The Number 2 Does Not Exist
And other p-removed primes

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Abstract

It is a common misconception that the number 2 exists. This startling belief has permeated through society since numbers were first discovered, and is still held by mathematical amateurs and professionals alike. We correct this enormous error and, along the way, give the *correct* definition of prime numbers, the *correct* statement of the prime number theorem, and *disprove* a theorem of Euclid.

1 Introduction

Imagine that the number 2 does not exist. That is, the natural numbers now look like this:

$$\{1, \quad 3, 4, \dots\}.$$

Consider the natural number 4, which surely still exists. Before, we could factor 4 as $4 = 2 \cdot 2$ and trivially as $4 = 4 \cdot 1$. Now, since 2 does not exist, we are left with only $4 = 4 \cdot 1$. Under any reasonable definition, 4 is now prime. The number 4 is not the only new prime. It is joined by 6, 8, 10, and many others. We obtain a collection of *new* primes, which we call *2-removed primes*. The first few are listed in Table 1.

This *2-removed* number system has various interesting properties. None, however, are particular to the number 2, and happen just as easily if we imagine that *any particular* prime does not exist. We call the resulting number systems the *p-removed* number systems. The remainder of this paper is dedicated to exploring these properties, in particular the p-removed primes that arise. First, we shall give a formal definition and a more convenient characterization.

Definition 1 (p-removed primes)**.** Let p be a prime number. A natural number $m > p$ is a *p-removed prime* if and only if every nontrivial positive integer factorization of m includes a p. That is, for every sequence of positive integers $\{d_k\}_{k=1}^{n}$ such that $m = d_1 d_2 \cdots d_n$ and $1 < d_k < m$, at least one d_k must equal p. A natural number $m > p$ is a *p-removed composite* provided that it is not a p-removed prime.

Note that p^2 and p^3 are always p-removed primes. Table 1 suggests that all powers of 2 are p-removed primes, but this is not the case, since $16 = 4 \cdot 4$ and $32 = 8 \cdot 4$ are both 2-removed composites. This generalizes into the following useful lemma.

Table 1: First few 2-removed primes

n	prime?	2-removed prime?
1		
3	✓	✓
4		✓
5	✓	✓
6		✓
7	✓	✓
8		✓
9		
10		✓
11	✓	✓
12		
13	✓	✓
14		✓

Lemma 1. *If p is a prime number, then p^k is a p-removed composite for every integer $k \geq 4$.*

Proof. Every integer $k \geq 4$ can be expressed as $k = 2n + 3m$ for nonnegative integers n and m. Thus, $p^k = p^{2n+3m} = (p^2)^n (p^3)^m$. □

Theorem 1. *A positive integer $k > p$ is a p-removed prime if and only if one of the following conditions hold:*

1. *k is prime;*
2. *$k = pq$, where q is a prime; or*
3. *$k = p^3$.*

(Note that this theorem lets us classify the “regular primes” as the 1-removed primes.)

Proof. Let k be a positive integer greater than p. If k is prime, then no nontrivial factorization exists. If $k = pq$ for some prime q, then its only nontrivial factorization includes a p. If $k = p^3$, then the only nontrivial factorizations are $k = (p^2)p$ and $k = p \cdot p \cdot p$, all of which require p. Thus k is a p-removed prime if any of the three conditions hold.

Inversely, suppose that none of the three conditions hold for some positive integer m. By (1), m has at least two prime factors. If m has *exactly* two prime factors, then m is factored nontrivially without p by (2). If m has at least three prime factors, write

$$m = p_1 p_2 p_3 \cdots p_n$$

for some sequence of primes $\{p_k\}$. If exactly one of these primes are p, say $p_1 = p$, then

$$m = (pp_2)p_3 \cdots p_n$$

is a nontrivial factorization without p. Likewise, if exactly two factors are p, say $p_1 = p_2 = p$, then

$$m = p^2 p_3 \cdots p_n$$

is a suitable nontrivial factorization. If three or more factors are p, then

$$m = p^x y$$

for some integers $x \geq 3$ and $y \geq 1$, where p is not a factor of y. If $y = 1$, condition (3) implies that $x > 3$, and Lemma 1 shows that m is p-removed composite. If $y > 1$, then this is a nontrivial factorization without p. In all cases, m is not a p-removed prime, proving the converse. □

Theorem 1 gives us a more convenient test for p-removed primality. If we were to use Definition 1 as a criterion, we would need to generate every possible nontrivial factorization of a number. Under Theorem 1 it suffices to compute just the prime factorization, for which many efficient computer implementations exist.

2 Frequency and density of p-removed primes

Table 1 suggests that 2-removed primes occur at a much higher rate than the old primes. As we progress through the natural numbers, does this rate slow down and begin to match the rate of the old primes, or do the 2-removed primes keep up their pace? Table 1 is insufficient to answer this question, but the pattern established in Theorem 1 gives us the tools we need for asymptotic and density analysis.

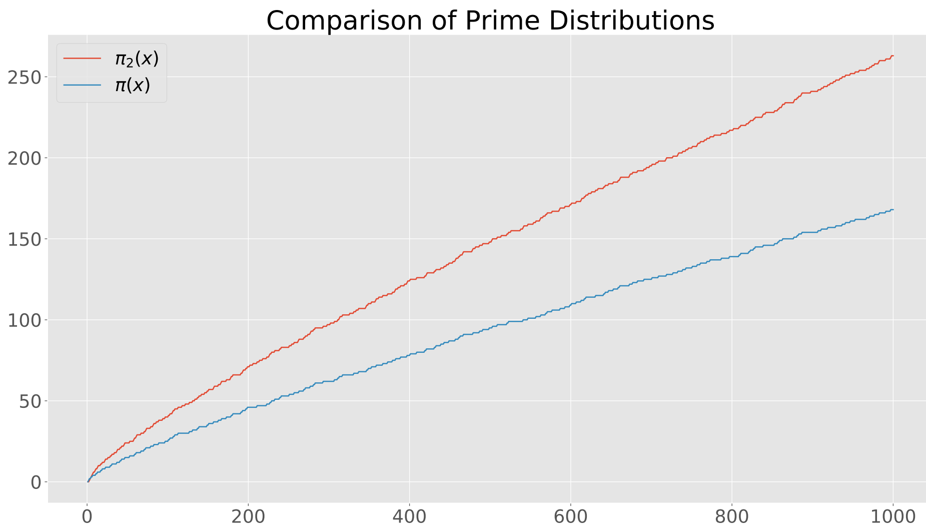

Figure 1: Comparison of prime distributions.

Definition 2. The *prime counting function* $\pi(x)$ gives the number of primes less than or equal to x. The *p-removed prime counting function* $\pi_p(x)$ gives the number of p-removed primes less than or equal to x.

For example,

$$\pi_2(1) = 0$$
$$\pi_2(3) = 1$$
$$\pi_2(14) = 10.$$

Definition 3. Let A be a subset of $\mathbb{N}$, and set $p_n = |A \cap [1, n]|/n$. The *natural density of* A is $\lim p_n$ if this limit exists.

For example, any finite set has natural density zero. The set of all even numbers has natural density $1/2$. The prime number theorem shows that the primes themselves have natural density zero.

Theorem 1 lets us implement π_p efficiently on a computer. In Figure 1, we have plotted both $\pi(x)$ and $\pi_2(x)$ for comparison. This figure provides evidence that 2-removed primes occur more frequently than regular primes. We can obtain more analytic results by using Theorem 1 to relate $\pi_p(x)$ to $\pi(x)$.

Theorem 2. *For any real x, the p-removed prime counting function satisfies*

$$\pi_p(x) = (\pi(x) - \pi(p))[x > p] + \pi\left(\frac{x}{p}\right) + [x \geq p^3]. \tag{1}$$

The brackets are Iverson brackets: for any statement P, $[P] = 1$ if P is true, and $[P] = 0$ if P is false.

Proof. The three terms in the sum correspond directly to the three conditions of Theorem 1. For instance, the second term comes from the number of prime multiples of p in $[1, x]$. The cardinality of this set is

$$|\{pq : pq \leq x,\ q \text{ prime}\}| = |\{q : q \leq x/p,\ q \text{ prime}\}| = \pi(x/p).$$

The remaining terms are verified similarly. $\square$

If we divide both sides of (1) by $\pi(x)$, we obtain

$$\frac{\pi_p(x)}{\pi(x)} = \left(1 - \frac{\pi(p)}{\pi(x)}\right)[x > p] + \frac{\pi(x/p)}{\pi(x)} + \frac{[x \geq p^3]}{\pi(x)}. \tag{2}$$

As $x \to \infty$, the first term tends to 1 and the third term tends to 0. The prime number theorem states that $\lim_{x\to\infty} \pi(x)/(x/\ln x) = 1$, which implies

$$\lim_{x\to\infty} \frac{\pi(x/p)}{\pi(x)} = \frac{1}{p}.$$

Applying this to (2) yields

$$\lim_{x\to\infty} \frac{\pi_p(x)}{\pi(x)} = 1 + \frac{1}{p}.$$

These remarks justify the following theorem.

Theorem 3. *The p-removed primes add roughly a factor of $1/p$ more primes. That is,*

$$\pi_p(x) \sim \left(1 + \frac{1}{p}\right)\pi(x).$$

The error of this approximation is $\pi(x/p) - \pi(x)/p + O(1)$.

To test this approximation, note that $\pi(1000) = 168$, so $(1 + .5)\pi(1000) = 252$. This is off by about ten, since $\pi_2(1000) = 263$. There are sharper estimates for $\pi(x)$ that we could use to improve our $\pi_p(x)$ approximation. See [1] for a brief overview.

To derive the error, note that Theorem 2 implies

$$\begin{aligned}\pi_p(x) - \pi(x) &= ([x > p] - 1)\pi(x) - \pi(p)[x > p] + \pi(x/p) + [x \geq p^3] \\ &= \pi(x/p) - [x \leq p]\pi(x) - \pi(p)[x > p] + [x \geq p^3].\end{aligned}$$

Every term except $\pi(x/p)$ is constant for large x, and therefore $O(1)$. Thus

$$\pi_p(x) - \pi(x) = \pi(x/p) + O(1).$$

Subtracting $\pi(x)/p$ yields the result.

Theorem 4. *The p-removed primes have natural density zero.*

Proof. By Theorem 3,

$$\begin{aligned}\frac{\pi_p(n)}{n} &= \frac{\pi(n) + \pi(n/p)}{n} + O(n^{-1}) \\ &\leq \frac{2\pi(n)}{n} + O(n^{-1}).\end{aligned}$$

In the limit, the first term vanishes by the prime number theorem, and the second term obviously vanishes. □

3 Anti-properties of p-removed primes

While the p-removed primes are in the same *spirit* as the primes, they do not enjoy many of the primes' familiar properties. By removing a single "building block" we have irrevocably altered the structure of the primes. As examples, below are two well-known elementary properties of primes.

Theorem (Euclid's Lemma)**.** *If a prime p divides ab, then p divides one of a or b.*

Theorem (Unique Factorization)**.** *Every natural number can be uniquely factored into a product of primes (up to the order of the factors).*

There are trivial counterexamples to both theorems in the p-removed primes. For Euclid's lemma, let q be the prime after p. Then p^2 (a p-removed prime) divides $p^2q^2 = (pq)(pq)$, but it does not divide pq. For unique factorization, note that

$$(p^3)^2 = p^6 = (p^2)^3,$$

so p^6 can be factored into a product of p-removed primes in two different ways.

There are likely other properties that fail to hold, but these two are the most common.

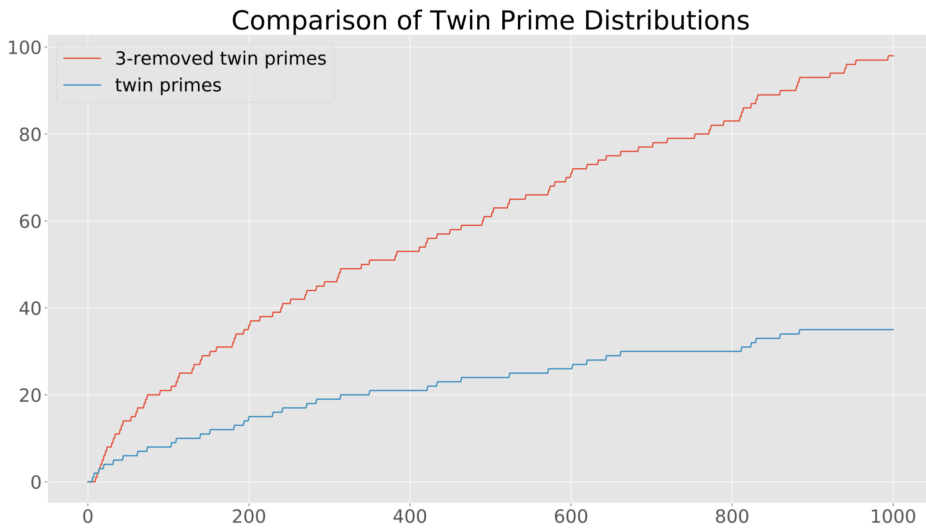

Figure 2: Comparison of twin-prime density.

4 Conclusion

Outside of our basic analysis, there are still questions to be answered about the p-removed primes:

- How badly does unique factorization fail in the p-removed primes? That is, given an integer $k > p$, how many ways can we express k as a product of p-removed primes?
- The definition of a p-removed prime generalizes readily to more than one prime. Imagine, for instance, the $\{p, q\}$-removed primes. What happens to the frequency analysis in this case? Are counterexamples as trivial to construct for theorems related to the regular primes?
- Are there infinitely many p-removed twin primes? For example, there are infinitely many 2-removed twin primes iff there are infinitely many twin primes. The $p = 3$ case is already not so obvious. We have plotted the number of twin-primes and 3-removed twin-primes up to 1000 in Figure 2.

However, now seems like a suitable place to stop.

References

[1] Weisstein, Eric. "Prime Counting Function," *MathWorld*. `http://mathworld.wolfram.com/PrimeCountingFunction.html`.