{2^k}) \underset{x\to+\infty}{\sim} \frac{1}{\log 2} \sum_{k\in Z} g^*(\frac{2ik\pi}{\log 2}) e^{-2ik\pi\log_2 x} + \sum_{k=0}^{\infty} \frac{d_k}{1-2^{\beta_k}} x^{-\beta_k}.$$

Thus under these conditions, $G_0(x)$ again fluctuates asymptotically.

Consider now the case where $g(\infty) \neq 0$; then a logarithmic term creeps in. For simplicity, we assume that the expansion of $g(x)$ is of a standard form.

Proposition 7.6 (Standard dyadic summation) *Assume that*

$$g(x) \underset{x\to+\infty}{\sim} \sum_{k=0}^{\infty} d_k x^{-k},$$

with $d_0 = g(\infty) \neq 0$ and that $g^*(s)$ is of fast decrease. Then:

$$G(x) \sim g(\infty)\log_2 x + \frac{1}{2} g(\infty) + \frac{\gamma[g]}{\log 2} + P(\log_2 x) + \sum_{k=1}^{\infty} \frac{d_k}{1-2^k} x^{-k},$$

with $\gamma[g]$ the Euler constant of g,

$$\gamma[g] = \int_0^1 g(x)\,\frac{dx}{x} + \int_1^{\infty} (g(x) - g(\infty))\,\frac{dx}{x},$$

and with an explicitly determined periodic function $P(.)$:

$$P(u) = \frac{-1}{\log 2} \sum_{k\neq 0} g^*(\frac{2ik\pi}{\log 2}) e^{-2ik\pi u}.$$

Proof. There is now a double pole of $G_0^*(s)$ at $s = 0$ so that a two term expansion is required for $g^*(s)$ there. By the fundamental splitting, one has

$$g^*(s) = -\frac{g(\infty)}{s} + \int_0^1 g(x)x^{s-1}\,dx + \int_1^\infty (g(x) - g(\infty))x^{s-1}\,dx,$$

so that

$$g^*(s) = -\frac{g(\infty)}{s} + \gamma[g] + \mathcal{O}(s).$$

The constant $\gamma[g]$ is called the *Euler constant* of g since $\gamma[e^{-x} - 1] = -\gamma$. □

This example illustrates in passing a general technique by which one can determine terms of the series expansion of a Mellin transform outside of its convergence strip; it may be viewed as an adaptation of the method of subtracted singularities of Theorem 7.7.

7.8 Combinatorial applications.

We discuss here three combinatorial applications of Mellin transform techniques. The sums in the first one involve the highly oscillating divisor function, but these oscillations are smoothed out in the final asymptotic estimate. The other 2 reveal periodicity phenomena not unlike those encountered in divide-and-conquer recurrences or the sum-of-digit function, though their amplitude is minute.

7.8.1 Catalan sums and the height of trees

What we call here Catalan sums are particular binomial sums,

$$S_n = \sum_{k=1}^{n} \lambda_k \frac{\binom{2n}{n-k}}{\binom{2n}{n}}, \tag{7.35}$$

where the λ_k are a fixed set of coefficients (often of an arithmetical character). Such sums occur as average values of combinatorial parameters for objects enumerated by the Catalan numbers,

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{7.36}$$

like plane trees, binary trees, or ballot sequences [6, 24].

The paper of De Bruijn, Knuth, and Rice [9] is the historical source of Mellin transform techniques applied to combinatorial enumeration. It treats the *expected height* $\bar{H}_n$ *of a random rooted plane trees of* n *nodes under the uniform distribution* (see Chapters 1, 3 for details). The sequence λ_k is in this case the divisor function $d(k)$. We briefly explain here the connection between this combinatorial problem and a Catalan sum like (7.35). (See also [41, §5.9] and [29, p. 135] for details.)

As seen in Chapter 1, a plane tree decomposes recursively as a root node to which is attached a sequence of trees. Let G_n be the number of trees with n nodes; the ordinary generating function of the sequence $\{G_n\}$ is defined by

$$G(z) = \sum_{n=1}^{\infty} G_n z^n.$$

This decomposition translates into a functional equation that admits an explicit solution

$$G(z) = \frac{z}{1 - G(z)} \quad \text{and} \quad G(z) = \frac{1 - \sqrt{1 - 4z}}{2}.$$

We have $G_{n+1} = C_n$ with C_n the Catalan number of (7.36).

Let similarly $G^{[h]}(z)$ be the generating function of trees of height at most h. As height is inherited from subtrees, one has then the basic recurrence

$$G^{[h+1]}(z) = \frac{z}{1 - G^{[h]}(z)} \quad \text{with} \quad G^{[0]}(z) = z.$$

This is an instance of a more general scheme discussed in Chapter 3. Here, the $G^{[h]}(z)$ are rational fractions that are also approximants to an infinite continued fraction representing $G(z)$. We have

$$G^{[h]}(z) = z\frac{F_{h+1}(z)}{F_{h+2}(z)},$$

where the $F_h(z)$ are a family of polynomials called Fibonacci polynomials and defined by the recurrence

$$F_0(z) = 0, \; F_1(z) = 1, \; F_{h+2}(z) = F_{h+1}(z) - zF_h(z).$$

Solving this linear recurrence in h (with z a parameter) yields the closed form

$$G^{[h]}(z) = 2z\frac{(1 + \sqrt{1 - 4z})^{h+1} - (1 - \sqrt{1 - 4z})^{h+1}}{(1 + \sqrt{1 - 4z})^{h+2} - (1 - \sqrt{1 - 4z})^{h+2}}. \qquad (7.37)$$

An alternative form of this last relation is

$$G(z) - G^{[h-2]}(z) = \sqrt{1-4z}\,\frac{u^h(z)}{1-u^h(z)} \quad \text{where} \quad u(z) = \frac{1-\sqrt{1-4z}}{1+\sqrt{1-4z}} = \frac{G(z)}{1-G(z)}. \tag{7.38}$$

There results that the $G^{[h]}(z)$ can be expressed in terms of $G(z)$ alone; the Taylor expansions then derive by the Lagrange-Bürmann inversion theorem for analytic functions (see [9] for details):

$$G_{n+1} - G_{n+1}^{[h-2]} = \sum_{j\geq 1} \Delta^2 \binom{2n}{n-jh}, \tag{7.39}$$

where

$$\Delta^2 \binom{2n}{n-m} = \binom{2n}{n+1-m} - 2\binom{2n}{n-m} + \binom{2n}{n-1-m}.$$

Thus, the number of trees of height $> h-2$ appears as a "sampled" sum of the $2n$th line of Pascal's triangle (upon taking second order differences).

By the well-known form of the expectations of discrete random variables, the mean height $\bar{H}_{n+1}$ satisfies

$$1 + \bar{H}_{n+1} = \frac{1}{G_{n+1}} \sum_{h\geq 1} \sum_{j\geq 1} \Delta^2 \binom{2n}{n-jh},$$

Grouping terms according to the value of jh then reduces this expression to a simple sum:

$$1 + \bar{H}_{n+1} = \frac{1}{G_{n+1}} \sum_{k} d(k) \Delta^2 \binom{2n}{n-k} \tag{7.40}$$

We are therefore lead to considering sums obeying a pattern similar to (7.35),

$$S_n^{(a)} = \sum_{k=1}^{n} d(k) \frac{\binom{2n}{n-k-a}}{\binom{2n}{n}},$$

since

$$1 + \bar{H}_{n+1} = (n+1)\left[S_n^{(1)} - 2S_n^{(0)} + S_n^{(-1)}\right]. \tag{7.41}$$

The treatment of the central sum is typical. Stirling's formula yields the Gaussian approximation of binomial numbers: for $k = w\sqrt{n}$, and with $k =$

$o(n^{3/4})$, one has

$$\frac{\binom{2n}{n-k}}{\binom{2n}{n}} \sim e^{-w^2}\left(1 - \frac{w^4+3w^2}{6n} + \frac{5w^8+6w^6-45w^4-60}{360n^2} + \cdots\right).$$

This leads to the continuous harmonic sum

$$F_0(x) = \sum_k d(k) e^{-k^2x^2},$$

and an elementary argument (domination of the central terms) justifies the use of the Gaussian approximation inside $S_n^{(0)}$:

$$S_n^{(0)} = F_0(\frac{1}{\sqrt{n}}) + o(1). \tag{7.42}$$

The asymptotic analysis of $F_0(x)$ for $x \to 0$ (here, $x = n^{-1/2}$) is typical of harmonic sums and closely follows the pattern encountered in the last section. From

$$F_0^*(s) = \frac{1}{2}\Gamma(\frac{s}{2})\zeta^2(s),$$

one gets

$$F_0(x) \sim \frac{1}{4}\frac{\sqrt{\pi}}{x}(-2\log(2x) + 3\gamma) + \frac{1}{4} - \frac{\sqrt{\pi}}{144}x + \cdots \tag{7.43}$$

The other sums $S_n^{(\pm 1)}$ are treated similarly. From (7.41), (7.42), (7.43) and their analogues, the expected height is found.

Proposition 7.7 *The expected height of a random plane rooted tree of n nodes is*

$$\sqrt{\pi n} - \frac{1}{2} + o(1).$$

Full asymptotic expansions could in principle be determined by this technique but a more global approach based on singularity analysis via Mellin transforms is presented below and it appears to be preferable.

The basic method here consists in approximating Catalan sums (7.35) by Gaussian sums of the form

$$F(x) = \sum_{k=1}^{\infty} \lambda_k e^{-k^2x^2},$$

and treating the latter by Mellin transforms. Related Catalan sums surface in the analysis of Batcher's odd-even merge sorting network [39] and in register allocation [28, 29], where the arithmetic function λ_k involved in (7.35) is either a function of the Gray code representation of k or the function $v_2(k)$ representing the exponent of 2 in the prime number decomposition of k.

7.8.2 The Bernoulli splitting process and dyadic sums

The Bernoulli splitting process is a general model of the random allocation of resource either in the time domain (like stations sharing a common communication channel) or in the space domain (like keys sharing some primary or secondary storage) whose analysis usually leads to a variety of dyadic sums. The abstract process takes a set G of individuals and splits them recursively as follows:

— If $\mathrm{card}(G) \leq 1$ then the process stops and no splitting occurs.

— Otherwise $\mathrm{card}(G) \geq 2$, and each $g \in G$ flips independently a fair coin. Let G_0 and G_1 be the two subsets of G corresponding to the the groups of individuals having flipped heads (0) and tails (1). Then the process is recursively applied to the two subsets G_0 and G_1.

A realisation of the process may be described by a tree $\tau(G)$ whose internal binary nodes correspond to splittings of more than 1 element; the external nodes either contain a single individual or the empty set.

If one views each elements of G as having determined in advance an infinite sequence of random bits (a "key") corresponding to coin flippings, the tree $\tau(G)$ appears to be just the *digital tree*, also known as *trie*, that is associated to G viewed as a set of "keys". Tries are basic data structures that dynamically support searches, insertions, and deletions whose description is found in standard treatises like [23, 31, 35, 40]. (Retrieval of an element g in $\tau(G)$ is achieved by following an access path dictated by g.) In this context, the Bernoulli splitting process models the characteristics of trees built from random uniform keys.

Given that the cardinality of the original group G is n, there are two random variables of major interest: the number I_n of splitting stages (where G is effectively split into G_0, G_1) that corresponds to the number of internal nodes in $\tau(G)$; the total number L_n of coin flippings that corresponds to internal path length in $\tau(G)$. The expectations $i_n = \mathrm{E}\{I_n\}$ and $\ell_n = \mathrm{E}\{L_n\}$ satisfy recurrences that reflect the nature of the splitting process; for $n \geq 2$, one has

$$i_n = 1 + \sum_{k=0}^{n} \pi_{n,k}(i_k + i_{n-k}), \quad \ell_n = n + \sum_{k=0}^{n} \pi_{n,k}(\ell_k + \ell_{n-k}), \quad \pi_{n,k} = \frac{1}{2^n}\binom{n}{k}, \tag{7.44}$$

with initial conditions $i_0 = i_1 = \ell_0 = \ell_1 = 0$. The splitting probabilities $\pi_{n,k}$ are specific of the Bernoulli splitting process and they represent the probability of turning k heads out of n coin flips. The solution to this problem was originally developed by Knuth and De Bruijn [31] and it constitutes a nice application of dyadic sums discussed in the previous section. An elementary but partial analysis is discussed in [41, Ch. VII].

The basic technique to solve (7.44) consists in introducing the exponential generating functions

$$I(z) = \sum_{n=0}^{\infty} i_n \frac{z^n}{n!}, \qquad L(z) = \sum_{n=0}^{\infty} \ell_n \frac{z^n}{n!}, \tag{7.45}$$

with which (7.44) transforms into

$$I(z) = 2e^{z/2} I(\frac{z}{2}) + (e^z - 1 - z), \quad L(z) = 2e^{z/2} L(\frac{z}{2}) + z(e^z - 1). \tag{7.46}$$

A functional equation of the form

$$\phi(z) = 2e^{z/2}\phi(\frac{z}{2}) + a(z), \tag{7.47}$$

with $a(z)$ a known function and $\phi(z)$ the unknown, is solved by iteration:

$$\begin{aligned} \phi(z) &= a(z) + 2e^{z/2}\phi(\frac{z}{2}) \\ &= a(z) + 2e^{z/2}a(\frac{z}{2}) + 4e^{3z/4}a(\frac{z}{4}) \\ &= \cdots \\ &= \sum_{k=0}^{\infty} 2^k e^{z(1-2^{-k})} a(\frac{z}{2^k}). \end{aligned} \tag{7.48}$$

This principle applies provided $a(z) = O(z^2)$, a condition satisfied by $I(z)$ and $L(z)$ for which $a(z) = e^z - 1 - z$ and $a(z) = z(e^z - 1)$, respectively. Upon expanding the exponentials, one finds the explicit forms

$$\begin{aligned} i_n &= \sum_{k=0}^{\infty} 2^k \left[1 - (1 - \frac{1}{2^k})^n - \frac{n}{2^k}(1 - \frac{1}{2^k})^{n-1}\right] \\ \ell_n &= n \sum_{k=0}^{\infty} \left[1 - (1 - \frac{1}{2^k})^{n-1}\right]. \end{aligned} \tag{7.49}$$

From there, the most direct route is the exponential approximation

$$(1-a)^n = e^{-n\log(1-a)} = e^{-na+\mathcal{O}(na^2)} \approx e^{-na}.$$

It is legitimate to use it in (7.49), see [31, p. 131] for a justification based on splitting the sums. With

$$F(x) = \sum_{k=0}^{\infty} 2^k \left[1 - (1 + \frac{x}{2^k})e^{-x/2^k}\right] \qquad G(x) = x\sum_{k=0}^{\infty}\left[1 - e^{-x/2^k}\right],$$

one finds elementarily $i_n = F(n) + \mathcal{O}(\sqrt{n})$ and $\ell_n = G(n) + \mathcal{O}(\sqrt{n})$. (We also discuss below an approach to full asymptotic expansions that avoids resorting to the exponential approximation.) The functions $F(x)$ and $G(x)$ are dyadic sums of a type already considered.

Proposition 7.8 *Consider a random tree grown from the Bernoulli splitting process with an initial group of size n. The number of internal nodes and the internal path length of the tree have averages that satisfy*

$$\begin{aligned} i_n &= \frac{n}{\log 2} + nP(\log_2 n) + \mathcal{O}(\sqrt{n}) \\ \ell_n &= n\log_2 n + \left(\frac{\gamma}{\log 2} + \frac{1}{2}\right) n + nQ(\log_2 n) + \mathcal{O}(\sqrt{n}), \end{aligned}$$

where $P(u)$ and $Q(u)$ are absolutely convergent Fourier series of variation less than 10^{-5}.

For tries, this result means that the number of binary nodes is on average about $1.44\,n$, *a 44% waste in storage* when compared to standard binary tree structures, while the average depth ℓ_n/n of a random external node is about $\log_2 n$, which corresponds to *an asymptotically optimal search cost* in an information-theoretic sense.

The Bernoulli splitting process is also an especially useful model in distributed algorithms. For intance, a sequential execution constitutes a way to regulate access to a common shared channel (groups consisting of single individuals may deliver their message without interference): this is the *tree communication protocol* of Capetanakis–Tsybakov discussed for instance in [27, Ch. 9]. Also, retaining only the leftmost branch of the tree leads to an efficient leader election algorithm [37].

Note. The example of the Bernoulli process is a nice application of the estimation of dyadic sums. Its importance owes to the fact that it illustrates a general paradigm corresponding to families of probability distribution functions $F_n(x)$ that obey a "dyadic law",

$$F_n(x) = \phi\left(\frac{n}{2^k}\right) \cdot (1 + o(1)). \tag{7.50}$$

A surprising number of instances have surfaced in the literature of algorithms, see [16, 31, 35]. For this reason, we re-examine this particular analysis together with closely related examples recurrently in the rest of this chapter.

In the next section, we show an approach based on singularity analysis of ordinary generating functions treated by means of Mellin transforms. In the last section of this chapter, we show how a full asymptotic expansion for trie sums can be obtained by a suitable treatment of intervening Dirichlet generating functions.

Exercise 38. Assume that the error term in (7.50) is $\mathcal{O}(n^{-1})$. State sufficient smoothness conditions on ϕ ensuring that the mean and variance associated with the probability distribution function F_n are $\log n + \mathcal{O}(1)$ and $\mathcal{O}(1)$, respectively. Provide dominant asymptotics for the variance.

7.8.3 Longest runs in binary strings.

Longest runs in random binary strings are treated by Knuth [32] in a paper that deals with the equivalent problem of carry propagation in parallel binary adders. There the problem requires an analysis of dominant poles of a family of rational functions eventually leading to an asymptotic approximation by dyadic sums. The final asymptotic part of the treatment is then well-suited to Mellin analysis.

Consider strings over a binary alphabet $\mathcal{A} = \{0, 1\}$. The problem is to estimate the expected length $\bar{L}_n$ of the longest run of 1's in a random string of length n, where all the 2^n possible strings are taken equally likely. The distribution was studied by Feller [13] and Knuth [32]. We can complete now an analysis already started in Chapters 1 (OGFs and constructions) and 4 (location of poles).

The probability that a random string of length n has no run of k consecutive 1's is

$$q_{n,k} = \frac{1}{2^n}\,[z^n]\,\frac{1 - z^k}{1 - 2z + z^{k+1}}. \tag{7.51}$$

The set of such strings is described by the regular expression $1^{<k} \cdot (01^{<k})^*$, where $1^{<k}$ denotes a sequence of less than k 1's and $(\,)^*$ denotes arbitrary repetition of a pattern; the general principles of Chapter 1 enable us to write the ordinary generating function of the set of strings under consideration as

$$\frac{1-z^k}{1-z} \cdot \frac{1}{1-z\frac{1-z^k}{1-z}},$$

which justifies (7.51).

Let ρ_k be the smallest positive root of the denominator of (7.51) that lies between $\frac{1}{2}$ and 1. An application of the principle of the argument shows such a root to exist with all other roots that are of a larger modulus (see Chapter 4). By dominant pole analysis, the $q_{n,k}$ satisfy

$$q_{n,k} \sim c_k (2\rho_k)^{-n} \quad \text{with} \quad c_k = \frac{1-\rho_k^k}{\rho_k(2-(k+1)\rho_k^k)}, \tag{7.52}$$

for large n but fixed k.

The denominator of the fraction in (7.51) behaves near $z = 1/2$ like a "perturbation" of $1 - 2z$ so that one expects ρ_k to be approximated by $\frac{1}{2}$ as $k \to \infty$. An elementary argument shows that

$$\rho_k = \frac{1}{2}\left(1 + 2^{-k-1} + \mathcal{O}(k2^{-2k})\right). \tag{7.53}$$

Accordingly $c_k = 1 + \mathcal{O}(k2^{-k})$.

By means of contour integration, one justifies the use of (7.53) inside (7.52) for a wide range of values of k and n, which results in the approximate formula:

$$q_{n,k} \approx (1 - 2^{-k-1})^n \approx e^{-n2^{-k-1}}.$$

Let $\hat{q}_{n,k}$ denote the approximation $e^{-n2^{-k-1}}$ to $q_{n,k}$. Following [32], one finds

$$\bar{L}_n \equiv \sum_{k=0}^{\infty}[1 - q_{n,k}] = \sum_{k=0}^{\infty}[1 - \hat{q}_{n,k}] + \mathcal{O}(\frac{1}{\sqrt{n}}) = \sum_{k=0}^{\infty}\left[1 - e^{-n2^{-k-1}}\right] + \mathcal{O}(\frac{1}{\sqrt{n}}).$$

This is a typical instance of dyadic sums studied already in great detail.

Proposition 7.9 *The length of the longest 1-run in a random binary string of length n has expectation*

$$\bar{L}_n = \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2} + \frac{1}{\log 2}\sum_{k\in Z\setminus\{0\}} \Gamma(\frac{2ik\pi}{\log 2}) e^{-2ik\pi \log_2 n} + \mathcal{O}(\frac{1}{\sqrt{n}}).$$

Thus, the longest run is expected to have length around $\log_2 n + 0.33274$.

An entirely similar analysis provides the expected size of the largest summand in a random composition of an integer n. More generally, Gourdon has shown these techniques to apply to estimates of largest components in combinatorial schemas of the type considered in §4.8.

EXERCISE 39. Estimate the variance of L_n.

EXERCISE 40. Discuss the periodicity phenomenon in connection with a limit probability distribution for L_n.

EXERCISE 41. Analyse the size of the largest image in a random surjection of size n.

7.9 Mellin analysis of generating functions

Mellin analysis has been developed so far with the purpose of estimating sums that are explicit expressions of combinatorial counts. In may cases, it can also be used to analyse directly generating functions, in particular in the vicinity of a singularity. We know from the singularity analysis method (Chapter 4) or from the saddle point method (Chapter 6) that such singular expansions have direct implications regarding the asymptotic form of coefficients. This suggests a class of "two-stage methods":

Mellin analysis of GFs. *Analyse a GF in the complex plane, near a singularity, as a Mellin transform of a harmonic sum when applicable. Use singularity analysis or the saddle point method to recover the asymptotic form of coefficients.*

A technical difficulty arises since the singular expansions need to be valid in some domain of the complex plane, not just the real line. In other words, one has to cope with *complex harmonic sums* that are of the the form

$$F(t) = \sum_k \lambda_k f(\mu_k t),$$

but where t may be complex.

In the particular case where the base function is the exponential, $f(t) = e^{-t}$, the problem can be solved simply by *analytic continuation*. Indeed, consider the inverse Mellin integral that represents $F(t)$,

$$F(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} F^*(s) t^{-s}\, dt, \tag{7.54}$$

valid for real t. Assume that the Dirichlet series $\sum \lambda_k \mu_k^{-s}$ of the harmonic sum is of moderate growth. Given the fast decrease of the Gamma function,

$$|\Gamma(c+iy)| = O(y^{c-1/2} e^{-\pi|y|/2}) \qquad (y \to \pm\infty),$$

the integral (7.54) still converges and represents an analytic function of t for any t such that $\Re(t) > 0$: to see it, set $r = re^{i\varphi}$, $s = c + iy$, so that

$$|t^{-s}| = r^{\sigma} e^{\varphi y},$$

and the integrand globally decreases as a negative exponential. This principle applies to the inverse Mellin integral itself and to the remainder integral obtained after the line of integration is shifted. This shows the persistence of asymptotic expansion in the domain of *complex* values of t inside any cone,

$$-\frac{\pi}{2} + \epsilon \le \operatorname{Arg}(t) \le \frac{\pi}{2} - \epsilon, \tag{7.55}$$

defined by an arbitrary fixed $\epsilon > 0$. We first illustrate this technique by the classical estimate of the number of integer partitions which resorts to saddle point analysis.

Two-stage saddle point analysis. The OGF of integer partitions (see Chapter 1) is

$$P(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-1},$$

and it admits the unit circle as a natural boundary. The behaviour near $z = 1$ is obtained simply by setting $z = e^{-t}$. We have

$$F(t) := \log(P(e^{-t})) = \sum_k \log(1 - e^{-kt})^{-1},$$

clearly a harmonic sum that needs to be studied near $t = 0$. The Mellin transform of $F(t)$, defined for $\Re(s) > 1$, is

$$F^*(s) = \zeta(s)\zeta(s+1)\Gamma(s).$$

The resulting singular expansion is

$$F^*(s) \asymp \frac{\pi^2}{6}\frac{1}{s-1} - \left[\frac{1}{2s^2} + \frac{\log\sqrt{2\pi}}{s}\right] - \frac{1}{24(s+1)},$$

in the whole of C, so that

$$F(t) \sim \frac{\pi^2}{6t} + \log\sqrt{\frac{t}{2\pi}} - \frac{1}{24}t \qquad (t \to 0^+).$$

This expansion translates back into an expansion of $P(z)$:

$$P(z) = \frac{e^{-\pi^2/12}}{\sqrt{2\pi(1-z)}} \exp\left(\frac{\pi^2}{6(1-z)}\right)(1 + \mathcal{O}((1-z))). \tag{7.56}$$

By the process described above, this expansion persists for complex t such that $|\text{Arg}(t)| < \frac{\pi}{4}$ (say), which in turn implies its truth in a wedge near 1, like $|\text{Arg}(1-z)| < \frac{\pi}{8}$.

The validity of the asymptotic expansion (7.56) is sufficient for a saddle point analysis of the integral

$$P_n \equiv [z^n]P(z) = \frac{1}{2i\pi}\int_{|z|=r} P(z)\frac{dz}{z^{n+1}}.$$

The simpler case of the function $\exp(z/(1-z))$ is similar. It was already discussed in Chapter 6 and the saddle point satisfies $r = 1 - \mathcal{O}(n^{-1/2})$. Now, since the range of the saddle point tends to 0, the asymptotic expansion (7.56) can be used to estimate the part of the integral that yields the dominant contribution. (The remainder of the contour is exponentially small, as in the case of $\exp(z/(1-z))$.)

The same approach applies almost verbatim to partitions into distinct parts corresponding to the infinite product

$$Q(z) = \prod_{k=1}^{\infty}(1 + z^k),$$

with

$$G(t) := \log Q(e^{-t}) = \sum_{k=1}^{\infty}\log(1 + e^{-kt}).$$

In that case, the Mellin transform is

$$G^*(s) = (1 - 2^{-s})\zeta(s)\zeta(s+1)\Gamma(s),$$

with singular expansion

$$G^*(s) \asymp \frac{\pi^2}{12}\frac{1}{s-1} - \frac{\log 2}{2s} + \frac{1}{24(s+1)} \qquad (s \in \langle -\infty, +\infty \rangle,$$

so that

$$\begin{cases} G(t) &= \dfrac{\pi^2}{12t} - \dfrac{1}{2}\log 2 + \mathcal{O}(t) \quad (t \to 0^+) \\ Q(z) &= \dfrac{e^{-\pi^2/24}}{\sqrt{2}} \exp\left(\dfrac{\pi^2}{12(1-z)}\right)(1+\mathcal{O}(1-z)) \quad (z \to 1^-). \end{cases}$$

Proposition 7.10 *The number P_n of partitions of integer n and the number Q_n of partitions into distinct summands satisfy*

$$P_n \sim \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}\,n}, \qquad Q_n \sim \frac{\exp(\pi\sqrt{n/3})}{4\,3^{1/4}\,n^{3/4}}.$$

The result of Prop. 7.10 is due to Hardy and Ramanujan [1] who obtained much more than simple asymptotic equivalence by means of the transformation theory of elliptic functions and consideration of infinitely many saddle points near the unit circle. The two-stage method with Mellin transforms was first introduced by De Bruijn [8] in a celebrated study of "Mahler's partition problem". This problem consists in finding the number of partitions of an integer as a sum of powers of 2 (with repetitions allowed!), which means the asymptotic analysis of

$$[z^n]\prod_{k=0}^{\infty}\left(1 - z^{2^k}\right)^{-1}.$$

The method was greatly generalized by Meinardus around 1954 and we refer the reader to the account given in Andrews' book [1, Ch. 6] for complete details and the full scope of the method.

Exercise 42. Determine the asymptotic number of partitions of integer n into squares, cubes, *etc*; into distinct squares, cubes, *etc*.

Exercise 43. Analyse the number of partitions of n whose summands are all congruent to a modulo a certain number m. Perform similarly the analysis for partitions into distinct summands.

EXERCISE 44. [De Bruijn [8]] The number of binary partitions,

$$B_n = [z^n] \prod_{k \geq 0} \left(1 - z^{2^k}\right)^{-1}$$

satisfies

$$\log(B_n) = \mathcal{O}((\log n)^2).$$

Find an asymptotic equivalent of $\log B_n$.

EXERCISE 45. Find the order of growth of the recurrent sequence

$$f_1 = 1, \quad f_n = f_{n-1} + f_{\lfloor n/2 \rfloor}.$$

Two-stage singularity analysis. Two-stage singularity analysis is an approach that may be used for the asymptotic analysis of sequences whose GFs involve "arithmetic" coefficients as well as powers of a fixed function.

We consider first the Catalan sum of Subsection 7.8.1 taken here under the form

$$A_n = \sum_k \lambda_k \binom{2n}{n-k}.$$

The height of general Catalan trees corresponds to $\lambda_k = d(k)$. The generating function of A_n is expressible in terms of an OGF of Catalan numbers as follows. Let $y(z)$ be the solution of the equation

$$y(z) = z(1 + y(z))^2 \quad \text{so that} \quad y(z) = \frac{1 - \sqrt{1-4z}}{1 + \sqrt{1-4z}} = \frac{1 - 2z - \sqrt{1-4z}}{2z}.$$

The Lagrange inversion theorem implies

$$[z^n]y(z)^k = \frac{k}{n}[w^{n-k}](1+w)^{2n} = \frac{k}{n}\binom{2n}{n-k}.$$

Thus, the OGF of A_n/n is

$$T(z) \equiv \sum_{n \geq 1} A_n \frac{z^n}{n} = \sum_{k \geq 1} \frac{\lambda_k}{k} y(z)^k.$$

The change of variables $y(z) = e^{-t}$, like in the integer partition example, puts $T(z)$ in the form of a harmonic sum with base function e^{-t} and with amplitude-frequency pair $(\lambda_k/k, k)$:

$$T(z) = \Phi(-\log(y(z))), \quad \text{where} \quad \Phi(t) = \sum_k \frac{\lambda_k}{k} e^{-kt}.$$

We assume from now on that the coefficient sequence λ_k is *"arithmetic"* in the sense that its DGF is meromorphic in the whole of C and is of polynomial growth at $\pm i\infty$ in any finite strip. The quantity t lies in a neighbourhood of 0 when z is near $1/4$, and the condition on the λ_k makes it possible to analyse the behaviour of $\Phi(t)$ as $t \to 0^+$ by Mellin transforms. Thus, the singular behaviour of $T(z)$, at least when z tends to $1/4$ from the left, is known.

The conditions of singularity analysis additionally require the expansion to be valid in an indented crown around 1/4. Now, since y(z) has a singularity of the square-root type,

$$y(z) \sim 1 - 2\sqrt{1-4z} \qquad (z \to \frac{1}{4}),$$

the mappings $z \mapsto y(z)$ and $z \mapsto t = -\log(y(z))$ *fold* angles by a factor of 2 near $z = 1/4$, see Fig. 7.9. Therefore, as $z \to 1/4$ in an indented crown, the corresponding value of $y(z)$ stays within the unit circle and t remains in the half-plane $\Re(t) > 0$ inside a cone of angle strictly less than π. This feature allows for the treatment of harmonic sums of a complex argument to apply. We have:

Proposition 7.11 *Consider the sum $A_n = \sum_k \lambda_k \binom{2n}{k}$ and assume that the sequence λ_k is arithmetic (in the sense defined above) and that $\lambda_k \geq 0$. Let $\Lambda(s) = \sum_k \lambda_k k^{-s}$ be the DGF of the sequence. Then, a full expansion of A_n is obtained by the following process:*

S_1. Locate the set $\mathcal{P}$ of poles of $\Lambda(s+1)\Gamma(s)$ left of the abscissa of convergence of $\Lambda(s+1)$, and form the asymptotic expansion

$$\Phi(t) \sim \sum_{s_0 \in \mathcal{P}} Res\left(\Lambda(s+1)\Gamma(s)t^{-s}\right)_{s=s_0}.$$

S_2. Compose the asymptotic expansion of $\Phi(t)$ with

$$t = -\log(y(z)) = \log\frac{1+\delta}{1-\delta} = 2\delta + 2\frac{\delta^3}{3} + \cdots,$$

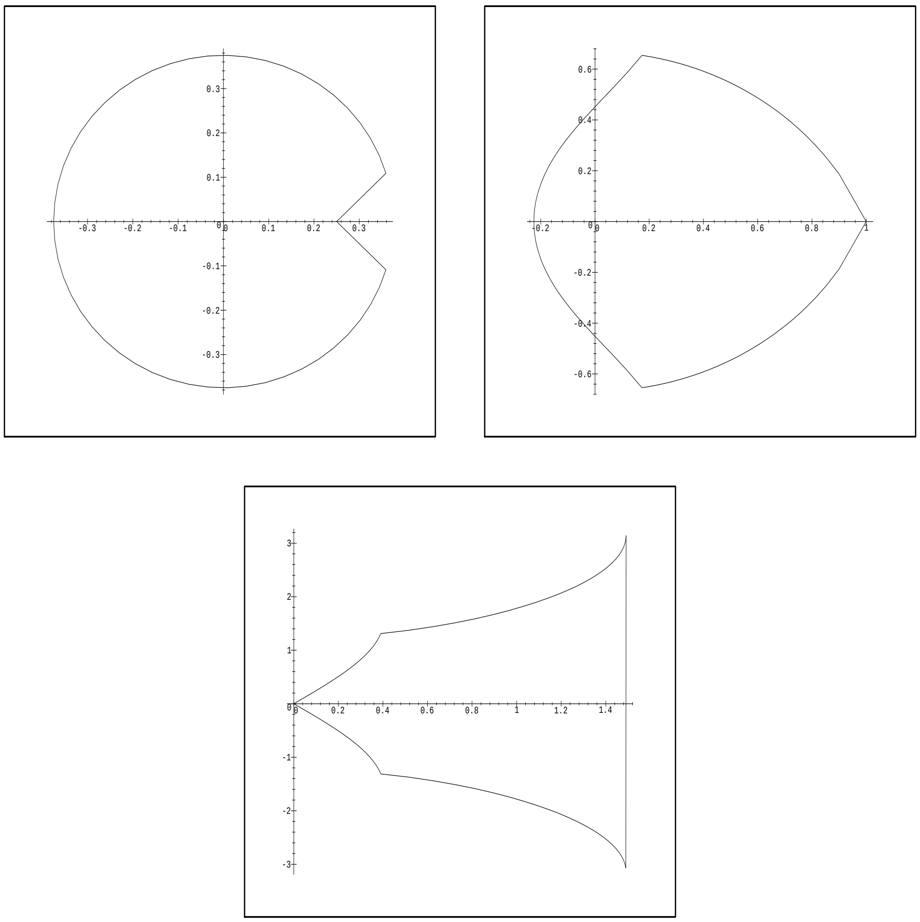

Figure 7.9: The domain of values of z, of $y(z)$ and of $t = -\log(y(z))$ when z varies inside an indented crown of external radius $3/4$ and half-angle $\pi/4$. (The principal value of the logarithm is taken here so that t exhibits a jump represented by the righmost segment of the 3rd diagram.)

where $\delta = \sqrt{1-4z}$*, which gives the singular expansion of* $T(z)$ *near* $z = 1/4$*.*

S_3*. Translate termwise the singular expansion of* $T(z)$ *by the singularity analysis theorem and recover the asymptotic expansion of* A_n/n *from the individual terms.*

Proof. Step S_1 is justified by the basic theorem of Mellin asymptotic summation, Thm. 7.9. The validity of step S_2 results from the general observation that $y(z)$ "folds angles" and from the fact that the Mellin analysis of harmonic sums extends to the positive real half-plane. This justifies the validity of the singular expansion in an indented crown around 1/4 singularity analysis applies. Finally, the nonnegativity assumption implies that 1/4 is the only singularity of $T(z)$ on $|z| \le 1/4$. These conditions validate the application of singularity analysis. □

The process applied to $\lambda(k) = d(k)$ produces the following expansion:

$$\Phi(t) \sim \frac{1}{2}(\log t - \gamma)^2 + \frac{\pi^2}{12} - 2\gamma_1 + \sum_{k=1}^{\infty}(-1)^k \frac{B_k^2}{k^2\, k!} t^k,$$

from which a full expansion of A_n derives.

This process applies to sequences

$$A_n = \sum_k \lambda_k \phi_n^{(k)} \quad \text{where} \quad \phi_n^{(k)} = [w^{n-k}](\phi(w))^n,$$

and $\phi(w)$ is a function analytic at 0 with nonnegative coefficients, satisfying the inversion conditions of Chapter 5 with the additional condition that $\phi(1) = \phi'(1)$. (This means that the function $y(z) = z\phi(y(z))$ attains value 1 at its singularity.) For instance, we may take for $\phi(w)$,

$$(1+w)^2,\ e^w,$$

which corresponds to the sums

$$\sum_k \lambda_k \binom{2n}{n-k},\ \sum_k \lambda_k \frac{n^{n-k}}{(n-k)!}.$$

The two-stage singularity analysis method presented here originates in the work of Flajolet and Prodinger [18] who used it to analyse the "order" function of binary trees that intervenes in register allocation problems [28, 29].

Exercise 46. Analyse asymptotically

$$\sum_k \sqrt{k}\binom{2n}{n-k}, \quad \sum_k \log k \binom{2n}{n-k},$$

$$\sum_k d(k)(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n}).$$

Exercise 47. Analyse asymptotically

$$\sum_k (-1)^k \sqrt{k}\binom{2n}{n-k}, \quad \sum_k (-1)^k \log k \binom{2n}{n-k},$$

$$\sum_k (-1)^k d(k)(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n}).$$

Exercise 48. Describe the shape of a full asymptotic expansion of

$$\sum d(k) H_k \binom{2n}{n-k}.$$

Exercise 49. Show that the coefficients of the full asymptotic expansion of the height of general Catalan trees have an explicit form as a sum of finite multiplicity over terms that involve only factorials, exponentials and rational functions.

Digital trees (tries) and digital search trees. This section is motivated by the model of digital *search* trees that are a hybrid of digital trees or tries and binary search trees. These trees are *per se* a useful data structure; additionally the underlying analytic model is also closely related to the behaviour of data compression algorithms like the celebrated Lempel-Ziv scheme. The analysis is however more difficult than that of tries: in the literature [31, 35], it is usually obtained by a study of difference-differential equations satisfied by EGF's in conjunction with Mellin transform asymptotics on coefficients.

In this section, we develop an analysis along the lines of a paper of Flajolet and Richmond [20]. The approach is based on singularities of OGFs rather than explicit (but sometimes intricate) coefficient forms of EGFs. This route is, in our view, more transparent and at the same time it lends itself to useful generalizations, like to the very general b-digital search trees, a "paged" data structure that motivates the treatment of [20]. Naturally, it would apply to many other problems like the maximum of geometric random variables.

We illustrate here the treatment of the usual trie. Thus, we present yet another derivation of the analysis of the expected number of nodes (splits) in a randomly grown digital trie (Bernoulli process). The starting point is the recurrence valid for all $n \geq 0$,

$$f_n = 1 - \delta_{n,0} - \delta_{n,1} + 2\sum_{k=0}^{n} \frac{1}{2^n} \binom{n}{k} f_k, \tag{7.57}$$

itself a simple rephrasing of Eq. (7.44), so that f_n coincides with i_n in the notations of (7.44). Let $f(z)$ be the OGF of f_n. Multiplication of both sides of (7.57) and formation of OGFs by the classical device of multiplying by z^n and summing yields:

$$f(2z) = \frac{(2z)^2}{1-2z} + \frac{2}{1-z} f(\frac{z}{1-z}). \tag{7.58}$$

In passing, we have used the Euler transformation of series (also known as a binomial transform)

$$a(z) = \frac{1}{1-z} b\left(\frac{z}{1-z}\right) \qquad \Longleftrightarrow \qquad a_n = \sum_{k=0}^{n} \binom{n}{k} b_k, \tag{7.59}$$

where $a_n = [z^n]a(z)$ and $b_n = [z^n]b(z)$. This equation reduces to the standard form,

$$f(z) = \frac{z^2}{1-z} + \frac{2}{1-z/2} f\left(\frac{z}{2-z}\right), \tag{7.60}$$

by the substitution $z \mapsto z/2$. Equation (7.60) is the basis of further developments.

A functional equation of the general form

$$f(z) = \alpha(z) + \beta(z) f(\sigma(z)), \tag{7.61}$$

where $f(z)$ is the unknown function, is solved formally by iteration (compare with (7.48-7.49)):

$$f(z) = \sum_{k=0}^{\infty} \alpha(\sigma^{(k)}(z)) \prod_{j=0}^{k-1} \beta(\sigma^{(j)}(z)), \tag{7.62}$$

where $\sigma^{(j)}(z)$ denotes the jth iterate of function σ. In the case at hand, the formal scheme (7.62) with

$$\alpha(z) = \frac{z^2}{1-z}, \quad \beta(z) = \frac{2}{1-z/2}, \quad \sigma(z) = \frac{z}{2-z}$$

is easily seen to be convergent in a neighbourhood of the origin since $\sigma(z) \sim z/2$ is there contracting. In addition, the iterates of σ —like those of any linear fractional transformation— have an explicit form:

$$\sigma^{(k)}(z) = \frac{2^k}{2^k - (2^k-1)z},$$

a fact that is easily verified by induction. This gives in turn an explicit form for $f(z)$:

$$f(z) = \frac{z^2}{1-z} \sum_{k=0}^{\infty} \frac{2^k}{(2^k - (2^k-1)z)^2}, \qquad (7.63)$$

as the products of (7.62) telescope.

Consideration of the explicit solution (7.63) shows that there is a singularity at $z = 1$, as expected, but also singularities at a set of points

$$\zeta_k = (1 - 2^{-k})^{-1}$$

that have 1 as limit with geometric convergence. Also a simple transformation puts (7.63) in the form

$$f(z) = \frac{z^2}{(1-z)^3} \sum_{k=0}^{\infty} \frac{2^{-k}}{(1+2^{-k}w)^2}, \quad w \equiv w(z) = \frac{z}{1-z}. \qquad (7.64)$$

Now, singularity analysis of $f(z)$ necessitates an asymptotic expansion of $f(z)$ near its singularity $z = 1$. When z tends to 1^-, we have $w \to +\infty$, so that an asymptotic expansion of the function

$$\Phi(w) = \sum_{k=0}^{\infty} 2^{-k} \frac{1}{(1+2^{-k}w)^2}$$

is required. This is by now a routine matter. The Mellin transform is

$$\Phi^*(s) = \frac{1}{1-2^{s-1}} \frac{\pi(1-s)}{\sin \pi s} \qquad \langle 0, 1 \rangle$$

There is a simple pole at $s = 1$ with residue $1/\log 2$ and each point $s = 1+\chi_k$, where $\chi_k = 2ik\pi/\log 2$ is also a simple pole. The next poles are at $s = 2, 3$, etc. This gives

$$\Phi(w) = \frac{1}{\log 2} w^{-1} + R(\log_2 w) w^{-1} + \mathcal{O}(w^{-2}) \qquad (w \to +\infty), \tag{7.65}$$

where the periodic function $R(u)$ is

$$R(u) := \frac{1}{\log 2} \sum_{k \in Z\setminus\{0\}} \frac{\pi \chi_k}{\sin \pi \chi_k} \exp(-2ik\pi u).$$

The expansion (7.65) provides the singular expansion of $f(z)$; to first order, we may replace $w = z/(1-z)$ by $1/(1-z)$ there, so that

$$f(z) = \frac{1}{(1-z)^2}\left[\frac{1}{\log 2} + R(\log_2(1-z)) + \mathcal{O}(1-z)\right],$$

and the asymptotic estimate of f_n follows straightforwardly by singularity analysis. We thus rederive in this way the result already stated in Prop. 7.8.

Digital search trees. The derivation given above for tries is admittedly not the most elementary possible. However, it has the great advantage of translating almost verbatim into an an analysis of digital search trees and their generalizations, see [20].

The *digital search tree process* is defined by a simple modification of the Bernoulli splitting process, where we allow "capture" of one element at each splitting stage. The process takes a group G of individuals and splits them as follows:

- If $\mathrm{card}(G) = 0$ then the process stops.
- Otherwise set $n = \mathrm{card}(G)$ so that $n \geq 1$. One element $\rho \in G$ (the root) is first selected in some arbitrary way. each of the remaining $(n-1)$ elements $g \in G \setminus \{\rho\}$ then flips independently a fair coin. Let G_0 and G_1 be the two subsets of $G \setminus \{\rho\}$ corresponding to the groups of individuals having flipped heads (0) and tails (1). Then the process is recursively applied to the two subsets G_0 and G_1.

A realisation of the process may be described by a tree $\tau(G)$ whose internal binary nodes contain selected elements and with the external nodes corresponding to empty subgroups. Such a tree may be used to support insertions, deletions and queries [31, 35, 40].

The basic recuurence describing the expectation ϕ_n of an additive parameter of the digital tree process then assumes the form (compare with (7.44))

$$\phi_n = e_n + \sum_{k=0}^{n-1} \pi^*_{n,k}(\phi_k + \phi_{n-1-k}), \qquad \pi^*_{n,k} = \frac{1}{2^{n-1}}\binom{n-1}{k}, \tag{7.66}$$

for some known "toll" function e_n; for instance path length is given by $e_n = n-1$. In terms of EGFs, this leads to a difference-differential equation instead of a plain difference equation for standard tries and the analyses are accordingly more difficult [21].

The treatment by means of OGF needs instead only the Euler transformation (7.59) and it leads to equations that are still very similar to those of tries [20], so that they can be derived in a similar manner. The nontrivial paramemeter here is path length (the number of binary nodes being plainly equal to n), and one has:

Proposition 7.12 *The expected path length in a digital search tree of n binary nodes is*

$$n\log_2 n + n\left(\frac{\gamma-1}{\log 2} + \frac{1}{2} - \alpha + \delta(\log_2 n)\right), \qquad \alpha = \sum_{k=1}^{\infty}\frac{1}{2^k-1} \doteq 1.6066995,$$

where $\delta(u)$ is aperiodic function of mean value 0 and amplitude less than 10^{-5}.

7.10 General Mellin summation

This section discusses general conditions under which the Mellin asymptotic summation process applies. It is included here in vue of the great potential of the method but being not essential to the rest of this book, it may be skipped on first reading.

We explore here the fact that Mellin transforms of functions with varying degrees of smoothness tend to be small (fast decrease condition), and that many natural coefficient sequences lead to Dirichlet series that are meromorphic and have moderate growth.

Smallness of Mellin transforms. Smallness of a Mellin transform is directly related to the degree of "smoothness" (differentiability, analyticity) of the original function.

First, let $f(x)$ be locally integrable with fundamental strip $\langle\alpha,\beta\rangle$. Then, uniformly with respect to σ in any closed subinterval of (α,β), one has

$$f^*(\sigma+it)=o(1) \qquad \text{as } t\to\pm\infty.$$

If in addition $f(x)$ is of class $\mathcal{C}^r$ and the fundamental strip of $\Theta^r f$ contains $\langle\alpha,\beta\rangle$, then

$$f^*(\sigma+it)=o(|t|^{-r}) \qquad \text{as } t\to\pm\infty.$$

To see it, put f^* under the form

$$f^*(\sigma+it)=\int_0^\infty f(x)e^{-\sigma x}e^{it\log x}\,dx;$$

this shows that $f^*(s)$ is an integrable function hashed by a complex exponential. By the Riemann-Lebesgue lemma [26, 42], $f^*(s)$ tends to 0 as $t\to\pm\infty$. Smallness is amplified in the case of higher differentiability properties since $\mathcal{M}[\Theta^r f(x);s]=(-1)^r s^r f^*(s)$.

Next, smallness extends beyond the fundamental strip for smooth functions with smooth derivatives. Let $f(x)$ be of class $\mathcal{C}^r$ with fundamental strip $\langle\alpha,\beta\rangle$. Assume that $f(x)$ admits an asymptotic expansion as $x\to 0^+$ (resp. $x\to+\infty$) of the form

$$f(x)=\sum_{(\xi,k)\in A} c_{\xi,k}\,x^{\xi}(\log x)^k+\mathcal{O}(x^{\gamma}) \tag{7.67}$$

where the ξ satisfy $-\alpha\le\xi<\gamma$ (resp. $\gamma<\xi\le-\beta$). Assume also that each derivative $\frac{d^j}{dx^j}f(x)$ for $j=1,\ldots,r$ satisfies an asymptotic expansion obtained by termwise differentiation of (7.67). Then the continuation of $f^*(s)$ satisfies

$$f^*(\sigma+it)=o(|t|^{-r}) \quad \text{as} \quad |t|\to\infty \tag{7.68}$$

uniformly for σ in any closed subinterval of $(-\gamma,\beta)$ (resp. of $(\alpha,-\gamma)$). To prove this, it suffices to consider extension to the left of $f^*(s)$. Choose some positive number $p>\gamma$ and define

$$a(x)=\left(\sum_{(\xi,k)\in A} c_{\xi,k}\,x^{\xi}(\log x)^k\right)\exp(-x^p).$$

The function $g(x)=f(x)-a(x)$ satisfies the assumptions of the previous case so that its transform $g^*(s)$ is $o(|s|^{-r})$ in its fundamental strip $\langle-\gamma,\beta\rangle$. The transform $a^*(s)$ is itself exponentially small given growth properties of the Gamma function and its derivatives. Thus $f^*(s) = a^*(s) + g^*(s)$ satisfies the stated bounds.

Finally, analyticity is the strongest possible form of smoothness for a function $f(x)$; in that case the transform $f^*(s)$ decays exponentially in a quantifiable way. Let $f(x)$ be analytic in S_θ where S_θ is the sector

$$S_\theta = \{z \in C \mid 0 < |t| < +\infty \text{ and } |\arg(z)| \leq \theta\} \quad \text{with} \quad 0 < \theta < \pi.$$

Assume that $f(x) = \mathcal{O}(x^{-\alpha})$ as $x \to 0$ in S_θ, and $f(x) = \mathcal{O}(x^{-\beta})$ as $x \to \infty$ in S_θ. Then,

$$f^*(\sigma + it) = O\left(e^{-\theta|t|}\right)$$

uniformly for σ in every closed subinterval of (α, β). The integral defining Mellin transforms is in this case applied to an analytic function. By Cauchy's theorem, the integration contour may be taken as the half-line of slope θ:

$$f^*(s) = \int_0^{e^{i\theta}\infty} f(t)t^{s-1}\, dt.$$

The change of variable $t = \rho e^{i\theta}$ gives

$$f^*(s) = e^{i\theta s} \int_0^\infty f(\rho e^{i\theta})\, \rho^{s-1}\, d\rho.$$

The result follows as the integral converges. (Smallness even extends outside of the fundamental strip by an argument similar to that encountered above and based on subtracting suitable combinations of exponentials.)

Arithmetic sequences and moderate growth of DGFs. We shall say that a pair of (amplitude-frequency) sequences $\{\lambda_k, \mu_k\}$ is *arithmetic* if: (i) the Dirichlet series $\Lambda(s) = \sum \lambda_k \mu_k^{-s}$ is meromorphic in the whole of C; (ii) the function $\Lambda(s)$ is of moderate growth. A single sequence λ_k is said to be arithmetic if $\{\lambda_k, k\}_{k=1}^\infty$ is arithmetic, that is to say if its standard DGF satisfies the two conditions above. A sequence or a pair of sequences is arithmetic *in the weak sense* if the growth condition is only required to hold on an infinite collection of horizontal lines whose distance to the real line tends to infinity. Many commonly encountered sequences are arithmetic as we now explain.

The sequences $\{1\}$ and $\{(-1)^k\}$ are arithmetic because of the basic growth property of the zeta function. Similarly, $\{k^\alpha\}$ for some fixed α is arithmetic as its DGF is $\zeta(k - \alpha)$. If $\{\lambda_k\}$ is arithmetic, so is $\{\lambda_k(\log k)^r$

for any integer r, since, as is well-known, asymptotic expansions holding in sectors of the complex plane can be differentiated. Arithmetic sequences are also clearly closed under sum and under multiplication by a scalar. For instance,

$$\sqrt{k},\ \frac{\log k}{\sqrt{k}},\ (-1)^k \log^2 k$$

are arithmetic.

An interesting class of arithmetic sequences corresponds to $\lambda_k = f(\frac{1}{k})$ where $f(w)$ admits an asymptotic expansion in ascending power of w at 0. Assume that

$$f(w) \sim \sum_{j=0}^{\infty} c_j w^{\alpha_j} \qquad (w \to 0^+).$$

Then, for any positive M, we may find m_0 such that

$$f(w) = \sum_{j=0}^{m_0} c_j w^{\alpha_j} + \mathcal{O}(w^{M+1}),$$

so that

$$\lambda_k = \sum_{j=0}^{m_0} c_j k^{-\alpha_j} + \mathcal{O}(k^{-M-1}).$$

Thus, summation over k yields, for $\Re(s) > -M$,

$$\Lambda(s) := \sum_{k \geq 1} \frac{\lambda_k}{k^{-s}} = \sum_{j=0}^{m_0} c_j \zeta(s + \alpha_j) + R(s), \tag{7.69}$$

where $R(s)$ is $\mathcal{O}(1)$ when s lies in a right half-plane $\Re(s) \geq \mu > -M$.

Equation (7.69) shows that $\Lambda(s)$ is meromorphic in $\Re(s) > -M$ and hence in the whole of C since M is arbitrary. Given the polynomial growth of the zeta function in any right half-plane, this also implies that λ_k is arithmetic. In addition, Eq. (7.69) shows that the singularities of $\Lambda(s)$ are simple poles at the points $s = 1 - \alpha_j$ with residue c_j. With a slight abuse of notations, we may summarize this property as

$$\Lambda(s) \asymp \sum_{j=0}^{\infty} c_j \zeta(s + \alpha_j).$$

As an application, let us show that the harmonic numbers are *arithmetic*. Take $\lambda_k = H_k - \log k$; we have

$$\lambda_k \sim \gamma + \frac{1}{2\,k} - \frac{1}{12\,k^2} + \frac{1}{120\,k^4} + \cdots.$$

Therefore

$$\begin{aligned}\Lambda(s) &\asymp \gamma\zeta(s) + \frac{1}{2}\zeta(s+1) - \frac{1}{12}\zeta(s+2) + \frac{1}{120}\zeta(s+4) + \cdots \\ &\asymp \frac{\gamma}{s-1} + \frac{1}{2s} - \frac{1}{12(s+1)} + \frac{1}{120(s+3)} + \cdots\end{aligned}$$

Thus, the DGF of the harmonic numbers has the singular expansion

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{H_k}{k^s} &= -\zeta'(s) + \Lambda(s) \\ &\asymp \left[\frac{1}{(s-1)^2} + \frac{\gamma}{s-1}\right] + \frac{1}{2s} - \frac{1}{12(s+1)} + \frac{1}{120(s+3)} + \cdots\end{aligned}$$

Several examples are presented in the problem section and in [16].

Exercise 50. Examine closure properties of arithmetic sequences. Show that $\{d(k)H_k\}$, $\{(H_k)^2\}$, $\binom{\alpha}{k}$ are arithmetic.

Maximum of geometric variables and skip lists. We consider here the problem of obtaining a full asymptotic expansion for the function

$$F(x) = \sum_{k=1}^{\infty} \left[1 - (1-2^{-k})^x\right].$$

For n an integer, $F(n)$ is the expectation of the maximum M_n of n independent random variables with a common distribution that is geometric of parameter $\frac{1}{2}$. Indeed, if G denotes such a geometrically distributed random variable; then

$$\Pr\{M_n \geq k\} = 1 - (\Pr\{G < k\})^n = (1-2^{-k})^{-n}.$$

The function F has an expression similar to the average depth of a node in a random digital tree and whose dominant asymptotics has been already treated by means of the exponential approximation.

The function $F(x)$ is a harmonic sum,

$$F(x) = \sum_{k=1}^{\infty} \left[1 - e^{-\mu_k x}\right] \quad \text{with} \quad \mu_k = -\log(1-2^{-k}),$$

and direct application of Mellin summation yields

$$F^*(s) = -\Gamma(s)\Lambda(s), \qquad \Lambda(s) = \sum_{k=1}^{\infty} \mu_k^{-s} \quad (s \in \langle -1, 0\rangle).$$

The expansion method that we have developed for harmonic numbers adapts and proves that the amplitude-frequency pair $\{1, \mu_k\}$ is arithmetic. We have

$$\begin{aligned} \Lambda(s) &= \sum_{k=1}^{\infty} \left(\log(1-2^{-k})^{-1}\right)^{-s} \\ &= \sum_{k=1}^{\infty} 2^{ks} \left(\frac{1}{2^{-k}} \log(1-2^{-k})^{-1}\right)^{-s}. \end{aligned} \tag{7.70}$$

For any s, the expansion

$$\left(\frac{1}{x}\log(1-x)^{-1}\right)^{-s} = 1 - \frac{1}{2}sx + \frac{1}{24}s(3s-5)x^2 - \frac{1}{48}s(s-2)(s-3)x^3 + \cdots \tag{7.71}$$

is valid as $x \to 0$. The use of (7.71) for the general term of (7.70) yields after summation over k an infinite collection of expansions, each of which like

$$\Lambda(s) = \frac{2^s}{2^s-1} - \frac{s}{2}\frac{2^{s-1}}{1-2^{s-1}} + \frac{s(3s-5)}{24}\frac{2^{s-2}}{1-2^{s-2}} - \frac{s(s-2)(s-3)}{48}\frac{2^{s-3}}{1-2^{s-3}} + \Omega(s), \tag{7.72}$$

provides a singular expansion valid in a larger strip (for instance, here $\Omega(s)$ is analytic in $\langle -\infty, 4\rangle$). At the same time, this process shows that $\Lambda(s)$ is of polynomial growth in any left half-plane.

Concerning the asymptotic expansion of $F(x)$ at ∞, there are singularities at each point $s = m + 2ik\pi/\log 2$ for m a nonnegative integer and $k \in Z$. Each line of poles on $\Re(s) = m$ contributes a periodic fluctuation. Thus:

Proposition 7.13 *The function* $F(x) = \sum_{k=1}^{\infty}[1-(1-2^{-k})^x]$ *admits as* $x \to +\infty$ *a full asymptotic expansion of the form*

$$F(x) \sim \log_2 x + P_0(\log_2 x) + \sum_{j=1}^{\infty} P_j(\log_2 x) x^{-j},$$

where each P_j *is a periodic function of period 1.*

Proof. The proof results from the developments indicated above. The form of each of the P_j is a direct reflection of the expansion (7.72). For instance, with $\chi_k = 2ik\pi/\log 2$:

$$\begin{aligned} P_0(x) &= \frac{\gamma}{\log 2} - \frac{1}{2} + \frac{1}{\log 2}\sum_{k\in Z\setminus\{0\}} \Gamma(\chi_k) e^{-2ik\pi \log_2 x} \\ P_1(x) &= -\frac{1}{2\log 2}\sum_{k\in Z}(\chi_k+1)\Gamma(\chi_k+1)e^{-2ik\pi\log_2 x}, \\ P_2(x) &= \frac{1}{24\log 2}\sum_{k\in Z}(\chi_k+2)(3\chi_k+1)\Gamma(\chi_k+2)e^{-2ik\pi\log_2 x}. \end{aligned}$$

□

Exercise 51. Discuss similarly the case of the maximum of n geometric RVs with mean p.

Exercise 52. Give a full asymptotic expansion for thge length of longest runs.

Skip lists due to Pugh [38] are a randomized data structure that constitutes an attractive alternative to many other tree structures like balanced trees. Assume that a collection $S = (s_1, \ldots, s_n)$ of elements in sorted order are to be kept in a structure that supports efficient retrieval. An idea that goes back to the first times of programming consists in building an index S' that contains only a fraction of the elements of S in sorted order together with pointers to the corresponding places where the indexed elements occur in S. For instance, if $|S'| = \frac{1}{2}|S|$, then this process will roughly divide the search cost in S by a factor of 2 since only $\mathcal{O}(1)$ operations need to be performed after the proper location in the index has been detected. A natural extension of this idea is then to build a second level index S'', and repeat the process. (Such ideas regarding indexed sequential files are also at the origin of balanced tree structures.)

Pugh's beautifully simple idea presented in [38] consists in building S' from S, S'' from S' etc., by mean of successive random samplings, where each elements is preserved in the next higher level index with probability 1/2. This solution has the great advantage of adapting to dynamically changing collections of data, so that it supports insertions, deletions, as well as queries. As is apparent, the index depth is exactly distributed like the maximum of n geometric random variables, and the analysis given above applies. Several other Mellin-based analyses of the cost of skip lists appear in [30, 36].

Notes

Dirichlet series are fundamental in analytic number theory, especially in connection with the distribution of prime numbers. Riemann is essentially responsible for the deeper aspects of this connection, and some of the techniques of the Mellin-Perron type were already known to him. It is starting from Riemann's works that Hadamard and De la Vallée-Poussin could eventually prove the prime number theorem. It came somewhat as a surprise and it constitutes perhaps yet another illustration, in Wigner's terms, of the "unreasonable effectiveness of mathematics" that methods developed a century and a half earlier for the purpose of quantifying regularities in the distribution of primes would prove so instrumental in analysing one of the most productive paradigms in the design of computer algorithms. In fact, the situation in the analysis of algorithms as discussed here is somewhat simpler since it does not depend upon knowledge of the location of the zeros of the zeta function.

The basic use of Dirichlet series and the Mellin-Perron formula is covered in almost any book on analytic number theory, see for instance Apostol [2] for a gentle introduction. The application to the fractal structure of divide-and-conquer recurrences and the companion periodicity phenomena is due to Flajolet and Golin [14, 15]. A systematic treatment of characteristics of number representation using these methods is developed by members of the "Vienna School" in [17]. The analysis of mergesort and of Delange's "digits theorem" [10] given here is typical.

Mellin transforms are close relatives of the integral transforms of Laplace and Fourier. As such, they play an important rôle in applied mathematics. Good general references that include a treatment of Mellin transforms are the books of Doetsch [11], Titchmarsh [43] and Widder [45]. The book by Wong [46] is in spirit especially close to us as it focuses on asymptotic analysis, in particular as applied to "harmonic integrals", a continuous analogue of our harmonic sums. Mellin himself formalized his transform for the purpose of analyzing both entire functions and special functions of the hypergeometric type. We refer to Lindelöf's notice [33] for a perspective on Mellin's research.

The first important applications of Mellin transforms in discrete mathematics are, to the best of our knowledge, an outcome of the cooperation of De Bruijn and Knuth in the mid 1960's. We have developed here the two historic examples of the height of trees (done jointly with Rice [9]) and of the analysis of digital trees that appeared in [31]. An account of tries in the wider context of random search trees is given by Mahmoud [35]; see also Hofri's book [27] where the connection with communication protocols is developed. The interesting application to longest runs is again due to Knuth [32]. (Warning: in early combinatorial applications, Mellin transforms asymptotics often appeared under the name of "Gamma function method", a term that is now abandonned.)

In the paper [19], we gave in outline a presentation of Mellin transforms asymptotics oriented towards problems of analytic combinatorics and the analysis of algorithms. There, for instance, we proposed the term "harmonic sum" and we assembled some of the basic techniques for dealing with harmonic sums. The work [19] was later expanded substantially and it resulted in the synthesis paper [16] by Flajolet, Gourdon and Dumas, from which we have borrowed heavily throughout the Mellin transform sections of this chapter.

The paradigm of dyadic sums is an important application of the general analysis of harmonic sums. It corresponds to infinite superpositions of a basic "process" scaled according to the powers of 2, either in the frequency or the amplitude domain. Perhaps some 50 papers have appeared over recent years analyzing various discrete probabilistic problems that fit into this category. Fortunately, the number of ways to conduct a Mellin analysis is much smaller. We have illustrated in this chapter the most important ones based on a direct analysis of harmonic sums (the Bernoulli splitting process and tries), a two-stage singularity analysis of generating functions (digital trees), or the complete expansion of associated Dirichlet series (maximum of geometric variables and skip lists).

Problems and Exercises

Dirichlet series are the central object of analytic number theory, especially as regards multiplicative properties related to the prime decomposition of integers.

Exercise 53. The DGF of numbers with prime factors in $\{2,3,5\}$ only is
$$\frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})}.$$
The DGF of numbers with *no* prime factor in $\{2,3,5\}$ is
$$(1-2^{-s})(1-3^{-s})(1-5^{-s})\zeta(s).$$

Exercise 54. The DGF of $\sigma_k(n)$, the sum of kth powers of the divisors of n is
$$\sum_{n\geq 1}\frac{\sigma_k(n)}{n^s}=\zeta(s-k)\zeta(s).$$

Exercise 55. Prove that the DGF of square-free numbers satisfies
$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s}=\frac{\zeta(s)}{\zeta(2s)}.$$
Give a number-theoretic interpretation for the coefficients of $\zeta(2s)/\zeta(s)$.

Exercise 56. [Rota] The coefficient of n^{-s} in the DGF
$$\frac{1}{2-\zeta(s)}$$
counts the number of orderered factorizations of n.

Find the DGF of the number of unordered factorizations of n.

Exercise 57. Use Moebius inversion to relate the two Dirichlet series
$$L(s)=\log\zeta(s) \qquad \text{and} \qquad P(s)=\sum_p \frac{1}{p^s},$$
where the last sum ranges overs all prime numbers p.

EXERCISE 58. The Euler totient function $\phi(n)$ counts the number of integers between 1 and $n-1$ that are relatively prime to n. Show that

$$\phi(p_1^{\alpha_1}\cdots p_r^{\alpha_r}) = p_1^{\alpha_1-1}(p_1-1)\cdots p_r^{\alpha_r-1}(p_r-1), \qquad \sum_{n=1}^{\infty}\frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

EXERCISE 59. Ascribe an arithmetic meaning to the coefficient of n^{-s} in $\zeta'(s)/\zeta(s)$.

Dirichlet series are special cases of Mellin transforms, and this connection explains some of the formal similarities between Mellin transforms and Dirichlet series.

EXERCISE 60. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and set $a_0{=}0$. Define the step function

$$S(x) = \sum_{n\le x} a_n.$$

Then,

$$S^*(s) = \sum_{m=1}^{\infty}\int_m^{m+1}(a_1+\cdots+a_m)x^{s-1}\,ds = -s\left[a_1 1^s + a_2 2^s + a_3 3^s + \cdots\right].$$

EXERCISE 61. Deduce the Mellin-Perron formula from the Mellin inversion formula.

EXERCISE 62. Find the functions of which $\zeta(s)$ and $(1-2^{-s})^{-1}$ are Mellin transforms.

Many Mellin transforms can be found by means of functional properties and/or Hankel contours.

EXERCISE 63. Show that, for suitable real r,

$$\int_0^{\infty}(1+x)^{-r}x^{s-1}\,dx = \frac{\Gamma(s)\Gamma(r-s)}{\Gamma(r)}.$$

[Hint. Use the Eulerian Beta integral [44]]

EXERCISE 64. Determine the transforms of

$$\frac{1}{(1+x)(1+2x)\cdots(1+mx)}, \qquad \frac{1}{(1+x)(1+qx)\cdots(1+q^{m-1}x)}.$$

Ramanujan's duality [3, 25] states, in essence, that the coefficients in the expansion of a function at 0 and $+\infty$ are often the same analytic function taken at the positive and negative integers.

EXERCISE 65. Give sufficient conditions on the complex function $\phi(s)$ ensuring that

$$\frac{1}{2i\pi}\int_{-1/2-i\infty}^{-1/2+i\infty} \phi(-s)x^{-s}\,\frac{\pi\,ds}{\sin \pi s} \sim \sum_{n=1}^{\infty} \phi(n)(-x)^n \quad (x \to 0)..$$

Give other conditions ensuring that

$$\frac{1}{2i\pi}\int_{1/2-i\infty}^{1/2+i\infty} \phi(-s)x^{-s}\,\frac{\pi\,ds}{\sin \pi s} \sim \sum_{n=1}^{\infty} \phi(-n)(-x)^{-n} \quad (x \to 0)..$$

EXERCISE 66. Give conditions for a qualified version of Ramanujan's duality: "if $r(n)$ is the coefficient of z^n in the expansion of $R(z)$ at 0, then $-r(-n)$ is the coefficient in the expansion of $R(z)$ at ∞."

EXERCISE 67. Discuss Ramanujan duality for the coefficients of the following functions

$$\frac{1}{1-z-z^2},\ e^{-z},\ \log(1+z),\ \arctan(z),$$
$$\frac{1}{\sqrt{1+z}},\ \int_0^\infty \frac{e^{-t}}{1+tz}\,dt,$$
$$\frac{\pi z}{e^{2\pi z}-1},\ \sum_{k\geq 0}\left[1-e^{-z/2^k}\right].$$

EXERCISE 68. Determine the asymptotic expansion at $+\infty$ of the analytic continuation of

$$R(z)=\sum_{n=1}^{\infty}\frac{(-z)^n}{\sqrt{n}}, \quad L(z)=\sum_{n=1}^{\infty}\log n(-z)^n.$$

EXERCISE 69. Show that the functions

$$R(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}, \quad L(z) = \sum_{n=1}^{\infty} (\log n)\, z^n$$

satisfy the conditions of singularity analysis. Generalize.

The Mellin asymptotics game in its more advanced versions often requires going back and forth between properties of Dirichlet generating functions and harmonic sums.

EXERCISE 70. Define

$$\Theta(x) = \sum_{n=1}^{\infty} e^{-n^2 x^2}.$$

Analyse asymptotically $\Theta(x)$ when $x \to 0$. Find the singularities of

$$\int_0^{\infty} \Theta^2(x) x^{s-1}\, dx.$$

EXERCISE 71. Consider the series

$$\rho(s) = \sum_{m,n \geq 1} \frac{1}{(m^2 + n^2)^s}.$$

Show that

$$\rho(s) = \sum_{n=2}^{\infty} \frac{r(n)}{n^s},$$

where $r(n)$ is the number of representations of n as a sum of two squares. Show that $\rho(s)$ is meromorphic in the whole of C, determine its singularities, and find its values at the negative integers.

Deduce an explicit expansion as $x \to 0$ of

$$\sum_{m,n} \left(1 + (m^2 + n^2)x^2\right)^2.$$

EXERCISE 72. Define

$$\beta(s) = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{1}{n^s}.$$

By relating $\beta(s)$ to $F(x) = (1 - e^{-x})^{-1/2} - 1$, show that $\beta(s)$ is meromorphic in the whole of C, that it satisfies

$$\beta(s) \sim \frac{1}{\sqrt{\pi}} \frac{1}{(s - \frac{1}{2})} \quad (s \to \frac{1}{2}),$$

and that $\beta(-m - \frac{1}{2})$ is a rational number.

EXERCISE 73. Use the information gathered on $\beta(s)$ to analyze asymptotically, as $x \to 0$,

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{1}{1 + n^2 x^2},$$

assuming that $\beta(s)$ is of at most of polynomial growth in any finite strip of the complex plane.

EXERCISE 74. Prove that $\beta(s)$ is of polynomial growth by using any finite form of the asymptotic expansion of the central binomial coefficients, and by relating it to sums of values of the zeta function.

EXERCISE 75. Find the asymptotic expansions of

$$\sum_{n=1}^{\infty} H_n e^{-n^2 x^2}, \qquad \sum_{n=1}^{\infty} (-1)^n H_n e^{-n^2 x^2}.$$

EXERCISE 76. The asymptotic expansion of

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-nx} \sqrt{1 + x}$$

involves a modified Euler's constant

$$\hat{\gamma} = \int_0^1 (e^{-x} \sqrt{1 + x} - 1) \frac{dx}{x} + \int_1^{\infty} (e^{-x} \sqrt{1 + x}) \frac{dx}{x}.$$

EXERCISE 77. The Mellin transform of $F(x) = \log(1 - e^{-x})^{-1}$ is $F^*(s) = \Gamma(s)\zeta(s+1)$. From the expansion of $F(x)$ at $x = 0$ which relates to the expansion of $F^*(s)$ at $s = 0$, one finds

$$\zeta(s) = \frac{1}{s-1} + c_1 + \mathcal{O}(s-1) \qquad (s \to 1)$$

with $c_1 = -\gamma$ the opposite of Eulers' constant.

EXERCISE 78. The singular expansion

$$\sum_n (1+n^2)^{-1/2} n^{-s} = \frac{1}{s} + \hat{\gamma} + \mathcal{O}(s) \qquad (s \to 0)$$

involves

$$\hat{\gamma} = -\gamma + \frac{1}{\sqrt{2}} + \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} \binom{2k}{k} (\zeta(2k+1) - 1).$$

EXERCISE 79. Find the singularities of the transform of

$$f(x) = e^{-x}\sqrt{1+x}.$$

Find the singularities of

$$\omega(s) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^2}} \frac{1}{n^s}.$$

Determine the asymptotic expansion as $x \to \infty$ of

$$F(x) = \sum_{n=1}^{\infty} e^{-n^2x^2} \sqrt{\frac{1+n^2x^2}{1+n^2}}.$$

(First assume that $F^*(s)$ is exponentially small at $\pm i\infty$.)

Harmonic integrals are the continuous analogue of harmonic sums. Their treatment along lines parallel to ours is detailed in Wong's book [46].

EXERCISE 80. Give conditions under which the Mellin transform of a Laplace transform factorizes:

$$\mathcal{M}\left[\int_0^\infty e^{-xt}\phi(t)\,dt; s\right] = \Gamma(s)\phi^*(1-s).$$

Treat similarly the case of Stieltjes transforms,

$$\int_0^\infty \phi(t)\,\frac{dt}{1+tx}.$$

EXERCISE 81. Discuss sufficient conditions for the factorization of transforms of harmonic integrals,

$$\mathcal{M}\left[\int_0^\infty a(t)b(xt)\,dt; s\right] = \int_0^\infty a(t)t^{-s}\,dt \cdot \int_0^\infty b(t)t^{s-1}\,dt.$$

EXERCISE 82. Find the asymptotic expansions at 0 and infinity of

$$\int_0^\infty e^{-t}\,\frac{dt}{1+tx}, \quad \int_0^\infty \frac{te^{-t}}{1-e^{-t}}\,\frac{dt}{1+tx}.$$

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