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ON SOME PROPERTIES OF QUASI-MV ALGEBRAS AND $\sqrt{'}$QUASI-MV ALGEBRAS. PART III

A b s t r a c t. In the present paper, which is a sequel to [14] and [3], we investigate further the structure theory of quasi-MV algebras and $\sqrt{'}$quasi-MV algebras. In particular: we provide an improved version of the subdirect representation theorem for both varieties; we characterise the Ursini ideals of quasi-MV algebras; we establish a restricted version of Jónsson's lemma, again for both varieties; we simplify the proof of standard completeness for the variety of $\sqrt{'}$ quasi-MV algebras; we show that this same variety has the finite embeddability property; finally, we investigate the structure of the lattice of subvarieties of $\sqrt{'}$quasi-MV algebras.

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1. Introduction

Quasi-MV algebras (for short, qMV algebras) were introduced in [12] in connection with quantum computation - namely, in an attempt to provide a convenient abstraction of the algebra over the set of all density operators of the Hilbert space $\mathbb{C}^2$, endowed with a suitable stock of quantum logical gates. Independently of their original quantum computational motivation, qMV algebras present an additional, purely algebraic, motive of interest as generalisations of MV algebras to the semisubtractive (in the sense of [15]) but not point regular case. Later, $\sqrt{\prime}$*quasi-MV algebras* (for short, $\sqrt{\prime}$qMV algebras) were introduced as term expansions of qMV algebras by an operation of square root of the inverse [9]. The above referenced papers contain the basics of the structure theory for these varieties, including appropriate standard completeness theorems w.r.t. the algebras over the complex numbers which constituted the motivational starting point of the whole investigation. In the subsequent papers [14], [3], [10] the algebraic properties of qMV algebras and $\sqrt{\prime}$qMV algebras were investigated in greater detail.

In the present paper, we try to gather some more results of the same kind. In the next section we improve on the results of [12], providing, for any qMV algebra $\mathbf{A}$, a classification of the pairs of congruences $\langle\theta_1, \theta_2\rangle$ for which $\mathbf{A}$ can be subdirectly embedded into the product $\mathbf{A}/\theta_1 \times \mathbf{A}/\theta_2$, with $\mathbf{A}/\theta_1$ an MV algebra and $\mathbf{A}/\theta_2$ a flat qMV algebra, and then we do something in a similar vein for $\sqrt{\prime}$qMV algebras. In § 3 we give a characterisation of Ursini ideals in qMV algebras. In § 4 we show that although the varieties $q\mathbb{MV}$ and $\sqrt{\prime}q\mathbb{MV}$ satisfy no nontrivial congruence identities - or even universal formulas - they nonetheless satisfy Jónsson's Lemma (with just a few easily surveyable exceptions for each variety). In § 5 we replace the standard completeness proof for $\sqrt{\prime}q\mathbb{MV}$ given in [9] by a simpler and more intuitive proof. In § 6 we settle an issue left open in [3] and show that $\sqrt{\prime}q\mathbb{MV}$ has the strong finite model property. Finally, in § 7 we provide a description of the lattice of subvarieties of $\sqrt{\prime}q\mathbb{MV}$.

With an eye to shrinking the paper down to an acceptable length, we assume familiarity with both the content and the notation of the above-referenced papers.

2. $q\mathbb{MV}$ and $\sqrt{'}q\mathbb{MV}$: Subdirect representation

The first, quite cursory, subsection below is devoted to an easy result yielding a complete classification of subdirectly irreducible flat $\sqrt{'}$qMV algebras which is going to be useful later.

2.1 Subdirectly irreducible flat $\sqrt{'}$qMV algebras

Lemma 1. $\mathbf{F}_{100}$, $\mathbf{F}_{020}$, $\mathbf{F}_{004}$ *are the only nontrivial subdirectly irreducible flat* $\sqrt{'}$*qMV algebras.*

Proof. Let $\mathbf{F}$ be a flat $\sqrt{'}$qMV algebra. We distinguish four jointly exhaustive cases:

1. $\mathbf{F}$ has at most 1 fixpoint for $\sqrt{'}$ beside 0, at most 2 fixpoints for $'$ which are not fixpoints for $\sqrt{'}$, and at most 4 other elements. It can be checked by inspection that $\mathbf{F}_{100}$, $\mathbf{F}_{020}$, $\mathbf{F}_{004}$ are the nontrivial subdirectly irreducible algebras with this property.

2. Let a, b be distinct fixpoints for $\sqrt{'}$. The congruences $\mathtt{Cg}^{\mathbf{F}}(0, a)$ and $\mathtt{Cg}^{\mathbf{F}}(0, b)$ correspond to partitions whose blocks are all singletons apart from, respectively, $\{0, a\}$ and $\{0, b\}$. Therefore, they are distinct atoms in the lattice of congruences Con($\mathbf{F}$).

3. Let $a, b, \sqrt{'}a, \sqrt{'}b$ be distinct elements which are fixpoints under $'$. The congruences $\mathtt{Cg}^{\mathbf{F}}(a, \sqrt{'}a)$ and $\mathtt{Cg}^{\mathbf{F}}(b, \sqrt{'}b)$ correspond to partitions whose blocks are all singletons apart from, respectively, $\left\{a, \sqrt{'}a\right\}$ and $\left\{b, \sqrt{'}b\right\}$. Therefore, they are distinct atoms in Con($\mathbf{F}$).

4. Let $a, b, \sqrt{'}a, \sqrt{'}b, a', b', \sqrt{'}a', \sqrt{'}b'$ be pairwise distinct elements which are not fixpoints under either operation. The congruences $\mathtt{Cg}^{\mathbf{F}}(a, a')$ and $\mathtt{Cg}^{\mathbf{F}}(a, b)$ correspond to partitions whose blocks are as follows:
$$\left\{\{a, a'\}, \left\{\sqrt{'}a, \sqrt{'}a'\right\}, \{c\} \text{ for every } c \in F - \left\{a, \sqrt{'}a, a', \sqrt{'}a'\right\}\right\}$$
$$\left\{\begin{array}{c} \{a, b\}, \left\{\sqrt{'}a, \sqrt{'}b\right\}, \{a', b'\}, \left\{\sqrt{'}a', \sqrt{'}b'\right\}, \\ \{c\} \text{ for every } c \in F - \left\{a, b, \sqrt{'}a, \sqrt{'}b, a', b', \sqrt{'}a', \sqrt{'}b'\right\} \end{array}\right\}.$$
It can be checked that they are distinct atoms in Con($\mathbf{F}$).

$\square$

2.2 Strictly meet irreducible congruences

Recall that an element a of a lattice $\mathbf{L}$ is *meet irreducible* if whenever $u \wedge w = a$, then $u = a$ or $w = a$. An element $a \in L$ is *strictly meet irreducible* if whenever $1 \neq a = \bigwedge X$ for some $X \subseteq L$, then $a \in X$. Strictly meet irreducible elements in congruence lattices are precisely those congruences whose quotient algebras are subdirectly irreducible. Since subdirectly irreducible qMV algebras are either MV algebras or flat algebras, strictly meet irreducibles in $\mathrm{Con}(\mathbf{A})$ for any qMV algebra $\mathbf{A}$ fall into two disjoint classes: those above χ and those above τ. Similarly, subdirectly irreducible $\sqrt{'}$qMV algebras fall into two disjoint classes: those above λ and those above μ. For the rest of this section we fix an $\mathbf{A}$, to serve as a generic example both for $q\mathbb{MV}$ and for $\sqrt{'}q\mathbb{MV}$. In particular, we assume that $\mathbf{A}$ in its qMV incarnation is neither an MV algebra nor a flat algebra and that it has at least one cloud with at least 2 irregular members, so that both subdirectly irreducible flat algebras be its quotients. Similarly, in its $\sqrt{'}$qMV incarnation, $\mathbf{A}$ is neither Cartesian nor flat and it has enough elements for all the three subdirectly irreducible flat $\sqrt{'}$qMV-algebras to be its quotients. The following lemmas gather some facts about subdirectly irreducible qMV algebras and $\sqrt{'}$qMV algebras (cp. [12], [3]) and restate them in the language of strictly meet irreducible congruences.

Lemma 2. *Strictly meet irreducible elements in the interval $[\tau, \nabla]$ in* $\mathrm{Con}(\mathbf{A})$ *fall into two classes: (i) the largest strictly meet irreducible element β, such that $\beta = \{\langle a, b\rangle \in A^2 \colon a, b \in \mathcal{R}(\mathbf{A})$ or $a, b \notin \mathcal{R}(\mathbf{A})\}$, and (ii) subcovers of β, corresponding in one-one manner to complement preserving bi-partitions of $A - \mathcal{R}(\mathbf{A})$. All other strictly meet irreducible elements of* $\mathrm{Con}(\mathbf{A})$ *are contained in $[\chi, \nabla]$.*

Proof. The interval $[\chi, \nabla]$ consists of $q\mathbb{MV} - \mathbb{MV}$ congruences, i.e. of congruences whose quotients are MV algebras. The interval $[\tau, \nabla]$ consists of $q\mathbb{MV} - \mathbb{F}q\mathbb{MV}$ congruences, i.e. of congruences whose quotients are flat algebras. In particular, $\mathbf{A}/\beta$ is the two-element simple flat algebra $\mathbf{F}_{10}$, and the characterisation of β in the Lemma follows from this. Now, if α is a strictly meet irreducible congruence in $[\tau, \nabla]$ and $\alpha \neq \beta$, then $\mathbf{A}/\alpha$

is the three-element s.i. flat algebra $\mathbf{F}_{02}$; hence, it partitions A into three classes: $\mathcal{R}(\mathbf{A})$, and two classes of irregular elements corresponding to a/α and a'/α for some $a \notin \mathcal{R}(\mathbf{A})$. Conversely, it is easy to verify that any complement preserving partition of the irregular elements of A into two classes, augmented with $\mathcal{R}(\mathbf{A})$, is a subcover of β in $\mathrm{Con}(\mathbf{A})$. □

Lemma 3. *Let* $\mathbf{A}$ *be a* $\sqrt{\prime}$*qMV algebra. Strictly meet irreducible elements in the interval* $[\mu, \nabla]$ *in* $\mathrm{Con}(\mathbf{A})$ *fall into three classes:*

(i) the largest strictly meet irreducible element β*, such that* $\beta = \{\langle a, b\rangle \in A^2 \colon a, b \in \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})$ *or* $a, b \notin \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})\}$*,*

(ii) subcovers of β*, corresponding in one-one manner to* $\sqrt{\prime}$ *preserving bipartitions of* $A - (\mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A}))$ *such that both partition classes are closed under* $'$*,*

(iii) subcovers of the above subcovers of β*, corresponding in one-one manner to partitions of* $A - (\mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A}))$ *into precisely four classes forming a four-cycle with respect to* $\sqrt{\prime}$*.*

All other strictly meet irreducible elements of $\mathrm{Con}(\mathbf{A})$ *are contained in* $[\lambda, \nabla]$*.*

Proof. Analogous to the proof of Lemma 2. The quotient $\mathbf{A}/\beta$ is isomorphic to $\mathbf{F}_{100}$. For any $\gamma < \beta$ the quotient $\mathbf{A}/\gamma$ is isomorphic to $\mathbf{F}_{020}$. For any δ, γ with $\delta < \gamma < \beta$ the quotient $\mathbf{A}/\delta$ is isomorphic to $\mathbf{F}_{004}$. □

The congruence lattice of a typical $\sqrt{\prime}$qMV algebra is shown in Fig. 1. The detail below β shows three meet irreducible subcovers of β, two of which meet below the third. This illustrates the situation in Lemma 3. The dotted line joining λ with η signals that not all congruences above a $\sqrt{\prime}q\mathbb{MV} - \mathbb{C}$ congruence are themselves Cartesian. Observe the two levels of meet irreducibles below β: subcovers of β and subcovers of these subcovers. This in turn is a reflection of the fact that there are three subdirectly irreducible flat $\sqrt{\prime}$qMV algebras (Lemma 1), as opposed to two subdirectly irreducible flat qMV algebras.

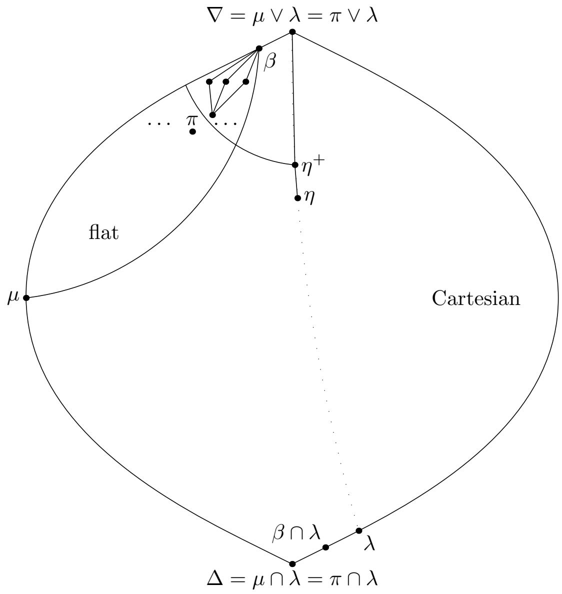

Figure 1: The congruence lattice of a typical $\sqrt{'}$qMV-algebra.

2.3 Subdirect products in $q\mathbb{MV}$

In [12] it was shown that every qMV algebra $\mathbf{A}$ is a subdirect product of an MV algebra and a flat algebra, namely, $\mathbf{A}$ is subdirectly embeddable into $\mathbf{A}/\tau \times \mathbf{A}/\chi$. Since subdirect representations are in general not unique, one can expect that there will be ”nonstandard” representations of qMV algebras as subdirect products with an MV algebra and a flat algebra as factors. This is indeed true, as we presently show, but somewhat surprisingly the MV factor is always going to be $\mathbf{A}/\chi$.

Let $\mathbf{M}$ stand for $\mathbf{A}/\chi$. Thus, $\mathbf{M}$ is the largest MV algebra that is a retract (both a subalgebra and a homomorphic image)[1] of $\mathbf{A}$. With each $m \in M$ we associate its cloud $cl(m)$. Then $\{cl(m) \colon m \in M\}$ consists precisely of congruence classes of χ. Suppose that P is a partition of A satisfying the following conditions:

1. $M \in P$;

[1] More precisely, $\mathbf{R_A}$ (the subalgebra) is isomorphic to $\mathbf{A}/\chi$ (the homomorphic image) via the mapping $f(a) = a/\chi$. In what follows we will disregard this subtlety, taking the label $\mathbf{M}$ as ambiguous between $\mathbf{A}/\chi$ and $\mathbf{R_A}$.

2. P preserves $'$;

3. each partition class contains at most one element of $cl(m)$, for every $m \in M$.

Lemma 4. *If there is a partition P of A satisfying 1-3 above, then the induced equivalence relation π_P is a congruence on $\mathbf{A}$.*

Proof. Immediate from the definition of P and the fact that $a \oplus b \in M$ holds for all $a, b \in A$. □

Before proceeding further, we recall a few facts about the structure of qMV algebras, in particular about the structure of clouds. For each cloud C we have a *twin cloud* $C' = \{c' : c \in C\}$ of cardinality equal to that of C, and $'$ is a bijection between C and C'. Moreover, C and C' are disjoint, except possibly for a single cloud, for which $C' = C$. This unique cloud, if there is any such, is called *median.* If a median cloud C exists, it contains at least one fixpoint for $'$, namely, the unique regular member of C. If any other fixpoints exist, they also belong to C.

We choose arbitrarily some maximal set S of clouds that contains at most one of each pair of twin clouds. In particular, the median cloud is not a member of S, but, by maximality, exactly one member of a pair of non-median twin clouds belongs to S. We well-order S arbitrarily and number its elements by ordinals α, with $0 < \alpha$, thus reserving 0 for the median cloud, if it exists. All non-median clouds not in S can then be dually well-ordered in a natural way by dualising the ordering of S. Informally, we think of the set of all clouds as indexed by "positive" and "negative" ordinals, with the median cloud indexed by 0.

Further, we well-order the median cloud so that its unique regular element is indexed by 0 and followed by all fixpoints, which in turn are followed by all non-fixpoint elements in such a way that if an element c is not a fixpoint, then the element c' is either the immediate successor of c or the immediate predecessor of c. Then, we well-order each non-median cloud in S arbitrarily, except that we require the regular element to be indexed by 0. Finally, each non-median cloud not in S gets well-ordered by mirroring via $'$ the ordering of its twin[2].

[2]The nitty-gritty of this procedure is exactly the same involved in the representation of qMV algebras as numbered MV algebras (see [3]).

Under these orderings each element a of A can be uniquely represented as a pair $(\pm\kappa, \lambda)$, where $\pm\kappa$ is the index of the cloud C that a belongs to (κ if $C \in S$, $-\kappa$ if $C' \in S$) and λ is the index of a within C. We will from now on write $a_{\pm\kappa\lambda}$ for elements of A, and $C_{\pm\kappa}$ for clouds. Thus, whenever this applies, C_0 is the median cloud and a_{00} is the unique regular fixpoint element. Moreover, for any β and any $\alpha > 0$ we have $a'_{\alpha\beta} = a_{-\alpha\beta}$.

Lemma 5. *For any qMV algebra* $\mathbf{A}$ *there exists a partition* P *of* A *with the properties stated just above Lemma 4.*

Proof. Let C_ν be a cloud of maximal cardinality, and let $\mu = |C_\nu|$. Further, let $\lambda = |S| + 1$, so that λ be the cardinality of the set of all "non-negative" clouds. For each $\alpha < \mu$ we will define a set P_α as follows. To begin with, we put $P_0 = M = \{a_{\pm\gamma 0} \colon \gamma < \lambda\}$. For any $\alpha > 0$ we have three cases to consider:

- If $c_{0\alpha}$ exists and is a fixpoint, we put $P_\alpha = \{a_{\pm\gamma\alpha} \colon \gamma < \lambda\}$.
- If $c_{0\alpha}$ exists and is not a fixpoint, we have two subcases:
  - If $c'_{0\alpha} = c_{0\alpha+1}$, then we put $P_\alpha = \{a_{\gamma\alpha} \colon \gamma < \lambda\} \cup \{a_{-\gamma\alpha+1} \colon 0 < \gamma < \lambda\}$.
  - If $c'_{0\alpha} = c_{0\alpha-1}$, then we put $P_\alpha = \{a_{\gamma\alpha} \colon \gamma < \lambda\} \cup \{a_{-\gamma\alpha-1} \colon 0 < \gamma < \lambda\}$.
- If $c_{0\alpha}$ does not exist, we put $P_\alpha = \{a_{\pm\gamma\alpha} \colon \gamma < \lambda\}$.

Each P_α is nonempty since $a_{\nu\alpha}$ exists for all $\alpha < \mu$, but there may be clouds C_γ such that $|C_\gamma| \leq \alpha$ and thus $P_\alpha \cap C_\gamma = \emptyset$.

To see that the sets P_α are pairwise disjoint, we first prove inductively that $P_\alpha \cap P_{\alpha+1} = \emptyset$, for each α. For the base case, we have $P_0 = M$ and if a_{01} exists and is not a fixpoint, we must have $a'_{01} = a_{02}$, because a_{00} is a fixpoint. Thus, $P_1 = \{a_{\gamma 1} \colon \gamma < \lambda\} \cup \{a_{-\gamma 2} \colon 0 < \gamma < \lambda\}$ and this is disjoint from M. In the other two cases the claim clearly holds. For the inductive step, observe that again a dubious case arises only if $c_{0\alpha}$ exists but is not a fixpoint. Suppose for contradiction that some b belongs to both P_α and $P_{\alpha+1}$. We have six cases to consider: (1) $b = a_{\gamma\alpha}$, (2) $b = a_{-\gamma\alpha+1}$, (3) $b = a_{-\gamma\alpha-1}$, (4) $b = a_{\gamma\alpha+1}$, (5) $b = a_{-\gamma\alpha+2}$, (6) $b = a_{-\gamma\alpha}$. It is however clear from the construction that $a_{\gamma\beta}$ can belong only to P_β for any β, so the cases (1) and (4) cannot happen. For (2) suppose $a_{-\gamma\alpha+1} \in P_\alpha \cap P_{\alpha+1}$. Then, by definition of P_α we get that $c'_{0\alpha} = c_{0\alpha+1}$. On the other hand, $a_{-\gamma\alpha+1} \in P_{\alpha+1}$ only if either $c_{0\alpha+1}$ exists and is a fixpoint, or $c_{0\alpha+1}$ does not exist; a contradiction. For (3) suppose $a_{-\gamma\alpha-1} \in P_\alpha \cap P_{\alpha+1}$. Observe first that α cannot be a limit ordinal in this case. Further, it follows immediately from the construction that $a_{\pm\gamma\beta} \notin P_{\beta+2}$, for any β. This also deals with case (5). For (6) suppose $a_{-\gamma\alpha} \in P_\alpha \cap P_{\alpha+1}$. Then, by definition of $P_{\alpha+1}$ we get that $c'_{0\alpha+1} = c_{0\alpha}$. But $a_{-\gamma\alpha} \in P_\alpha$ only if either $c_{0\alpha}$ exists and is a fixpoint, or $c_{0\alpha}$ does not exist; a contradiction again.

¿From the remarks about cases (3) and (5) it now follows that $P_\alpha \cap P_\beta = \emptyset$ for $\alpha \neq \beta$. It is also clear from construction that $P = \bigcup_{\alpha<\mu} P_\alpha$ exhausts A.

Finally, to show that P preserves $'$, observe first that if $c_{0\alpha}$ does not exists, or exists and is a fixpoint, then P_α is closed under $'$. Suppose $c_{0\alpha}$ exists but is not a fixpoint and let $a \neq b \in P_\alpha$. We have two cases, according to whether $c'_{0\alpha} = c_{0\alpha+1}$ or $c'_{0\alpha} = c_{0\alpha-1}$. Let us only deal with the second case. Then, $b = a_{\gamma\alpha}$ or $b = a_{-\gamma\alpha-1}$, for some $0 < \gamma < \mu$. We also have $c'_{0\alpha} \in P_{\alpha-1}$. But since $c_{0\alpha-1}$ exists, is not a fixpoint, and $c'_{0\alpha-1} = c_{0\alpha} = c_{0\alpha-1+1}$, we get that $P_{\alpha-1} = \{a_{\gamma\alpha-1} \colon \gamma < \lambda\} \cup \{a_{-\gamma\alpha} \colon 0 < \gamma < \lambda\}$. Now, b' can be $a_{-\gamma\alpha}$ or $a_{\gamma\alpha-1}$, but in either case it belongs to $P_{\alpha-1}$. □

Theorem 6. *Let π_P be the congruence on $\mathbf{A}$ induced by the partition P of Lemma 5. Then, π_P is a maximal element in $\mathrm{Con}(\mathbf{A})$ with the property $\pi_P \cap \chi = \Delta$. Thus, for every congruence $\phi \in [\tau, \pi_P]$, the algebra $\mathbf{A}$ is a subdirect product of an MV algebra $\mathbf{A}/\chi$ and a flat algebra $\mathbf{A}/\phi$.*

Proof. That $\pi_P \cap \chi = \Delta$ is readily seen from the construction of π_P. Since $\tau \cap \chi = \Delta$ as well, any congruence $\phi \in [\tau, \pi_P]$ yields subdirect representation of $\mathbf{A}$ into $\mathbf{A}/\chi \times \mathbf{A}/\phi$. To see that π_P is maximal with this property, take any $\psi > \pi_P$ and a pair of elements $\langle a, b\rangle \in \psi - \pi_P$. Then $a \in cl(m)$ and $b \in cl(n)$ for some $m, n \in M$. By construction, a/π_P has (precisely) one element in common with $cl(n)$, say c, and since $a \notin b/\pi_P$, we have $c \neq b$. Therefore, $\langle c, b\rangle \in \chi$ and so $\psi \cap \chi > \Delta$ as required. □

Notice that the construction of π_P depends on the initial choice of a suitable partition. This partition is not unique in general and thus π_P is only a maximal, not the largest, $q\mathbb{MV}-\mathbb{F}q\mathbb{MV}$ congruence that intersects to

Δ with χ. The next result provides something of a contrast to the previous one.

Theorem 7. *Let ϕ be a* $q\mathbb{MV} - \mathbb{F}q\mathbb{MV}$ *congruence. If for some* $q\mathbb{MV} - \mathbb{MV}$ *congruence ψ we have $\phi \cap \psi = \Delta$, then $\psi = \chi$.*

Proof. Since $\mathbf{A}/\phi \in \mathbb{F}q\mathbb{MV}$, we have $\phi \geq \tau$ and so $\tau \cap \psi = \Delta$ by assumption. Suppose $\psi > \chi$ and take $\langle a, b\rangle \in \psi - \chi$. Therefore $a \in cl(n)$ and $b \in cl(m)$ for some distinct $n, m \in M$. It follows that $a \oplus 0 \neq b \oplus 0$ and $\langle a \oplus 0, b \oplus 0\rangle \in \tau$. This contradicts the assumption, and therefore $\psi = \chi$ as claimed. $\square$

2.4 Subdirect products in $\sqrt{'}q\mathbb{MV}$

In (almost) perfect analogy with qMV algebras, each $\sqrt{'}$qMV algebra $\mathbf{A}$ is a subdirect product of a Cartesian algebra and a flat algebra. Namely, $\mathbf{A}$ subdirectly embeds into $\mathbf{A}/\lambda \times \mathbf{A}/\mu$. Extending the terminology from $q\mathbb{MV}$, we can say that elements a and b belong to the same *$\sqrt{'}qMV$ cloud* (hereafter simply *cloud* whenever it is clear from context that the setting is $\sqrt{'}q\mathbb{MV}$), if $a \oplus 0 = b \oplus 0$ and $\sqrt{'}a \oplus 0 = \sqrt{'}b \oplus 0$. A cloud is *regular* if it contains a regular element, and *coregular* if it contains a coregular one. A regular (coregular) cloud contains precisely one regular (coregular) element. The cloud containing k is a unique cloud that is both regular and coregular, we will call it *median*. Again, analogously to $q\mathbb{MV}$, the median cloud is the only cloud that can contain fixpoints for $\sqrt{'}$ and/or $'$. Moreover, each fixpoint for $\sqrt{'}$ is a fixpoint for $'$, but not vice versa.

Let $\mathbf{M}$ stand for $\mathbf{A}/\lambda$. Thus, $\mathbf{M}$ is the largest Cartesian algebra that is a retract of $\mathbf{A}$. With each $m \in M$ we associate its cloud $cl(m)$. Then $\{cl(m) \colon m \in M\}$ consists precisely of congruence classes of λ. Suppose that P is a partition of A satisfying the following conditions:

1. M is a single partition class;
2. P preserves $\sqrt{'}$;
3. each partition class contains at most one element of $cl(m)$, for every $m \in M$

The following lemma is immediate.

Lemma 8. *The equivalence relation π_P induced by P is a congruence on $\mathbf{A}$.*

Reasoning as in the previous section, we can now construct a suitable partition P. The construction is hardly more than a two-dimensional version of the construction from the previous subsection; however, describing it in full might confuse rather than clarify matters, so we only offer a sketch, from which the willing reader can easily extract the required details. We begin by numbering the regular clouds just as in the previous section. Then we do the same for coregular clouds. Observe that C_0 is the median cloud in both cases, so the numberings are consistent and can be extended to a single numbering putting $C_{\alpha 0}$ for the regular cloud numbered α and $C_{0\beta}$ for the coregular cloud numbered β. Then we number all other clouds coordinatewise, i.e., by pairs of numbers $\langle \alpha, \beta \rangle$ such that $a \oplus 0 \in C_{\alpha 0}$ and $\sqrt{'}a \oplus 0 \in C_{0\beta}$. Now we need to number elements within clouds. We do it systematically, beginning from some cloud $C_{\alpha\beta}$ of largest cardinality and keeping track of $\sqrt{'}$ and $'$ so that appropriate fixpoints agreed.

Lemma 9. *For any $\sqrt{'}qMV$ algebra $\mathbf{A}$ there exists a partition P of A with the properties stated just above Lemma 8.*

Proof. By the remarks above the lemma. □

Theorem 10. *Let π_P be the congruence on $\mathbf{A}$ described above. Then, π_P is a maximal element in $\mathrm{Con}(\mathbf{A})$ with the property $\pi_P \cap \lambda = \Delta$. Thus, for every congruence $\phi \in [\mu, \pi_P]$, the algebra $\mathbf{A}$ is a subdirect product of a Cartesian algebra $\mathbf{A}/\lambda$ and a flat algebra $\mathbf{A}/\phi$.*

Proof. Exactly parallel to the proof of Lemma 6. □

Here again one should notice that the construction of π depends on the initial choice of a suitable partition and is in general not unique. Therefore π_P is only a maximal, not the largest, flat congruence that intersects to Δ with λ. So, as before, μ is by far not unique. But also as before, λ is.

Theorem 11. *Let ϕ be a $\sqrt{'}q\mathbb{MV}-\mathbb{F}$ congruence. If for some $\sqrt{'}q\mathbb{MV}-\mathbb{C}$ congruence ψ we have $\phi \cap \psi = \Delta$, then $\psi = \lambda$.*

Proof. Since $\mathbf{A}/\phi$ is flat, we have $\phi \geq \mu$ and so $\mu \cap \psi = \Delta$ by assumption. Suppose $\psi > \lambda$ and take $\langle a, b \rangle \in \psi - \lambda$. Therefore $a \in C_{\alpha\beta}$ and

$b \in C_{\gamma\delta}$ for some distinct pairs of (numbers of) elements $\langle\alpha,\beta\rangle, \langle\gamma,\delta\rangle \in \mathcal{R}(\mathbf{A}) \times \mathcal{COR}(\mathbf{A})$. So we have either $\alpha \neq \gamma$ or $\beta \neq \delta$. If the former, then $a \oplus 0 \neq b \oplus 0$ and thus $\langle a \oplus 0, b \oplus 0\rangle \in \mu \cap \psi$. This contradicts the assumption. If the latter, then $\sqrt{\prime}a \oplus 0 \neq \sqrt{\prime}b \oplus 0$ and thus $\left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0 \right\rangle \in \mu \cap \psi$. This contradicts the assumption as well. Thus the claim is proved. $\square$

3. $q\mathbb{MV}$: Ideals and deductive filters of the 0-assertional logics

Recall from [11] that, if $\mathbb{K}$ is a class of similar algebras whose similarity type includes a constant 0, a term $p(\overrightarrow{x}, \overrightarrow{y})$ in the language of $\mathbb{K}$ is a $\mathbb{K}$-*ideal term* in $\overrightarrow{y}$ if $\mathbb{K} \vDash p(\overrightarrow{x}, 0, ..., 0) \approx 0$, and that a nonempty subset J of the universe of $\mathbf{A} \in \mathbb{K}$ is a $\mathbb{K}$-*ideal* of $\mathbf{A}$ (w.r.t. 0) if for any $\mathbb{K}$-ideal term $p(\overrightarrow{x}, \overrightarrow{y})$ we have that $p^{\mathbf{A}}\left(\overrightarrow{a}, \overrightarrow{b}\right)$ whenever $\overrightarrow{a} \in A$, $\overrightarrow{b} \in J$. 0-*ideal determined varieties* (i.e. varieties which are both 0-subtractive and 0-regular) are especially well-behaved since the notion of $\mathbb{K}$-ideal can suitably replace the notion of congruence (as the corresponding lattices are isomorphic). MV algebras, for example, are 0-ideal determined [6].

In [12] it was observed that $q\mathbb{MV}$ is not 0-ideal determined. In the same paper, however, we borrowed from the structure theory of MV algebras two equivalent characterisations of the notion of $\mathbb{MV}$-ideal, hereafter reproduced for the reader's convenience:

Definition 12. *Let* $\mathbf{A}$ *be a quasi-MV algebra and let* $J \subseteq A$. *We say that* J *is an* ideal *of* $\mathbf{A}$ *iff for all* $a, b \in A$ *the following conditions are satisfied:*

I1 $0 \in J$;

I2 $a, b \in J \Rightarrow a \oplus b \in J$;

I3 $a \in J, b \leq a \Rightarrow b \in J$.

Definition 13. *Let* $\mathbf{A}$ *be a quasi-MV algebra and let* $J \subseteq A$. *We say that* J *is a* weak ideal *of* $\mathbf{A}$ *iff for all* $a, b \in A$ *the following conditions are satisfied:*

W1 $0 \in J$;

W2 $a, b \in J \Rightarrow a \oplus b \in J$;

W3 $a \in J, b \in A \Rightarrow a \otimes b \in J$.

In any MV algebra $\mathbf{A}$, a subset $J \subseteq A$ is an ideal iff it is a weak ideal; in an arbitrary qMV algebra, however, the former notion is stronger (all ideals are weak ideals but not conversely). It makes sense to try and investigate the relationship between these concepts and the concept of $q\mathbb{MV}$-ideal; a first result was obtained in [3], where it was shown that ideals do *not* coincide with $q\mathbb{MV}$-ideals. The aim of this section is twofold: on the one hand, proving that $q\mathbb{MV}$-ideals coincide with *weak* ideals, and, on the other hand, giving alternative characterisations of ideals.

We first improve slightly on Lemma 40 of [12]:

Lemma 14. *Condition W3 in Definition 13 can be equivalently replaced by any of the following:*

$$\begin{array}{ll} \textit{W3'}. & a \in J, b \preceq a \Rightarrow b \in J \\ \textit{W3''}. & a \in J, b \leq a \Rightarrow b \oplus 0 \in J \end{array}$$

Proof. W3→W3'. See [12], Lemma 40.

W3'→W3". Suppose that for every $a \in J, b \in A$, we have that $a \in J$ and $b \preceq a$ imply $b \in J$. Let $c \in J$ and $d \leq c$. Since $d \oplus 0 \leq d$, by Lemma 39 in [12] it follows that $d \oplus 0 \preceq c$ and thus $d \oplus 0 \in J$.

W3"→W3. Suppose that for every $a \in J, b \in A$, $a \in J$ and $b \leq a$ imply $b \oplus 0 \in J$. Let $c \in J$ and $d \in A$. By results in [12], $c \otimes d \leq c$, whence $c \otimes d = (c \otimes d) \oplus 0 \in J$. □

Next, we give necessary and sufficient conditions for a weak ideal to be an ideal.

Lemma 15. *If* $\mathbf{A}$ *is a quasi-MV algebra,* J *is an ideal of* $\mathbf{A}$ *iff (i) it is a weak ideal of* $\mathbf{A}$ *and (ii) for any* $a \in A$*,* $a \in J$ *iff* $a \oplus 0 \in J$*.*

Proof. In order to prove that any ideal is a weak ideal, all we have to show is that I1-I3 imply W3. Thus, let J be an ideal of $\mathbf{A}$ and let $a \in J$; since $a \otimes b \leq a$, we are done by I3. As $\langle a, a \oplus 0 \rangle \in \chi$, by I3 in any ideal $a \in J$ iff $a \oplus 0 \in J$. Conversely, suppose that J is a weak ideal of $\mathbf{A}$ and that $a \in J$ iff $a \oplus 0 \in J$. Our conclusion follows from W3" in Lemma 14. $\square$

Finally, we prove the main result of this section. Recall from [4] that a term $t(\overrightarrow{x})$ in the similarity type of $q\mathbb{MV}$ is called *regular* just in case for any $\overrightarrow{a}$ in $\mathbf{A} \in q\mathbb{MV}$ we have that $t^{\mathbf{A}}(\overrightarrow{a}) \in \mathcal{R}(\mathbf{A})$. In other words, regular terms are either constants or contain at least an occurrence of $\oplus$. It was proved in the same paper that:

Lemma 16. *If $t(\overrightarrow{x})$ is a regular $q\mathbb{MV}$ term, then*

$$q\mathbb{MV} \vDash t(\overrightarrow{x}) \approx 0 \text{ iff } \mathbb{MV} \vDash t(\overrightarrow{x}) \approx 0$$

Theorem 17. *Let $\mathbf{A}$ be a qMV algebra, and let $J \subseteq A$. Then J is a weak ideal of $\mathbf{A}$ iff J is a $q\mathbb{MV}$-ideal of $\mathbf{A}$ (w.r.t. 0).*

Proof. Left to right. We first prove that J is closed w.r.t. all *regular* $q\mathbb{MV}$-ideal terms. Observe that $p(\overrightarrow{x}, \overrightarrow{y})$ is a regular $q\mathbb{MV}$-ideal term in $\overrightarrow{y}$ iff it is a regular $\mathbb{MV}$-ideal term in $\overrightarrow{y}$: in fact, in virtue of Lemma 16,

$$q\mathbb{MV} \vDash p(\overrightarrow{x}, 0, ..., 0) \approx 0 \text{ iff } \mathbb{MV} \vDash p(\overrightarrow{x}, 0, ..., 0) \approx 0.$$

Thus, suppose that $p(x_1, ..., x_n, y_1, ..., y_m)$ is a regular $q\mathbb{MV}$-ideal term in $y_1, ..., y_m$, that $a_1, ..., a_n \in A$ and that $b_1, ..., b_m \in J$. By Lemma 14, $b_1 \oplus 0, ..., b_m \oplus 0 \in J \cap \mathcal{R}(\mathbf{A})$ and, since p is regular,

$$p^{\mathbf{A}}(a_1, ..., a_n, b_1, ..., b_m) = p^{\mathbf{A}}(a_1 \oplus 0, ..., a_n \oplus 0, b_1 \oplus 0, ..., b_m \oplus 0).$$

Next, consider the MV algebra $\mathbf{R}_{\mathbf{A}}$. As we observed in the previous discussion, Definition 13 characterises $\mathbb{MV}$-ideals, whence $J \cap \mathcal{R}(\mathbf{A})$ is an $\mathbb{MV}$-ideal of $\mathbf{R}_{\mathbf{A}}$. Since p is an $\mathbb{MV}$-ideal term in $\overrightarrow{y}$,

$$\begin{aligned} & p^{\mathbf{A}}(a_1, ..., a_n, b_1, ..., b_m) \\ = \;& p^{\mathbf{A}}(a_1 \oplus 0, ..., a_n \oplus 0, b_1 \oplus 0, ..., b_m \oplus 0) \in J \cap \mathcal{R}(\mathbf{A}) \subseteq J, \end{aligned}$$

and we get our conclusion. To round off our proof, simply observe that all nonregular $q\mathbb{MV}$ terms have the form $x^{\prime(\ldots)\prime}$ (the variable x followed by zero or more occurrences of $'$) and that none of them is a $q\mathbb{MV}$-ideal term.

Right to left. Obviously $x \oplus y$ is a $q\mathbb{MV}$-ideal term in x, y, and $x \otimes y$ is a $q\mathbb{MV}$-ideal term in y, both w.r.t. 0, and this suffices to establish our claim. $\square$

Some results in [4] imply that J is a weak ideal iff it is a deductive filter on $\mathbf{A}$ of the 0-assertional logic of the quasivariety generated by the standard qMV algebra $\mathbf{S}$ (for short, a $S(\mathbf{Q}(\mathbf{S}),0)$-*filter* on $\mathbf{A}$), a result from which the preceding theorem follows rather easily. However, the proof of that theorem is rather long and convoluted compared to the short and easy proof of Theorem 17.

In the next theorem, we characterise ideals as the deductive filters on qMV algebras of *dual Łukasiewicz logic*, i.e. of the 0-assertional logic $\mathcal{S}(\mathbb{MV},0)$ of $\mathbb{MV}$. This logic can be axiomatised by taking as axioms the negations of the axioms of Łukasiewicz logic, and by taking as sole inference rule *dual modus ponens*:

$$t, s \otimes t' \vdash s.$$

We denote by $\downarrow K$ the set $\{a \in A : a \leq b \text{ for some } b \in K\}$.

Theorem 18. *Let $\mathbf{A}$ be a qMV algebra, and let $J \subseteq A$. The following are equivalent:*

1. *J is an ideal of $\mathbf{A}$;*
2. *$J = \downarrow K$, for some weak ideal K of $\mathbf{A}$;*
3. *J is a $\mathcal{S}(\mathbb{MV},0)$-filter of $\mathbf{A}$.*

Proof. 1.$\leftrightarrow$2. If J is an ideal of $\mathbf{A}$, then $J \cap \mathcal{R}(\mathbf{A})$ is a weak ideal of $\mathbf{A}$: it is closed w.r.t. $\oplus$ and downwards closed w.r.t. $\preceq$. Clearly, $J = \downarrow (J \cap \mathcal{R}(\mathbf{A}))$. Conversely, given a weak ideal K of $\mathbf{A}$, $0 \in \downarrow K$. By isotony of $\oplus$, $\downarrow K$ is also closed w.r.t. $\oplus$ and by transitivity of the preordering relation $\leq$ it is downwards closed w.r.t. it.

1.$\to$3. If t is the dual of any axiom of Łukasiewicz logic, then for any $\overrightarrow{a} \in A^n$, $t^A(\overrightarrow{a}) = 0 \in J$. Now, suppose that $t, s \otimes t' \in J$. Then

$$t \Cup s = t \oplus \left(s \otimes t'\right) \in J,$$

whence $s \in J$ as $s \leq t \Cup s$.

3.$\to$1. The rules (i) $t, s \vdash t \oplus s$, (ii) $s \vdash t \otimes s$ and (iii) $t \dashv\vdash t \oplus 0$ are sound rules of $\mathcal{S}(\mathbb{MV},0)$. Our conclusion follows then from Lemma 15. $\square$

4. $q\mathbb{MV}$ and $\sqrt{'}q\mathbb{MV}$: Jónsson's Lemma

Several restricted versions of Jónsson's Lemma are known for varieties that fail to be congruence distributive ([8], [5]). We will prove yet two more such results for varieties of qMV algebras and $\sqrt{'}$qMV algebras. Namely, Jónsson's Lemma turns out to work fine for any variety $\mathbb{V}$ of qMV algebras, as long as we are interested in subdirectly irreducible members of $\mathbb{V}$ that are MV algebras. Similarly, for any variety $\mathbb{W}$ of $\sqrt{'}$qMV algebras, Jónsson's Lemma works as long as subdirectly irreducible Cartesian algebras are concerned. Since all other subdirectly irreducible algebras in the respective varieties are few and their presence easy to detect, for practical reasons these restricted versions are as good as the full version. Some very slight generalisations of what we have just stated are also possible, but a simple example shows that no significantly better result can be expected. Although our arguments for $q\mathbb{MV}$ and $\sqrt{'}q\mathbb{MV}$ are very similar at a generic level, certain specific differences would make a uniform presentation cumbersome. Thus, we will present the proofs for $q\mathbb{MV}$ first, and in a possibly detailed way. Next we will deal with $\sqrt{'}q\mathbb{MV}$, and then we will skip such generalities as are common to both cases, focusing on detailed proofs of the little extras we need to make the arguments work.

4.1 Two observations for qMV algebras

Let $\mathbb{K}$ be a class of qMV algebras and let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathbf{V}(\mathbb{K})$. Then $\mathbf{A} \in \mathbf{HSP}(\mathbb{K})$ so $\mathbf{A} = \mathbf{B}/\phi$ for some strictly meet irreducible congruence ϕ on an algebra $\mathbf{B}$ such that $\mathbf{B} \leq \mathbf{C} = \prod_{i\in I} \mathbf{C}_i$, for some algebras $\mathbf{C}_i \in \mathbb{K}$.

Lemma 19. *If $\mathbf{A}$ is an MV algebra, then the algebra $\mathbf{B}$ above can be taken to be an MV algebra as well.*

Proof. Suppose $\mathbf{A} = \mathbf{D}/\phi$ for some congruence ϕ on a qMV algebra $\mathbf{D}$. Since χ is the smallest $q\mathbb{MV} - \mathbb{MV}$ congruence on $\mathbf{D}$, we have $\mathbf{A} = \mathbf{D}/\phi = (\mathbf{D}/\chi)/(\phi/\chi)$. Now, since $\mathbf{D}/\chi$ is isomorphic to $\mathbf{R_D}$, it is isomorphic to a subalgebra of $\mathbf{D}$. Thus, if $\mathbf{D} \leq \mathbf{C}$, for any qMV algebra $\mathbf{C}$, then $\mathbf{D}/\chi \leq \mathbf{C}$ as well. Put $\mathbf{B} = \mathbf{D}/\chi$ to obtain the desired MV algebra. □

Lemma 20. *If* $\mathbf{A}$ *is the algebra* $\mathbf{F}_{10}$*, then the congruence* $\phi = \beta$*; hence it is meet prime.*

Proof. Take congruences $\theta_1, \theta_2 \in \mathrm{Con}(\mathbf{B})$ such that $\theta_1 \cap \theta_2 \leq \beta$. Since for $i = 1, 2$, the algebra $\mathbf{B}/\theta_i$ is a subdirect product of an MV algebra and a flat algebra, we have $\theta_i = \tau_i \cap \chi_i$ for some $q\mathbb{MV} - \mathbb{MV}$ congruence χ_i and $q\mathbb{MV} - \mathbb{F}q\mathbb{MV}$ congruence τ_i. Then, $\tau_1 \cap \chi_1 \cap \tau_2 \cap \chi_2 \leq \phi$ and so it suffices to show that $\tau_1 \leq \beta$ or $\tau_2 \leq \beta$. Since $\tau_1, \tau_2 \in [\tau, \nabla]$ and β is a unique coatom in this interval (cf. Lemma 2) it suffices to show that at least one of τ_1, τ_2 is strictly below ∇. This, however, is obvious as $\mathbf{B} \notin \mathbb{MV}$. □

4.2 Jónsson's Lemma for $q\mathbb{MV}$ with one exception

Now we are ready for the proof of our version of Jónsson's Lemma. We need some setup first. Let $\mathbf{A}$ be a subdirectly irreducible qMV algebra different from $\mathbf{F}_{02}$. Let the algebras $\mathbf{B}$, $\{\mathbf{C}_i\}_{i \in I}$, $\mathbf{C}$ and the congruence ϕ be as in the previous subsection. For $J \subseteq I$, define $\theta_J = \big\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \supseteq J\big\}$. It is easy to see that θ_J is a congruence on $\mathbf{C}$. Define further a family $G = \{J \subseteq I \colon \theta_J|_B \subseteq \phi\}$, where $\theta_J|_B$ stands for $\theta_J \cap B^2$, as usual.

Lemma 21. *There is an ultrafilter* U *on* I *such that* $U \subseteq G$*.*

Proof. It suffices to show that $I \in G$, and G is upward closed and maximal, i.e., $J \cup K \in G$ implies $K \in G$ or $J \in G$, for any $J, K \subseteq I$. It is not difficult to show that $I \in G$ and G is upward closed. Maximality requires some work. Suppose $J \cup K \in G$. This means $\theta_{J \cup K}|_B \subseteq \phi$. Since $\big\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \in J \cup K\big\} = \big\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \in J\big\} \cap \big\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \in K\big\}$ we get $\theta_{J \cup K} = \theta_J \cap \theta_K$. Therefore $\theta_{J \cup K}|_B = \theta_J|_B \cap \theta_K|_B$. It follows that $\theta_J|_B \cap \theta_K|_B \subseteq \phi$. Now, since ϕ is strictly meet irreducible, we have either $\phi \geq \chi$ or $\phi \geq \tau$. Moreover, if $\phi \geq \tau$, then by our special assumption $\mathbf{A}$ must be $\mathbf{F}_{10}$, and so $\phi = \beta$. We will consider the two cases in turn. If $\phi \geq \chi$, then by Lemma 19 we can assume $\mathbf{B}$ is congruence distributive. Then, since ϕ is meet irreducible, it is also meet prime and therefore $\theta_J|_B \subseteq \phi$ or $\theta_K|_B \subseteq \phi$. If $\phi \geq \tau$, then Lemma 20 applies and thus also $\theta_J|_B \subseteq \phi$ or $\theta_K|_B \subseteq \phi$. It follows that $J \in G$ or $K \in G$ as desired. □

Theorem 22. *Let $\mathbb{K}$ be a class of qMV algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible algebra different from $\mathbf{F}_{02}$, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

Proof. Let U be an ultrafilter on I with $U \subseteq G$. Consider the ultraproduct $\prod_{i\in I} \mathbf{C}_i/U$. Since $\prod_{i\in I} \mathbf{C}_i/U$ is a quotient of $\mathbf{C}$ by the congruence $\nu = \big\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \in U\big\}$, we have $\nu = \bigvee\{\theta_J \colon J \in G\}$ and therefore $\nu|_B \leq \phi$. Let $\mathbf{D}$ be the homomorphic image of $\mathbf{B}$ by the homomorphism corresponding to ν. Then $\mathbf{D} \leq \mathbf{C}/\nu$. Since $\nu|_B \leq \phi$, by homomorphism theorems we get that the quotient $\mathbf{D}/\phi$ is well defined and isomorphic to $\mathbf{A}$. Thus $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$ as claimed. $\square$

Corollary 23. *Let $\mathbb{K}$ be a class of qMV algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible MV algebra, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

4.3 Jónsson's Lemma for $q\mathbb{MV}$: expanded generating class

We can avoid the exception altogether, if we cheat just a little with the choice of the generating class of algebras. For a class $\mathbb{K}$ of qMV algebras, define $\mathbb{K}^2$ to be $\mathbb{K} \cup \{\mathbf{K_1} \times \mathbf{K}_2 \colon \mathbf{K}_1, \mathbf{K}_2 \in \mathbb{K}\}$.

Lemma 24. *If $\mathbf{F}_{02} \in \mathbf{V}(\mathbb{K})$, then $\mathbf{F}_{02} \in \mathbf{HS}(\mathbb{K}^2)$.*

Proof. If $\mathbf{F}_{02} \in \mathbf{V}(\mathbb{K})$, then $\mathbb{K}$ must contain an algebra $\mathbf{K}$ with at least one irregular element. If for some such irregular element a, $a' \neq a$, then the subalgebra $\mathbf{S}$ of $\mathbf{K}$ generated by a contains precisely two irregular elements. Thus, $\mathbf{S}/\tau = \mathbf{F}_{02}$. Now we suppose that any irregular element of $\mathbf{K}$ is a fixpoint and we distinguish two cases. Pick some $a = a' \neq a \oplus 0$; if $\mathbf{K}$ contains a regular element $b \neq a \oplus 0$, then $\mathbf{K}$ is not flat and therefore, in $\mathbf{K}^2$, we have $(1, a) \neq (0, a) = (1, a)'$, so the situation reduces to the previous case. If $\mathbf{K}$ is flat and $\mathbb{K} = \{\mathbf{K}\}$, then $\mathbf{V}(\mathbb{K}) = \mathbf{V}(\mathbf{F}_{10})$ and therefore $\mathbf{F}_{02} \notin \mathbf{V}(\mathbb{K})$. Thus, $\mathbb{K}$ must also contain an algebra $\mathbf{L}$ nonisomorphic to $\mathbf{K}$ and then either $\mathbf{L}$ has an irregular element a with $a \neq a'$, or $\mathbf{L}$ has $0 \neq 1$. If the former, $\mathbf{F}_{02} \in \mathbf{HS}(\mathbf{L})$, if the latter $\mathbf{F}_{02} \in \mathbf{HS}(\mathbf{K} \times \mathbf{L})$. $\square$

Theorem 25. *Let $\mathbb{K}$ be a class of qMV algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible algebra, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$.*

Proof. If $\mathbf{A}$ is different from $\mathbf{F}_{02}$, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$ by Theorem 22, so $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$. If $\mathbf{A}$ is $\mathbf{F}_{02}$, then $\mathbf{A} \in \mathbf{HS}(\mathbb{K}^2)$ by Lemma 24 and therefore $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$ as well. $\square$

4.4 A property of $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruences

Let now $\mathbb{K}$ be a class of $\sqrt{'}$qMV algebras and let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathbf{V}(\mathbb{K})$. We have $\mathbf{A} \in \mathbf{HSP}(\mathbb{K})$ so $\mathbf{A} = \mathbf{B}/\phi$ for some strictly meet irreducible congruence ϕ on an algebra $\mathbf{B}$ such that $\mathbf{B} \leq \mathbf{C} = \prod_{i \in I} \mathbf{C}_i$, for some algebras $\mathbf{C}_i \in \mathbb{K}$. The following two lemmas have proofs that are *mutatis mutandis* the same as the proofs of Lemmas 19 and 20.

Lemma 26. *If* $\mathbf{A}$ *is a Cartesian algebra, then the algebra* $\mathbf{B}$ *above can be taken to be Cartesian as well.*

Lemma 27. *If* $\mathbf{A}$ *is the algebra* $\mathbf{F}_{100}$*, then the congruence* $\phi = \beta$*; hence it is meet prime.*

For qMV algebras the analogues of the above two lemmas sufficed. For $\sqrt{'}$qMV algebras we need a little more, because quotients of Cartesian algebras need not be Cartesian and so, although Cartesian algebras are relatively congruence distributive, we cannot use this fact in a straightforward way. We can, however, do the following. For any congruence ϕ on a $\sqrt{'}$qMV algebra $\mathbf{A}$, define the *Cartesian closure* of ϕ to be the relation $\overline{\phi}$ defined by putting

$$\overline{\phi} = \{\langle a, b\rangle \in A^2 \colon \langle a \oplus 0, b \oplus 0\rangle \in \phi \text{ and } \left\langle \sqrt{'}a \oplus 0, \sqrt{'}b \oplus 0 \right\rangle \in \phi\}$$

The next lemma justifies the terminology.

Lemma 28. *For any congruence* ϕ*, its Cartesian closure* $\overline{\phi}$ *is the smallest* $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ *congruence containing* ϕ*.*

Proof. It is clear that $\overline{\phi}$ is an equivalence relation, since ϕ is. Congruence properties follow from ϕ being a congruence, together with the relevant properties of the operations, such as associativity and commutativity of $\oplus$. We will leave the details to the reader, proving only one case as an example. Suppose $\langle a, b\rangle \in \overline{\phi}$; we want to show that $\left\langle \sqrt{'}a, \sqrt{'}b \right\rangle \in \overline{\phi}$. By definition of $\overline{\phi}$ we have $\langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{'}a \oplus 0, \sqrt{'}b \oplus 0 \right\rangle \in \phi$. As the identity $(x \oplus 0)' \approx x' \oplus 0$ holds in all $\sqrt{'}$qMV algebras, we get $\langle a' \oplus 0, b' \oplus 0\rangle \in \phi$. Since $\sqrt{'}\sqrt{'}x \approx x'$ holds as well, we have $\left\langle \sqrt{'}\sqrt{'}a \oplus 0, \sqrt{'}\sqrt{'}b \oplus 0 \right\rangle \in \phi$. By definition of Cartesian closure, $\left\langle \sqrt{'}a, \sqrt{'}b \right\rangle \in \overline{\phi}$ follows.

It is not difficult to show that $\sqrt{\prime}q\mathbb{MV}-\mathbb{C}$ congruences are closed under arbitrary intersections, so there exist the smallest $\sqrt{\prime}q\mathbb{MV}-\mathbb{C}$ congruence containing ϕ, say, η. We will show that $\eta = \overline{\phi}$. Clearly $\overline{\phi} \geq \eta$, so we only need to show the converse. Take $\langle a, b\rangle \in \overline{\phi}$. Then

$$\langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle \in \phi$$

and since $\eta \geq \phi$ we get

$$\langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle \in \eta.$$

But $\mathbf{A}/\eta$ is Cartesian, so $\langle a, b\rangle \in \eta$ closing the argument. □

A property of Cartesian closures we will need in what follows is that they commute with intersections.

Lemma 29. *Let ϕ, ψ be arbitrary congruences on $\mathbf{A}$. Then $\overline{\phi \cap \psi} = \overline{\phi} \cap \overline{\psi}$.*

Proof. We calculate

$$\begin{aligned}
\langle a, b\rangle \in \overline{\phi \cap \psi} \text{ iff } & \langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle \in \phi \cap \psi \\
\text{iff } & \langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle \in \phi \text{ and} \\
& \langle a \oplus 0, b \oplus 0\rangle, \left\langle \sqrt{\prime}a \oplus 0, \sqrt{\prime}b \oplus 0\right\rangle \in \psi \\
\text{iff } & \langle a, b\rangle \in \overline{\phi} \text{ and } \langle a, b\rangle \in \overline{\psi} \\
\text{iff } & \langle a, b\rangle \in \overline{\phi} \cap \overline{\psi}
\end{aligned}$$

□

4.5 Jónsson's Lemma for $\sqrt{\prime}$qMV algebras: two exceptions

Let $\mathbb{K}$ be a class of $\sqrt{\prime}$qMV algebras and $\mathbf{A}$ be a subdirectly irreducible algebra, different from $\mathbf{F}_{020}$ and $\mathbf{F}_{004}$, belonging to $\mathbf{V}(\mathbb{K})$. Then $\mathbf{A} = \mathbf{B}/\phi$, with $\mathbf{B} \leq \mathbf{C} = \prod_{i\in I} \mathbf{C}_i$, for some $\mathbf{C}_i \in \mathbb{K}$ and some congruence ϕ on $\mathbf{B}$. As before, for $J \subseteq I$ define the congruence $\theta_J = \left\{\langle a, b\rangle \in C^2 \colon \{i \in I \colon a(i) = b(i)\} \supseteq J\right\}$. Then let $G = \{J \subseteq I \colon \theta_J|_B \subseteq \phi\}$.

Lemma 30. *There is an ultrafilter U on I such that $U \subseteq G$.*

Proof. The proof begins exactly as for Lemma 21. We show that $I \in G$, and G is upward closed and maximal. In the course of proving maximality, we arrive at the situation where for congruences θ_J and θ_K we have $\theta_J|_B \cap \theta_K|_B \subseteq \phi$. We want to show $\theta_J|_B \leq \phi$ or $\theta_K|_B \leq \phi$. As ϕ is strictly meet irreducible, we have either $\phi \geq \lambda$ or $\phi \geq \mu$. Moreover, if $\phi \geq \mu$, then by our special assumption $\mathbf{A}$ must be $\mathbf{F}_{100}$ and so $\phi = \beta$. As before this leaves us with two cases. If $\phi \geq \lambda$, then $\mathbf{A}$ is a Cartesian algebra and so by Lemma 26 we can assume $\mathbf{B}$ is Cartesian. Then, since $\mathbf{A}/\phi$ is Cartesian, we have $\overline{\theta_J|_B \cap \theta_K|_B} \leq \phi$. Lemma 29 then yields $\overline{\theta_J|_B} \cap \overline{\theta_K|_B} \leq \phi$. Now we have three $\sqrt{'}q\mathbb{MV}-\mathbb{C}$ congruences, and so we can make use of the fact that they distribute. We have $\phi = \phi \vee (\overline{\theta_J|_B} \cap \overline{\theta_K|_B}) = (\phi \vee \overline{\theta_J|_B}) \cap (\phi \vee \overline{\theta_K|_B})$. Since ϕ is meet irreducible, we get $\phi = \phi \vee \overline{\theta_J|_B}$ or $\phi = \phi \vee \overline{\theta_K|_B}$. Hence $\theta_J|_B \leq \phi$ or $\theta_K|_B \leq \phi$ as we needed. In the other case, with $\phi \geq \mu$, our exception assumption guarantees that Lemma 27 applies and thus also $\theta_J|_B \subseteq \phi$ or $\theta_K|_B \subseteq \phi$. In either case, maximality of G follows. $\square$

The following result is then proved exactly as Theorem 22.

Theorem 31. *Let $\mathbb{K}$ be a class of $\sqrt{'}qMV$ algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible algebra different from $\mathbf{F}_{020}$ and $\mathbf{F}_{004}$, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

Corollary 32. *Let $\mathbb{K}$ be a class of $\sqrt{'}qMV$ algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible Cartesian algebra, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

4.6 Jónsson's Lemma for $\sqrt{'}$qMV algebras: expanded generating class

Exactly as for qMV algebras, if we allow ourselves products of just two members of the generating class, we can avoid the exceptions. Recall that $\mathbb{K}^2$ is defined as $\mathbb{K} \cup \{\mathbf{K_1} \times \mathbf{K}_2 \colon \mathbf{K}_1, \mathbf{K}_2 \in \mathbb{K}\}$.

Lemma 33. *If $\mathbf{F}_{020} \in \mathbf{V}(\mathbb{K})$, then $\mathbf{F}_{020} \in \mathbf{HS}(\mathbb{K}^2)$. If $\mathbf{F}_{004} \in \mathbf{V}(\mathbb{K})$, then $\mathbf{F}_{004}, \mathbf{F}_{020} \in \mathbf{HS}(\mathbb{K}^2)$.*

Proof. We will first show that if $\mathbb{K}$ contains a nontrivial Cartesian algebra, then $\mathbf{F}_{004}, \mathbf{F}_{020} \in \mathbf{HS}(\mathbb{K}^2)$. Let $\mathbf{C}$ be a nontrivial Cartesian algebra in $\mathbb{K}$, so that $\sqrt{'}1 \neq 1$ in $\mathbf{C}$. Consider the element $\left\langle \sqrt{'}1, 1 \right\rangle \in C^2$. This element is neither regular, nor coregular. Applying $\sqrt{'}$ successively, we get $\{\left\langle\sqrt{'}1,1\right\rangle,\left\langle 0,\sqrt{'}1\right\rangle,\left\langle\sqrt{'}0,0\right\rangle,\left\langle 1,\sqrt{'}0\right\rangle\}$. Now this set, together with $\langle 0,0\rangle/\mu$ is the universe of a subalgebra of $\mathbf{C}^2/\mu$, which is isomorphic to $\mathbf{F}_{004}$. As $\mathbf{F}_{020}$ is a homomorphic image of $\mathbf{F}_{004}$, we obtain $\mathbf{F}_{004} \in \mathbf{SH}(\mathbb{K}^2)$ and $\mathbf{F}_{020} \in \mathbf{HSH}(\mathbb{K}^2)$. By congruence extension property [14] $\mathbf{SH}(\mathbb{K}^2) = \mathbf{HS}(\mathbb{K}^2)$ and so $\mathbf{F}_{004}, \mathbf{F}_{020} \in \mathbf{HS}(\mathbb{K}^2)$ as we claimed.

Now, if $\mathbb{K}$ contains no nontrivial Cartesian algebras, that is, if all of its members are flat, we have three cases. (1) If all elements of all members of $\mathbb{K}$ are fixpoints for $\sqrt{'}$, then neither $\mathbf{F}_{004}$, nor $\mathbf{F}_{020}$ belongs to $\mathbf{V}(\mathbb{K})$. (2) If all elements of all members of $\mathbb{K}$ are fixpoints for $'$, but not all are fixpoints for $\sqrt{'}$, then $\mathbf{V}(\mathbb{K}) = \mathbf{V}(\mathbf{F}_{020})$. In this case $\mathbf{F}_{020} \in \mathbf{S}(\mathbb{K})$. (3) If some members of $\mathbb{K}$ contain elements that are not fixpoints for $\sqrt{'}$ and some members of $\mathbb{K}$ contain elements that are not fixpoints for $'$, then $\mathbf{V}(\mathbb{K}) = \mathbf{V}(\mathbf{F}_{004})$. In this case $\mathbf{F}_{004} \in \mathbf{S}(\mathbb{K})$ and $\mathbf{F}_{020} \in \mathbf{HS}(\mathbb{K})$. □

Theorem 34. *Let $\mathbb{K}$ be a class of $\sqrt{'}$qMV algebras. If $\mathbf{A} \in \mathbf{V}(\mathbb{K})$ is a subdirectly irreducible algebra, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$.*

Proof. If $\mathbf{A} \notin \{\mathbf{F}_{020}, \mathbf{F}_{004}\}$, then $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K})$ by Theorem 31, so $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$. If $\mathbf{A} \in \{\mathbf{F}_{020}, \mathbf{F}_{004}\}$, then $\mathbf{A} \in \mathbf{HS}(\mathbb{K}^2)$ by Lemma 33 and therefore $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{K}^2)$ as well. □

4.7 Two lightweight applications

We will present two curiosities that do not seem to fit anywhere else. Their common theme is that they were discovered in the course of investigating Jónsson's Lemma for qMV algebras and $\sqrt{'}$qMV algebras. First, we show that some form of restriction of Jónsson's Lemma is indeed necessary. That may seem an obvious corollary of the fact that neither $q\mathbb{MV}$ nor $\sqrt{'}q\mathbb{MV}$ is congruence distributive, but it is not entirely so. There exist varieties for which Jónsson's Lemma holds without restrictions, yet satisfying no congruence identities [5]. We will show that for $q\mathbb{MV}$ (or $\sqrt{'}q\mathbb{MV}$) this is not the case. Let $\mathbf{K}$ be any non-flat algebra with a single irregular element (which has perforce to be a fixpoint). One example of such an algebra is the 4-element Diamond algebra (cf. Example 3 in [12]), but in general $\mathbf{K}$ can be infinite.

Theorem 35. *The algebra $\mathbf{F}_{02}$ belongs to $\mathbf{V}(\mathbf{K})$ but $\mathbf{F}_{02} \notin \mathbf{HSP}_U(\mathbf{K})$.*

Proof. That $\mathbf{F}_{02}$ belongs to $\mathbf{V}(\mathbf{K})$ follows from the proof of Lemma 24. To show the second part, notice that having a single irregular element is expressible by the first order sentence $\exists! x\,(x \oplus 0 \neq x)$ which carries over to ultraproducts, so every algebra from $\mathbf{P}_U(\mathbf{K})$ has a unique irregular element. Therefore every algebra from $\mathbf{HSP}_U(\mathbf{K})$ has at most one irregular element and so $\mathbf{F}_{02} \notin \mathbf{HSP}_U(\mathbf{K})$. □

Similar examples can be constructed for varieties of $\sqrt{'}$qMV algebras. Let for instance $\mathbf{A}$ be the 5-element Cross of [9] ($\mathtt{Rt}\,(\text{Ł}_3)$, in the notation of Section 7 below). Then $\mathbf{F}_{004}, \mathbf{F}_{020} \in \mathbf{V}(\mathbf{A})$, but neither belongs to $\mathbf{HSP}_U(\mathbf{A})$.

Our next observation is, if not quite a consequence of Lemma 24, then at least a side-effect of its proof. Let us come back to the algebra $\mathbf{K}$ of Theorem 35, demanding this time that the MV part of $\mathbf{K}$ be the standard MV algebra.

Theorem 36. *The variety $q\mathbb{MV}$ is generated by* $\mathbf{K}$.

Proof. Since the standard MV algebra is a subalgebra of $\mathbf{K}$, all subdirectly irreducible MV algebras belong to $\mathbf{V}(\mathbf{K})$. By Lemma 24, the two subdirectly irreducible flat qMV algebras also belong to $\mathbf{V}(\mathbf{K})$. Thus, $\mathbf{V}(\mathbf{K}) = q\mathbb{MV}$. □

5. $\sqrt{'}q\mathbb{MV}$: A new proof of standard completeness

$\sqrt{'}q\mathbb{MV}$ is generated as a variety (although not as a quasivariety) by the standard $\sqrt{'}$qMV algebra $\mathbf{S}_r$. The first proof of this standard completeness theorem was given in [9] by means of a rather complex argument, involving in an essential way a translation procedure. In this subsection we considerably simplify such a proof, using three results established in our previous papers on the subject:

1. Every $\sqrt{'}$qMV algebra is (subdirectly) embeddable into the product of a Cartesian algebra and a flat algebra (Theorem 36 in [9]);
2. Cartesian algebras generate $\sqrt{'}q\mathbb{MV}$ as a variety (Theorem 41 in [9]);
3. The standard algebra $\mathbf{S}_r$ generates $\mathbb{C}$ as a quasivariety (Lemma 43 in [3]).

To prevent any possible charge of circularity, we point out that no one of such results depends in any way on the standard completeness theorem for $\sqrt{'}q\mathbb{MV}$.

Theorem 37. *Let t, s be terms of type $\langle 2,1,0,0,0\rangle$. Then $\mathbf{S}_r \vDash t \approx s$ iff $\sqrt{'}q\mathbb{MV} \vDash t \approx s$.*

Proof. Let $\mathbf{A}$ be an arbitrary $\sqrt{'}$qMV algebra. By 1. above, $\mathbf{A} \in \mathbf{SP}(\mathbb{C} \cup \mathbb{F})$. Furthermore, by 3. $\mathbb{C} \subseteq \mathbf{ISPP}_u(\mathbf{S}_r)$, whereas by 2. $\mathbb{F} \subseteq \mathbf{HSP}(\mathbb{C})$. Summing up,

$$\mathbf{A} \in \mathbf{SP}(\mathbf{ISPP}_u(\mathbf{S}_r) \cup \mathbf{HSPISPP}_u(\mathbf{S}_r)).$$

However, $\mathbf{HSPISPP}_u(\mathbf{S}_r)$ simplifies to $\mathbf{HSPP}_u(\mathbf{S}_r)$, while obviously $\mathbf{ISPP}_u(\mathbf{S}_r) \subseteq \mathbf{HSPP}_u(\mathbf{S}_r)$. We conclude that

$$\mathbf{A} \in \mathbf{SPHSPP}_u(\mathbf{S}_r) = \mathbf{HSPP}_u(\mathbf{S}_r).$$

Since for any class $\mathbb{K}$, $\mathbf{P}_u(\mathbb{K}) \subseteq \mathbf{HP}(\mathbb{K})$, it follows that
$\mathbf{A} \in \mathbf{HSPHP}(\mathbf{S}_r) = \mathbf{HSP}(\mathbf{S}_r)$. □

6. $\sqrt{'}q\mathbb{MV}$: Strong finite model property

A quasivariety $\mathbb{Q}$ has the *finite model property* (FMP) if it is included in the variety generated by its finite members, whereas it has the *strong finite model property* (SFMP) if it is generated as a quasivariety by its finite members. It was shown in [7] that the SFMP is equivalent to the *finite embeddability property* (FEP): $\mathbb{Q}$ has the FEP if every finite partial subalgebra of an algebra $\mathbf{A} \in \mathbb{Q}$ can be embedded into a finite algebra $\mathbf{B} \in \mathbb{Q}$. Examples of quasivarieties with the FEP arising in algebraic logic are BCK algebras [2] and MV algebras [1].

The FMP was established both for $q\mathbb{MV}$ and for $\sqrt{'}q\mathbb{MV}$[3] in [14]. In [3], moreover, the SFMP was shown to hold for $q\mathbb{MV}$ and for the variety

[3]The proof of the FMP for $\sqrt{'}q\mathbb{MV}$ relies on the partly *wrong* Theorem 37 of [14] (see Theorem 46 below for a correction of the wrong item of this result). Anyway, such a bug can be fixed as the proof of Lemma 38 therein can be reformulated so as to show that the standard $\sqrt{'}$qMV algebra with rational coordinates (rather than its subalgebra of regular and coregular elements) is locally finite.

$\mathbb{F}$ of *flat* $\sqrt{\prime}$qMV algebras, yet the issue whether $\sqrt{\prime}q\mathbb{MV}$ (or even the subquasivariety $\mathbb{C}$ of Cartesian algebras) has the SFMP was left unanswered. The aim of this section is to settle the issue in the positive. We will adopt the following strategy. We will first establish the FEP for $\mathbb{C}$, whence the subquasivariety at issue has the SFMP as well. Then we will avail ourselves of the decomposition results in [9] to extend the property to the whole of $\sqrt{\prime}q\mathbb{MV}$.

Theorem 38. *$\mathbb{C}$ has the FEP.*

Proof. Let $\mathbf{A}=\left\langle A,\oplus^{\mathbf{A}},\sqrt{\prime}^{\mathbf{A}},0^{\mathbf{A}},1^{\mathbf{A}},k^{\mathbf{A}}\right\rangle$ be a Cartesian $\sqrt{\prime}$qMV algebra, and let $D\subseteq A$ be a finite set which, w.l.g., contains $k^{\mathbf{A}}$. Call $\mathbf{D}$ the partial subalgebra of $\mathbf{A}$ with universe D. By Theorem 36 in [9], any $d\in D$ can be unambiguously identified via its image in the pair algebra representation of $\mathbf{A}$, $\left\langle d\oplus^{\mathbf{A}}0,\sqrt{\prime}^{\mathbf{A}}d\oplus^{\mathbf{A}}0\right\rangle$. Recall, moreover, that $\mathbf{A}$ has an MV subreduct $\mathbf{R}_{\mathbf{A}}=\left\langle \mathcal{R}\left(\mathbf{A}\right),\oplus^{\mathbf{A}},{}^{\prime\mathbf{A}},0^{\mathbf{A}},1^{\mathbf{A}}\right\rangle$ containing both $d\oplus 0$ and $\sqrt{\prime}d\oplus 0$ for any $d\in D$. It follows that

$$\mathbf{E}=\left\langle E_1\cup E_2,\oplus^{\mathbf{E}},{}^{\prime\mathbf{E}},0^{\mathbf{E}},1^{\mathbf{E}}\right\rangle$$

where:

- $E_1=\left\{d\oplus^{\mathbf{A}}0:d\in D\right\},E_2=\left\{\sqrt{\prime}^{\mathbf{A}}d\oplus^{\mathbf{A}}0:d\in D\right\}$;
- for any operation symbol f,

$$f^{\mathbf{E}}(a_1,...,a_n)=\begin{cases} f^{\mathbf{A}}(a_1,...,a_n) \text{ if } f^{\mathbf{A}}(a_1,...,a_n)\in E_1\cup E_2;\\ \text{undefined, otherwise}\end{cases}$$

is a finite partial subalgebra of $\mathbf{R}_{\mathbf{A}}$. By the FEP for MV algebras, $\mathbf{E}$ can be embedded into a finite MV algebra $\mathbf{B}$ containing a fixpoint $k^{\mathbf{B}}$ (as $k^{\mathbf{A}}\in D$). Now, construct the pair algebra $\wp\left(\mathbf{B}\right)$ out of $\mathbf{B}$, which is clearly finite. What remains to be shown, therefore, is the fact that $\mathbf{D}$ embeds into $\wp\left(\mathbf{B}\right)$. Thus, for any $\left\langle d\oplus^{\mathbf{A}}0,\sqrt{\prime}^{\mathbf{A}}d\oplus^{\mathbf{A}}0\right\rangle\in D$, let

$$f\left(\left\langle d\oplus^{\mathbf{A}}0,\sqrt{\prime}^{\mathbf{A}}d\oplus^{\mathbf{A}}0\right\rangle\right)=\left\langle g\left(d\oplus^{\mathbf{A}}0\right),g\left(\sqrt{\prime}^{\mathbf{A}}d\oplus^{\mathbf{A}}0\right)\right\rangle$$

where g is the embedding of $\mathbf{E}$ into $\mathbf{B}$. f is one-one by the injectivity of g. It clearly preserves $k^{\mathbf{A}}$ and $0^{\mathbf{A}},1^{\mathbf{A}}$, whenever the latter are members of D. It also preserves the operations provided they are defined; in fact:

$$
\begin{aligned}
& f\left(\left\langle d \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right\rangle \oplus^{\wp(\mathbf{R}_{\mathbf{A}})}\left\langle c \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} c \oplus^{\mathbf{A}} 0\right\rangle\right) \\
& = f\left(\left\langle d \oplus^{\mathbf{A}} c, k^{\mathbf{A}}\right\rangle\right) \\
& \quad \text{def. } \oplus^{\wp(\mathbf{R}_{\mathbf{A}})} \\
& = \left\langle g\left(d \oplus^{\mathbf{A}} c\right), g\left(k^{\mathbf{A}}\right)\right\rangle \\
& \quad \text{def. } f \\
& = \left\langle g\left(d \oplus^{\mathbf{A}} c\right), k^{\mathbf{B}}\right\rangle \\
& \quad g \text{ preserves } k^{\mathbf{A}} \\
& = \left\langle g\left(d \oplus^{\mathbf{A}} 0\right) \oplus^{\mathbf{B}} g\left(c \oplus^{\mathbf{A}} 0\right), k^{\mathbf{B}}\right\rangle \\
& \quad g \text{ preserves } \oplus^{\mathbf{A}} \\
& = \left\langle g\left(d \oplus^{\mathbf{A}} 0\right), g\left(\sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right)\right\rangle \oplus^{\wp(\mathbf{B})}\left\langle g\left(c \oplus^{\mathbf{A}} 0\right), g\left(\sqrt{\prime}^{\mathbf{A}} c \oplus^{\mathbf{A}} 0\right)\right\rangle \\
& \quad \text{def. } \oplus^{\wp(\mathbf{B})} \\
& = f\left(\left\langle d \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right\rangle\right) \oplus^{\wp(\mathbf{B})} f\left(\left\langle c \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} c \oplus^{\mathbf{A}} 0\right\rangle\right) \\
& \quad \text{def. } f
\end{aligned}
$$

$$
\begin{aligned}
& f\left(\sqrt{\prime}^{\wp(\mathbf{R}_{\mathbf{A}})}\left\langle d \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right\rangle\right) & \\
& = f\left(\left\langle \sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0, \left(d \oplus^{\mathbf{A}} 0\right)^{\prime}\right\rangle\right) & \text{def. } \sqrt{\prime}^{\wp(\mathbf{R}_{\mathbf{A}})} \\
& = \left\langle g\left(\sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right), g\left(\left(d \oplus^{\mathbf{A}} 0\right)^{\prime}\right)\right\rangle & \text{def. } f \\
& = \left\langle g\left(\sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right), \left(g\left(d \oplus^{\mathbf{A}} 0\right)\right)^{\prime}\right\rangle & g \text{ preserves } {}^{\prime\mathbf{A}} \\
& = \sqrt{\prime}^{\wp(\mathbf{B})}\left\langle g\left(d \oplus^{\mathbf{A}} 0\right), g\left(\sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right)\right\rangle & \text{def. } \sqrt{\prime}^{\wp(\mathbf{B})} \\
& = \sqrt{\prime}^{\wp(\mathbf{B})} f\left(\left\langle d \oplus^{\mathbf{A}} 0, \sqrt{\prime}^{\mathbf{A}} d \oplus^{\mathbf{A}} 0\right\rangle\right) & \text{def. } f
\end{aligned}
$$

□

Corollary 39. $\mathbb{C}$ *has the SFMP.*

Theorem 40. $\sqrt{\prime}q\mathbb{MV}$ *has the SFMP.*

Proof. Let $\&_{i \leq n} t_i \approx s_i \Rightarrow t \approx s$ be a quasiequation which fails in $\sqrt{\prime}q\mathbb{MV}$. Recalling that quasiequations carry over to subalgebras and products, by the direct decomposition theorem for $\sqrt{\prime}q\mathbb{MV}$ there are a Cartesian $\sqrt{\prime}$qMV algebra $\mathbf{C}$ and a flat $\sqrt{\prime}$qMV algebra $\mathbf{F}$ s.t. $\&_{i \leq n} t_i \approx s_i \Rightarrow t \approx s$ fails in $\mathbf{C} \times \mathbf{F}$, hence either in $\mathbf{C}$ or in $\mathbf{F}$. If the former, then our quasiequation fails in a finite member of $\mathbb{C}$ by Corollary 39; if the latter, our result follows from Lemma 40 in [3]. □

7. $\sqrt{\prime}q\mathbb{MV}$: The lattice of subvarieties

The lattice of subvarieties of $q\mathbb{MV}$ was given a complete description in [3]. Not much is known, on the other hand, about the structure of the lattice $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$ of subvarieties of $\sqrt{\prime}q\mathbb{MV}$. In the present section we will provide a fairly complete descripton of this lattice as well.

7.1 The flat part

As in [3], we start with the easiest subtask: characterising the sublattice of *flat* subvarieties. This much is readily done, once we know that there are only three nontrivial subdirectly irreducible flat $\sqrt{\prime}$qMV algebras (Lemma 1):

Lemma 41. *There are just three nontrivial varieties of flat $\sqrt{\prime}qMV$ algebras:*

- $\mathbb{F} = \mathbf{V}(\mathbf{F}_{004})$
- $\mathbf{V}(\mathbf{F}_{100})$, *axiomatised by* $x \approx \sqrt{\prime}x$
- $\mathbf{V}(\mathbf{F}_{020})$, *axiomatised by* $x \approx x'$.

Proof. Every nontrivial flat $\sqrt{\prime}$qMV algebra contains either $\mathbf{F}_{100}$, or $\mathbf{F}_{020}$, or $\mathbf{F}_{004}$ as a subalgebra, whence for any subvariety $\mathbb{V}$ of $\mathbb{F}$ either $\mathbb{F} \subseteq \mathbb{V}$ or $\mathbf{V}(\mathbf{F}_{100}) \subseteq \mathbb{V}$ or $\mathbf{V}(\mathbf{F}_{020}) \subseteq \mathbb{V}$. It is easily seen that $\mathbf{V}(\mathbf{F}_{100})$ is axiomatised by $x \approx \sqrt{\prime}x$ and that $\mathbf{V}(\mathbf{F}_{020})$ is axiomatised by $x \approx x'$. $\square$

Of course, these three varieties form a chain in $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$: $\mathbf{V}(\mathbf{F}_{100}) \subset \mathbf{V}(\mathbf{F}_{020}) \subset \mathbb{F}$.

7.2 Varieties generated by strongly Cartesian algebras

We next proceed to tackle the problem of describing the structure of the rest of the lattice. A question which naturally arises in this context is: where do *Cartesian* algebras sit? We know from [9] that the varietal closure of $\mathbb{C}$ is the variety of all $\sqrt{\prime}$qMV algebras, but the whereabouts within $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$ of the proper subquasivarieties of $\mathbb{C}$ which happen to be varieties remain to be explored. The next Lemma gives a partial answer to this question.

Lemma 42. *Let $\mathbb{V}$ be a variety of $\sqrt{'}$qMV algebras. Then $\mathbb{V}$ is a subquasivariety of $\mathbb{C}$ iff it contains only strongly Cartesian algebras.*

Proof. For the nontrivial direction, suppose $\mathbb{V} \subseteq \mathbb{C}$ and $\mathbf{A} \in \mathbb{V}$ is not strongly Cartesian. Then $\mu \neq \Delta$ and $\mathbf{A}/\mu$ is a nontrivial flat algebra. Since $\mathbb{C} \cap \mathbb{F} = \{0\}$, it follows that $\mathbb{V}$ is not closed with respect to quotients, a contradiction. □

Throughout this section, by "MV* algebras" we will mean expansions of MV algebras by an additional constant k, satisfying the axiom $k \approx k'$. This variety has been investigated by Lewin and his colleagues [13], who proved that: i) the category of such algebras is equivalent to the category of MV algebras; ii) the variety itself is generated as a quasivariety by the standard algebra over the $[0,1]$ interval. Although e.g. all nontrivial Boolean algebras are ruled out by this definition, in virtue of the above-mentioned results the two concepts can be considered, for many purposes, interchangeable. We now present a general construction (to some extent implicit in our previous papers on the subject) to obtain a $\sqrt{'}$qMV algebra out of an MV* algebra.

Definition 43. *Let $\mathbf{A} = \left\langle A, \oplus^{\mathbf{A}}, '^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}}, k^{\mathbf{A}} \right\rangle$ be an MV* algebra. The* pivoted rotation *of $\mathbf{A}$ is the structure*

$$\mathtt{Rt}(\mathbf{A}) = \left\langle A \cup f(A), \oplus^{\mathtt{Rt}(\mathbf{A})}, \sqrt{'}^{\mathtt{Rt}(\mathbf{A})}, 0^{\mathtt{Rt}(\mathbf{A})}, 1^{\mathtt{Rt}(\mathbf{A})}, k^{\mathtt{Rt}(\mathbf{A})} \right\rangle$$

where:

- $f(A) = \left\{ f(x) : x \in A - \left\{ k^{\mathbf{A}} \right\} \right\}$ *is a disjoint bijective copy of* $A - \left\{ k^{\mathbf{A}} \right\}$;
- $$a \oplus^{\mathtt{Rt}(\mathbf{A})} b = \begin{cases} a \oplus^{\mathbf{A}} b, \text{ if } a, b \in A; \\ a \oplus^{\mathbf{A}} k^{\mathbf{A}}, \text{ if } a \in A \text{ and } b \in f(A); \\ k^{\mathbf{A}} \oplus^{\mathbf{A}} b, \text{ if } a \in f(A) \text{ and } b \in A; \\ 1^{\mathbf{A}}, \text{ if } a, b \in f(A). \end{cases}$$
- $$\sqrt{'}^{\mathtt{Rt}(\mathbf{A})} a = \begin{cases} f(a), \text{ if } a \in A - \left\{ k^{\mathbf{A}} \right\}; \\ \left(f^{-1}(a) \right)^{'\mathbf{A}}, \text{ if } a \in f(A); \\ k^{\mathbf{A}}, \text{ if } a = k^{\mathbf{A}}. \end{cases}$$
- $0^{\mathtt{Rt}(\mathbf{A})} = 0^{\mathbf{A}}$; $1^{\mathtt{Rt}(\mathbf{A})} = 1^{\mathbf{A}}$; $k^{\mathtt{Rt}(\mathbf{A})} = k^{\mathbf{A}}$.

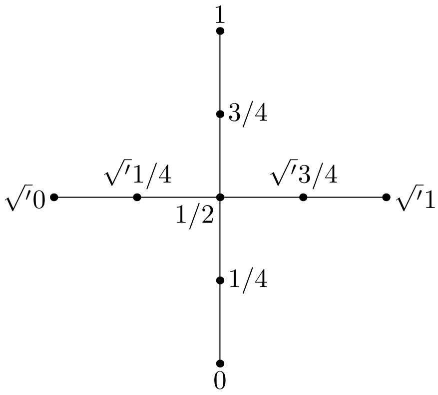

Figure 2: Pivoted rotation.

Example 44. *The pivoted rotation* $\mathtt{Rt}\,(\textbf{\L}_5)$ *of the 5-element Łukasiewicz chain* $\textbf{\L}_5$ *is depicted in Fig. 2.*

It is easy to check that such a construction always yields a strongly Cartesian $\sqrt{\prime}$qMV algebra.

Of course, the smallest nontrivial MV algebra to which a pivoted rotation can be applied is the 3-element Łukasiewicz chain $\textbf{\L}_3$. Remarkably enough, the variety of $\sqrt{\prime}$qMV algebras generated by $\mathtt{Rt}\,(\textbf{\L}_3)$ *includes* all the flat subvarieties.

Lemma 45. $\mathbb{F} \subset \mathbf{V}(\mathtt{Rt}\,(\textbf{\L}_3))$.

Proof. Clearly, $\mathtt{Rt}\,(\textbf{\L}_3) \notin \mathbb{F}$, whence it suffices to show that $\mathbf{F}_{004} \in \mathbf{HSP}(\mathtt{Rt}\,(\textbf{\L}_3))$. Let us consider $\mathtt{Rt}\,(\textbf{\L}_3) \times \mathtt{Rt}\,(\textbf{\L}_3)$ (Fig. 3). The set

$$\mathtt{Rt}\,(\textbf{\L}_3) \times \mathtt{Rt}\,(\textbf{\L}_3) - \left\{ \left\langle 0, \sqrt{\prime}0 \right\rangle, \left\langle \sqrt{\prime}0, 1 \right\rangle, \left\langle 1, \sqrt{\prime}1 \right\rangle, \left\langle \sqrt{\prime}1, 0 \right\rangle \right\}$$

is a subuniverse of $\mathtt{Rt}\,(\textbf{\L}_3) \times \mathtt{Rt}\,(\textbf{\L}_3)$. Call $\mathbf{D}$ the corresponding subalgebra; thus $\mathbf{D} \in \mathbf{SP}(\mathtt{Rt}\,(\textbf{\L}_3))$. Now, $\mathbf{F}_{004} = \mathbf{D}/\mu$. □

Strongly Cartesian algebras form a proper positive universal class - hereafter called $\mathbb{S}$. What about its varietal closure? In [14], Theorem 37, an erroneous claim was made to the effect that the algebra $\mathtt{Rt}\left(\mathbf{MV}_{[0,1]}\right)$ (hence, a fortiori, the class of all strongly Cartesian algebras) generates $\sqrt{\prime}q\mathbb{MV}$. We will correct it now.

Theorem 46. *Strongly Cartesian algebras do not generate* $\sqrt{\prime}q\mathbb{MV}$.

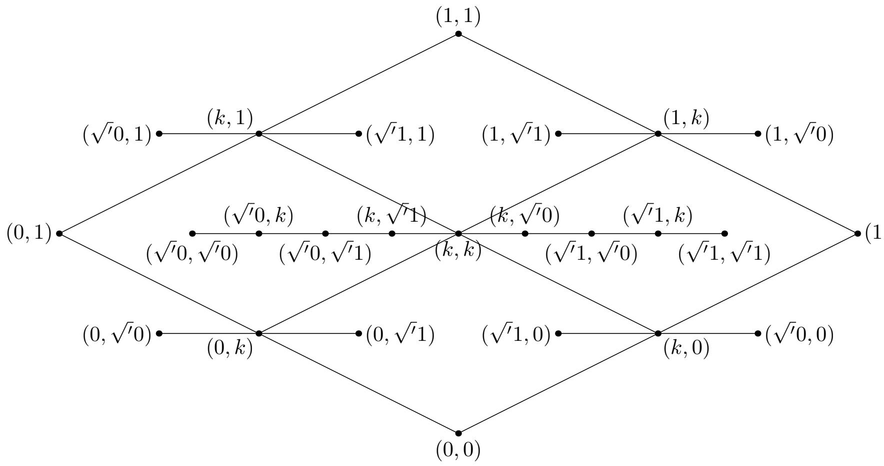

Figure 3: $\mathtt{Rt}(\mathbf{Ł}_3) \times \mathtt{Rt}(\mathbf{Ł}_3)$

Proof. By Lemma 45 we know that $\mathbb{S}$ generates all subdirectly irreducible flat $\sqrt{'}$qMV algebras. Thus, the question reduces to whether $\mathbb{S}$ generates all subdirectly irreducible Cartesian members of $\sqrt{'}\mathbb{qMV}$. By Corollary 32 we know that all subdirectly irreducible Cartesian $\sqrt{'}$qMV algebras in $\mathbf{V}(\mathbb{S})$ belong to $\mathbf{HSP}_U(\mathbb{S})$. But, since $\mathbb{S}$ is a positive universal class, it is closed under quotients, subalgebras and ultraproducts, so $\mathbf{HSP}_U(\mathbb{S}) = \mathbb{S}$. However, there are subdirectly irreducible Cartesian algebras outside $\mathbb{S}$, for example every pair algebra $\mathcal{P}(\mathbf{I})$ with $\mathbf{I}$ a subdirectly irreducible MV algebra is such (see [9], [10] for more on pair algebras). It follows that $\mathbf{V}(\mathbb{S})$ is a proper subvariety of $\sqrt{'}q\mathbb{MV}$. $\square$

We will show that $\mathbf{V}(\mathbb{S})$ has a rather natural finite base. Recall that the derived operation symbol $\Cup$ is defined as follows:

$$x \Cup y = \big(x' \oplus y\big)' \oplus y.$$

Consider the following identity:

$$(\sqrt{'}x \oplus k) \Cup (x \oplus k) \approx 1 \qquad (S)$$

Interpreted over Cartesian algebras whose regular elements are linearly ordered, S says that any element a is either greater or equal than k or such that its square root of the inverse is greater or equal than k. Because of the properties of $\sqrt{'}$, this is equivalent (over Cartesian algebras with linearly ordered regular elements) to every element being either regular or coregular. We will prove that S suffices for a base of $\mathbf{V}(\mathbb{S})$ relative to $\sqrt{'}q\mathbb{MV}$. First, however, we establish two auxiliary Lemmas.

Lemma 47. *If* $\mathbf{A}$ *is a subdirectly irreducible Cartesian* $\sqrt{'}$ *qMV algebra, then* $\mathbf{R_A}$ *is subdirectly irreducible too.*

Proof. Let $\mathbf{B}$ be any strongly Cartesian $\sqrt{'}$ qMV algebra. By Theorem 47 in [9], the lattice of congruences of $\mathbf{R_B}$ is isomorphic to the lattice of relative congruences of $\mathbf{B}$; however, in a strongly Cartesian algebra all congruences are relative, whence $\mathrm{Con}(\mathbf{B})$ is isomorphic to $\mathrm{Con}(\mathbf{R_B})$. Therefore, $\mathbf{B}$ is subdirectly irreducible iff so is $\mathbf{R_B}$.

Now, for any congruence $\varphi \in \mathrm{Con}(\mathtt{Rt}\,(\mathbf{R_A})\,)$, let

$$\varphi_{\mathbf{A}} = \{\langle a, b\rangle \in A : \langle a, b\rangle \in \varphi \text{ or } a = b\}\,.$$

Observe that if $\langle a, b\rangle \in \varphi$, with $a \neq b$, then $a, b \in \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})$, whence $\varphi_{\mathbf{A}}$ is a congruence on $\mathbf{A}$. Since $\mathbf{A}$ is subdirectly irreducible, moreover, its lattice of congruences contains a monolith η. Clearly, $\eta\restriction_{\mathtt{Rt}(\mathbf{R_A})}$ is nontrivial. Let ψ be any nontrivial congruence on $\mathtt{Rt}\,(\mathbf{R_A})$. Then $\eta \leq \psi_{\mathbf{A}}$ and thus

$$\Delta < \eta\restriction_{\mathtt{Rt}(\mathbf{R_A})} \leq \psi_{\mathbf{A}}\restriction_{\mathtt{Rt}(\mathbf{R_A})} = \psi.$$

Therefore $\eta\restriction_{\mathtt{Rt}(\mathbf{R_A})}$ is the monolith in $\mathrm{Con}(\mathtt{Rt}\,(\mathbf{R_A})\,)$ and so $\mathtt{Rt}\,(\mathbf{R_A})$ is subdirectly irreducible. By our previous observation, $\mathbf{R_A}$ is subdirectly irreducible too. $\square$

Lemma 48. *Let* $\mathbf{A}$ *be a subdirectly irreducible Cartesian, but not strongly Cartesian algebra. Then there is an element* $a \in A$ *with* $a \oplus k \neq 1$ *and* $\sqrt{'}a \oplus k \neq 1$.

Proof. Since $\mathbf{A}$ is Cartesian, but not strongly Cartesian, there is an element $u \in A$ that is neither regular nor coregular. Consider the set $U = \{u, \sqrt{'}u, u', \sqrt{'}u'\}$. All members of U are distinct, moreover $U \cap (\mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})) = \emptyset$. As $\mathbf{A}$ is subdirectly irreducible, by Lemma 47 $\mathbf{R_A}$ is subdirectly irreducible and thus linearly ordered, so each $a \in U$ has either $a \oplus 0 > k$ or $a \oplus 0 < k$.

If $u \oplus 0 < k$, then obviously $u \oplus k < 1$, so if $\sqrt{'}u \oplus k < 1$ we can take $a = u$. Suppose $\sqrt{'}u \oplus k = 1$, i.e., $\sqrt{'}u \oplus 0 \geq k$. By the remark at the end of last paragraph, then $\sqrt{\prime}u \oplus 0 > k$. Therefore, $\sqrt{\prime}u' \oplus 0 < k$. Putting $a = \sqrt{\prime}u'$ we obtain $a \oplus k < 1$ and $\sqrt{\prime}a \oplus k = \sqrt{\prime}\sqrt{\prime}u' \oplus 0 = u \oplus 0 < 1$.

If $u \oplus 0 > k$, then $u' \oplus 0 < k$ and we can repeat the above argument with u' in place of u, obtaining the desired conclusion. □

Theorem 49. $\mathbf{V}(\mathbb{S})$ *is axiomatised relative to* $\sqrt{\prime}q\mathbb{MV}$ *by* S.

Proof. First we show that all strongly Cartesian algebras satisfy S. Let $\mathbf{A}$ be strongly Cartesian and let $a \in \mathcal{R}(\mathbf{A})$. Then $\sqrt{\prime}a \oplus k = 1$, and we are done. Similarly, if $a \in \mathcal{COR}(\mathbf{A})$, $a \oplus k = 1$, and S likewise follows.

To show the converse it suffices to prove that any subdirectly irreducible algebra $\mathbf{A}$ not in $\mathbf{V}(\mathbb{S})$ falsifies S. In fact, since all subdirectly irreducible flat algebras belong to $\mathbf{V}(\mathbb{S})$, we can assume $\mathbf{A}$ is Cartesian but not strongly Cartesian. Observe first that by the Łukasiewicz axiom S is equivalent to

$$(x \oplus k) \Cup (\sqrt{\prime}x \oplus k) \approx 1 \qquad (S')$$

Now, by Lemma 48 there is an $a \in A$ with $a \oplus k \neq 1$ and $\sqrt{\prime}a \oplus k \neq 1$. Since $\mathcal{R}(\mathbf{A})$ is linearly ordered, we get that $a \oplus k \leq \sqrt{\prime}a \oplus k$ or $\sqrt{\prime}a \oplus k \leq a \oplus k$.

If the former, then $(a \oplus k)' \oplus (\sqrt{\prime}a \oplus k) = 1$. Therefore, $((a \oplus k)' \oplus (\sqrt{\prime}a \oplus k))' \oplus (\sqrt{\prime}a \oplus k) = \sqrt{\prime}a \oplus k \neq 1$ falsifying S'.

If the latter, then $(\sqrt{\prime}a \oplus k)' \oplus (a \oplus k) = 1$. Therefore, $((\sqrt{\prime}a \oplus k)' \oplus (a \oplus k))' \oplus (a \oplus k) = a \oplus k \neq 1$ falsifying S. □

The next lemma would be a standard corollary of Jónsson's Lemma, if that lemma held in full generality. As it does not, we will supply a proof.

Lemma 50. *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be varieties of* $\sqrt{\prime}qMV$ *algebras. Then,* $(\mathbb{V} \vee \mathbb{W})_{SI} = \mathbb{V}_{SI} \cup \mathbb{W}_{SI}$.

Proof. If at least one of $\mathbb{V}$, $\mathbb{W}$ is flat, then we have $\mathbb{V} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{V}$, and the claim holds trivially. If both $\mathbb{V}$ and $\mathbb{W}$ are non-flat, then they have exactly the same flat subdirectly irreducible members, so it suffices to show that the claim holds for Cartesian subdirectly irreducible algebras. So let $\mathbf{A}$ be a Cartesian subdirectly irreducible member of $\mathbb{V} \vee \mathbb{W}$. By Corollary 32 we get $\mathbf{A} \in \mathbf{HSP}_U(\mathbb{V} \cup \mathbb{W})$ and thus $\mathbf{A} \in \mathbf{HS}(\mathbf{C})$ for some ultraproduct $\mathbf{C}$ of algebras from $\mathbb{V} \cup \mathbb{W}$. Therefore $\mathbf{C}$ itself belongs to $\mathbb{V}$ or $\mathbb{W}$, and then so does $\mathbf{A}$. □

By Lemma 50 the lattice whose elements are nontrivial $\sqrt{\prime}$ qMV varieties generated by strongly Cartesian algebras - henceforth denoted as $\mathcal{L}^V(\mathbb{S})$ -

is a sublattice of $\mathcal{L}^V(\sqrt{'}q\mathbb{MV})$. Our next goal is showing that it is in one-one correspondence with the lattice of nontrivial MV* varieties. To this purpose, we extend in a natural way the concept of rotation of an MV* algebra to whole *varieties* of such algebras.

Definition 51. *Let* $\mathbb{V}$ *be a variety of MV* algebras. We define* $\mathtt{Rt}(\mathbb{V})$ *as* $\mathbf{V}(\{\mathtt{Rt}(\mathbf{A}) : \mathbf{A} \in \mathbb{V}\})$.

Putting at once this definition to good use, we observe that the bottom of $\mathcal{L}^V(\mathbb{S})$, i.e. $\mathbf{V}(\mathtt{Rt}(\mathbf{Ł}_3))$, is nothing but the rotation of the bottom of $\mathcal{L}^V(\mathbb{MV}^*)$, which is $\mathbf{V}(\mathbf{Ł}_3)$. But there is more to it: $\mathcal{L}^V(\mathbb{S})$ is, actually, isomorphic to $\mathcal{L}^V(\mathbb{MV}^*)$.

Theorem 52. $\mathcal{L}^V(\mathbb{MV}^*)$ *is isomorphic to* $\mathcal{L}^V(\mathbb{S})$ *via the mapping* $\varphi(\mathbb{V}) = \mathtt{Rt}(\mathbb{V})$.

Proof. Order preservation is obvious. The only tricky parts of our theorem are injectivity and surjectivity.

We first prove that our mapping is one-one. If $\mathbb{V} \neq \mathbb{W}$, then without loss of generality there is an equation $t \approx s$, with t, s regular terms, which holds in $\mathbb{V}$ but fails in $\mathbb{W}$. Then $t \approx s$ also fails in the class of qMV reducts of $\mathtt{Rt}(\mathbb{W})$. Now, suppose by contradiction that $t \approx s$ fails in $\mathtt{Rt}(\mathbf{A})$, for some $\mathbf{A} \in \mathbb{V}$. Consequently, for some $\overrightarrow{a} \in \mathtt{Rt}(A)$, $t^{\mathtt{Rt}(\mathbf{A})}\left(\overrightarrow{a \oplus 0}\right) = t^{\mathtt{Rt}(\mathbf{A})}(\overrightarrow{a}) \neq s^{\mathtt{Rt}(\mathbf{A})}(\overrightarrow{a}) = s^{\mathtt{Rt}(\mathbf{A})}\left(\overrightarrow{a \oplus 0}\right)$, a contradiction with $\mathbb{V} \vDash t \approx s$. Then $\mathtt{Rt}(\mathbb{W}) \neq \mathtt{Rt}(\mathbb{V})$.

Finally, we prove that our mapping is onto. We have to prove that, if $\mathbb{R}$ is a $\sqrt{'}$qMV variety generated by strongly Cartesian algebras, then $\mathbb{R} = \mathtt{Rt}(\mathbb{V})$, for some MV* variety $\mathbb{V}$. It suffices to prove that, for any strongly Cartesian $\mathbf{B} \in \mathbb{R}$ (which w.l.g. can be taken to be subdirectly irreducible and then linearly preordered by Lemma 47) there are $\mathbb{V} \subseteq \mathbb{MV}^*$ and $\mathbf{C} \in \mathtt{Rt}(\mathbb{V})$ such that $\mathbf{B}$ and $\mathbf{C}$ have the same equational theory. So, let $\mathbb{V} = \mathbf{V}(\{\mathbf{R_A} : \mathbf{A} \in \mathbb{R}\})$ and $\mathbf{C} = \mathtt{Rt}(\mathbf{R_B})$. Then:

- Obviously $\mathbb{V} \subseteq \mathbb{MV}^*$;
- $\mathbf{C} = \mathtt{Rt}(\mathbf{R_B}) \in \mathtt{Rt}(\mathbb{V}) = \mathtt{Rt}(\mathbf{V}(\{\mathbf{R_A} : \mathbf{A} \in \mathbb{R}\}))$;
- $\mathbf{B}$ and $\mathbf{C}$ not only have the same equational theory, but are indeed isomorphic, since $\mathbf{B}$ is strongly Cartesian and linearly preordered.

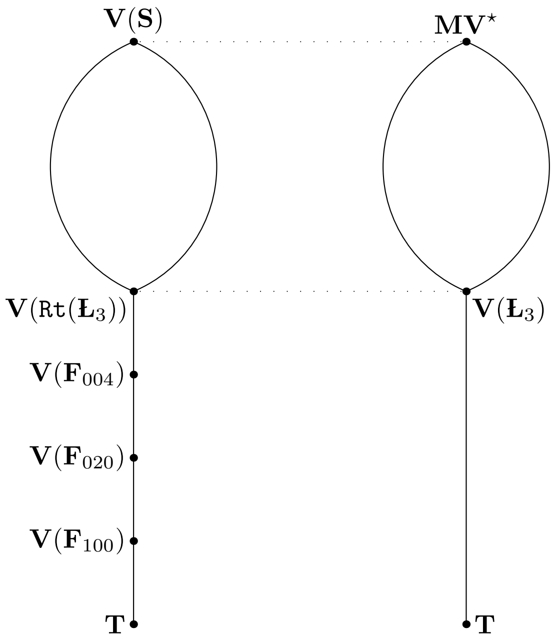

Figure 4: Subvarieties generated by strongly Cartesian algebras.

□

The structure of $\mathcal{L}^V(\mathbb{S})$ is depicted in Fig. 4.

7.3 A glimpse on the structure of the lattice

In this subsection we will look at the subvariety lattice of $\sqrt{\prime}q\mathbb{MV}$ in some more detail. Two new pieces of notation will be convenient. For a Cartesian algebra $\mathbf{A}$, we will write $\mathbf{A}^\times$ for the algebra $\mathtt{Rt}(\mathbf{R_A})$ and $\mathbf{A}^\Diamond$ for the algebra $\mathcal{P}(\mathbf{R_A})$. This notation extends naturally to subvarieties of $\sqrt{\prime}q\mathbb{MV}$, so for a variety $\mathbb{V}$, we will write $\mathbb{V}^\times$ for the variety generated by the class $\{\mathbf{A}^\times : \mathbf{A} \in \mathbb{V}_\mathbb{C}\}$ and $\mathbb{V}^\Diamond$ for the variety generated by the class $\{\mathbf{A}^\Diamond : \mathbf{A} \in \mathbb{V}_\mathbb{C}\}$, where $\mathbb{V}_\mathbb{C}$ denotes the class of all Cartesian members of $\mathbb{V}$. The following two lemmas extend very slightly Lemma 47. Although they are not necessary to establish the results in this section, we include them because they justify natural intuitions about $\mathbb{V}^\times$ and $\mathbb{V}^\Diamond$.

Lemma 53. *A non-flat $\sqrt{'}$ qMV algebra* $\mathbf{A}$ *is subdirectly irreducible iff* $\mathbf{A}^{\times}$ *is subdirectly irreducible iff* $\mathbf{A}^{\Diamond}$ *is subdirectly irreducible. Moreover,* $\{\mathbf{A}^{\times}\colon \mathbf{A} \in \mathbb{V}_{\mathbb{C}}\}$ *and* $\{\mathbf{A}^{\Diamond}\colon \mathbf{A} \in \mathbb{V}_{\mathbb{C}}\}$ *are closed under ultraproducts.*

Proof. The equivalences follow from Lemma 47. For the moreover part, let $\mathbf{B} = \prod_{i\in I} \mathbf{A}_i^{\times}/U$ be an ultraproduct of algebras from $\{\mathbf{A}^{\times}\colon \mathbf{A} \in \mathbb{V}\}$. It is straightforward to verify that $\mathbf{B}$ embeds into $(\prod_{i\in I} \mathbf{A}_i/U)^{\times}$ via the quotient map id/U, where id is the identity map. To establish the embedding of $\prod_{i\in I} \mathbf{A}_i^{\Diamond}/U$ into $(\prod_{i\in I} \mathbf{A}_i/U)^{\Diamond}$, we use the fact that the pair algebra operator commutes with ultraproducts, established in [3]. □

If $\mathbb{K}$ is any class of $\sqrt{'}$qMV algebras, the operators $\mathbf{H}_C$ (defined as the operator whose output for the argument $\mathbb{K}$ is the class of all Cartesian homomorphic images of algebras in $\mathbb{K}$) and $\mathbf{Q} = \wp\mathbf{SR}$ (defined as the operator whose output for the argument $\mathbb{K}$ is the class of all pair algebras over subalgebras of the term subreducts of regular elements of algebras in $\mathbb{K}$) are well-defined class operators. We obtain the following results:

Lemma 54. *The class* $\{\mathbf{A}^{\times}\colon \mathbf{A} \in \mathbb{V}_{\mathbb{C}}\}$ *is closed under the operators* $\mathbf{H}$ *and* $\mathbf{S}$*. The class* $\{\mathbf{A}^{\Diamond}\colon \mathbf{A} \in \mathbb{V}_{\mathbb{C}}\}$ *is closed under the operators* $\mathbf{H}_C$ *and* $\mathbf{Q}$*.*

Proof. For the first statement, closure under $\mathbf{S}$ is clear. For closure under quotients, it suffices to observe that $\mathbf{A}^{\times}/\phi$ is isomorphic to $\left(\mathbf{A}/\overline{\phi}\right)^{\times}$ via the mapping $f(a/\phi) = a/\overline{\phi}$.

Let us proceed to establish the second statement. Let $\mathbf{A}$ be a Cartesian algebra from $\mathbb{V}$. Consider $\mathbf{A}^{\Diamond}$. That $\mathbf{A}^{\Diamond}/\theta$ is Cartesian for any $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruence θ is a definitional tautology, so we only need to show that $\mathbf{A}^{\Diamond}/\theta$ is isomorphic to $\mathbf{B}^{\Diamond}$ for some Cartesian $\mathbf{B}$. Taking $\mathbf{B}$ to be $\mathbf{A}/\theta|_{\mathbf{A}^{\times}}$, we get that in the pair representation of $\mathbf{B}^{\Diamond}$ every element is of the form $\langle a/\theta|_{\mathbf{A}^{\times}}, b/\theta|_{\mathbf{A}^{\times}}\rangle$ for some $a, b \in \mathcal{R}(\mathbf{A})$. Now taking $\langle a, b\rangle$ in the pair representation of $\mathbf{A}^{\Diamond}$ it is straightforward to show that the map $\langle a, b\rangle/\theta \mapsto \langle a/\theta|_{\mathbf{A}^{\times}}, b/\theta|_{\mathbf{A}^{\times}}\rangle$ establishes the desired isomorphism. Finally, closure under $\mathbf{Q}$ follows directly from the fact that $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^{\Diamond})$, which in turn follows directly from the relevant definitions. □

Lemma 55. *Let* $\mathbb{V}$ *be a non-flat variety of $\sqrt{'}$ qMV algebras. The varieties* $\mathbb{V}$*,* $\mathbb{V}^{\times}$*, and* $\mathbb{V}^{\Diamond}$ *have precisely the same strongly Cartesian and flat subdirectly irreducible members. Moreover, all s.i. members of* $\mathbb{V}$ *(and* a fortiori *of* $\mathbb{V}^{\Diamond}$*) are superalgebras of s.i. members of* $\mathbb{V}^{\times}$*.*

Proof. Since $\mathbb{V}$ is a non-flat variety, $\mathbb{V}^{\times}$ is also non-flat, and so all flat subdirectly irreducible algebras belong to $\mathbb{V}^{\times}$. Hence, $\mathbb{V}$, $\mathbb{V}^{\times}$, and $\mathbb{V}^{\Diamond}$ have the same flat subdirectly irreducible members. So, if $\mathbf{A}$ is a strongly Cartesian subdirectly irreducible algebra in $\mathbb{V}^{\Diamond}$, then $\mathbf{A}^{\times} = \mathbf{A}$ belongs to $\mathbb{V}^{\times}$. Since $\mathbb{V}^{\Diamond} \supseteq \mathbb{V} \supseteq \mathbb{V}^{\times}$, it shows that $\mathbb{V}$, $\mathbb{V}^{\times}$, and $\mathbb{V}^{\Diamond}$ have the same strongly Cartesian s.i. members. The remaining assertion follows by observing that if $\mathbf{A}$ is a Cartesian subdirectly irreducible algebra in $\mathbb{V}$, then $\mathbf{A}^{\times}$ is a subalgebra of $\mathbf{A}$ and belongs to $\mathbb{V}^{\times}$. $\square$

Lemma 56. *Every non-flat variety $\mathbb{V}$ belongs to the interval $[\mathbb{V}^{\times}, \mathbb{V}^{\Diamond}]$. Moreover, we have $\mathbb{V}^{\times\times} = \mathbb{V}^{\times} = \mathbb{V}^{\Diamond\times}$ and $\mathbb{V}^{\times\Diamond} = \mathbb{V}^{\Diamond} = \mathbb{V}^{\Diamond\Diamond}$.*

Proof. Since any non-flat variety of $\sqrt{\prime}$ qMV algebras is generated by its Cartesian members, it suffices to establish the equalities for $\mathbb{V}_C$. Since $\mathbf{A}^{\times\times} = \mathbf{A}^{\times} = \mathbf{A}^{\Diamond\times}$, the classes of $\mathbb{V}_C^{\times\times}$, $\mathbb{V}_C^{\times}$, and $\mathbb{V}_C^{\Diamond\times}$ coincide, so we obtain the first pair of equalities. The second pair follows similarly from the fact that $\mathbf{A}^{\times\Diamond} = \mathbf{A}^{\Diamond} = \mathbf{A}^{\Diamond\Diamond}$. $\square$

Corollary 57. *No nontrivial variety $\mathbb{V}$ in $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$ satsifies any nontrivial congruence identity.*

Proof. By Theorems 52, 45 and Lemma 56 $\mathbf{V}(\mathbf{F}_{100})$ is the single atom of $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$. However, the class of congruence lattices of algebras in $\mathbf{V}(\mathbf{F}_{100})$ coincides with the class of all equivalence lattices over some set. The result follows then by Whitman's Theorem. $\square$

Lemma 58. *Let $\mathbb{V}$ and $\mathbb{W}$ be non-flat varieties. The following hold:*

1. $(\mathbb{V} \vee \mathbb{W})^{\Diamond} = \mathbb{V}^{\Diamond} \vee \mathbb{W}^{\Diamond}$
2. $(\mathbb{V} \cap \mathbb{W})^{\times} = \mathbb{V}^{\times} \cap \mathbb{W}^{\times}$
3. $(\mathbb{V} \cap \mathbb{W})^{\Diamond} = \mathbb{V}^{\Diamond} \cap \mathbb{W}^{\Diamond}$
4. $(\mathbb{V} \vee \mathbb{W})^{\times} = \mathbb{V}^{\times} \vee \mathbb{W}^{\times}$

In particular, $\mathbb{X}^{\Diamond}$ is a topological closure operator and $\mathbb{X}^{\times}$ a topological interior operator.

Proof. The equalities (2) and (3) are obvious. For (1) and (4) the right-to-left direction is clear in both cases, so it remains to show the converse.

Let $\mathbf{A}$ be a Cartesian subdirectly irreducible member of $(\mathbb{V} \vee \mathbb{W})^{\Diamond}$. Then $\mathbf{A} = \mathbf{B}^{\Diamond}$ for some subdirectly irreducible $\mathbf{B} \in \mathbb{V} \vee \mathbb{W}$. Thus, by Lemma 50 we obtain that $\mathbf{B} \in \mathbb{V}$ or $\mathbf{B} \in \mathbb{W}$ and therefore $\mathbf{A} \in \mathbb{V}^{\Diamond}$ or $\mathbf{A} \in \mathbb{W}^{\Diamond}$. Hence, all Cartesian subdirectly irreducible members of $(\mathbb{V} \vee \mathbb{W})^{\Diamond}$ belong to $\mathbb{V}^{\Diamond} \cup \mathbb{W}^{\Diamond}$ and this suffices for the claim. The same argument with $\Diamond$ replaced by $\times$ establishes (4). Lemma 56 together with (1) and (2) establish the remaining claim. □

We propose to call the intervals $[\mathbb{V}^{\times}, \mathbb{V}^{\Diamond}]$ by a rather suggestive name of *slices*. The following quite obvious lemma shows that this is not a misnomer.

Lemma 59. *Let $\mathbb{X}$ and $\mathbb{Y}$ be non-flat varieties of $\sqrt{'}$ qMV algebras. Suppose $\mathbb{X} \in [\mathbb{V}^{\times}, \mathbb{V}^{\Diamond}]$ and $\mathbb{Y} \in [\mathbb{W}^{\times}, \mathbb{W}^{\Diamond}]$. Then $\mathbb{X} \cap \mathbb{Y} \in [(\mathbb{V} \cap \mathbb{W})^{\times}, (\mathbb{V} \cap \mathbb{W})^{\Diamond}]$ and $\mathbb{X} \vee \mathbb{Y} \in [(\mathbb{V} \vee \mathbb{W})^{\times}, (\mathbb{V} \vee \mathbb{W})^{\Diamond}]$.*

Proof. Follows immediately from Lemma 58. □

Consider now an operator σ acting on varieties of $\sqrt{'}$ qMV algebras: $\sigma(\mathbb{V}) = \mathbb{V}^{\times}$ if $\mathbb{V}$ is a non-flat variety, and $\sigma(\mathbb{V}) = \mathbb{V}$ otherwise.

Theorem 60. *The operator σ is a lattice homomorphism mapping the subvariety lattice $\mathcal{L}^{V}(\sqrt{'}q\mathbb{MV})$ onto its sublattice $\mathcal{L}^{V}(\mathbb{S})$.*

Proof. By Lemma 56, the map σ is well-defined and total. Since each strongly Cartesian variety $\mathbb{V}$ has $\mathbb{V}^{\times} = \mathbb{V}$, the map σ is onto. By Lemma 59 it is a lattice homomorphism. □

To sum up, $\mathcal{L}^{V}(\sqrt{'}q\mathbb{MV})$ has a quite well-behaved sublattice core consisting of flat and strongly Cartesian varieties. In particular this core is countable and, except for the flat part, isomorphic to the lattice of subvarieties of $\mathrm{MV}^{\star}$. The core is surrounded by a halo composed of slices (Fig. 5). We will analyse their structure in some detail in another paper, here let us only announce that some slices contain uncountably many varieties.

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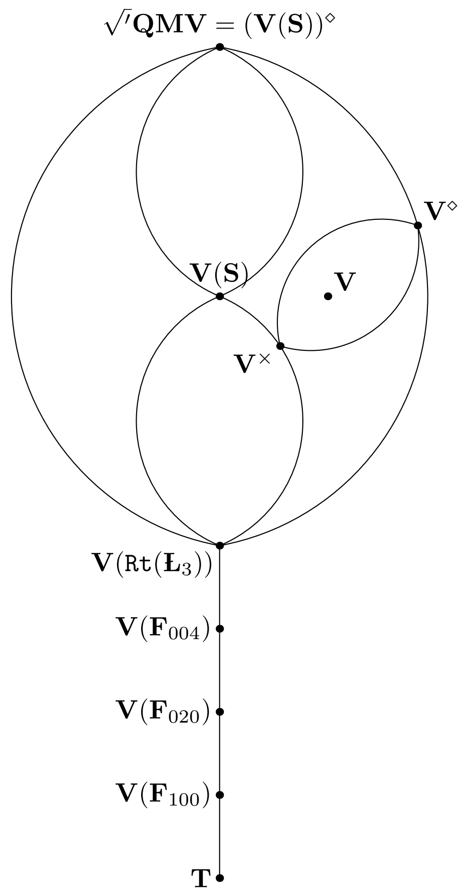

Figure 5: Structure of the lattice $\mathcal{L}^V(\sqrt{\prime}q\mathbb{MV})$.

References

[1] W.J. Blok and I.M.A. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** (2000), pp. 233–257.

[2] W.J. Blok and C.J. van Alten, *The finite embeddability property for residuated lattices, pocrims and BCK algebras*, Algebra Universalis **48** (2002), pp. 253–271.

[3] F. Bou, F. Paoli, A. Ledda and H. Freytes, *On some properties of quasi-MV algebras and $\sqrt{'}$quasi-MV algebras. Part II*, Soft Computing **12**, 4 (2008), pp. 341–352.

[4] F. Bou, F. Paoli, A. Ledda, M. Spinks and R. Giuntini, *The logic of quasi-MV algebras*, Journal of Logic and Computation, forthcoming.

[5] I. Chajda, *Jónsson's Lemma for normally presented varieties*, Mathematica Bohemica **122**, 4 (1997), pp. 381–382.

[6] R. Cignoli, I.M.L. D'Ottaviano and D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Kluwer, Dordrecht, 1999.

[7] I.M.A. Ferreirim, *On Varieties and Quasivarieties of Hoops and Their Reducts*, PhD Thesis, University of Chicago, 1992.

[8] R. Freese and R. McKenzie, Commutator Theory for Congruence Modular Varieties, London Mathematical Society Lecture Notes, vol. 125, Cambridge University Press, Cambridge, 1987.

[9] R. Giuntini, A. Ledda and F. Paoli, *Expanding quasi-MV algebras by a quantum operator*, Studia Logica **87**, 1 (2007), pp. 99–128.

[10] R. Giuntini, A. Ledda and F. Paoli, *Categorical equivalences for $\sqrt{'}$quasi-MV algebras*, Journal of Logic and Computation, forthcoming.

[11] H.P. Gumm and A. Ursini, *Ideals in universal algebras*, Algebra Universalis **19** (1984), pp. 45–55.

[12] A. Ledda, M. Konig, F. Paoli and R. Giuntini, *MV algebras and quantum computation*, Studia Logica **82**, 2 (2006), pp. 245–270.

[13] R. Lewin, M. Sagastume, P. Massey, *MV* algebras*, Logic Journal of the IGPL **12**, 6 (2004), pp. 461–483.

[14] F. Paoli, A. Ledda, R. Giuntini and H. Freytes, *On some properties of quasi-MV algebras and $\sqrt{'}$quasi-MV algebras. Part I*, Reports on Mathematical Logic **44** (2008), pp. 53–85.

[15] A. Salibra, *Topological incompleteness and order incompleteness in lambda calculus*, ACM Transactions on Computational Logic **4**, 3 (2003), pp. 379–401.

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