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MINIMAL NON-RELEVANT LOGICS WITHOUT THE K AXIOM II. NEGATION INTRODUCED AS A PRIMITIVE UNARY CONNECTIVE

A b s t r a c t. In the first part of this paper (RML No. 42) a spectrum of constructive logics without the K axiom is defined. Negation is introduced with a propositional falsity constant. The aim of this second part is to build up logics definitionally equivalent to those displayed in the first part, negation being now introduced as a primitive unary connective. Relational ternary semantics is provided for all logics defined in the paper.

1. Introduction

This paper is a sequel to [6]. Let us expose its results briefly. As is well known, B_+ is Routley and Meyer's basic positive logic in the ternary relational semantics with a set of designated points (see, e.g., [7]). Now, B_{K+}

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is the result of adding the K rule

$$\text{(a).}\ \vdash A \Rightarrow \vdash B \rightarrow A$$

to B_{+}. The logic $\mathrm{B}_{\mathrm{K}+}$ is the basic positive logic in the ternary relational semantics without a set of designated points (see [6]). In [6] a series of logics extending $\mathrm{B}_{\mathrm{K}+}$ and included in "intuitionistic" modal logic S4 are defined. It is proved that all the logics in this series lack the K axiom

$$\text{(b).}\ A \rightarrow (B \rightarrow A)$$

Negation is introduced in these logics with a propositional falsity constant F. Firstly, the logic B_{Km} is defined. It is the result of introducing a minimal negation in $\mathrm{B}_{\mathrm{K}+}$. Secondly, the logic $\mathrm{B}_{\mathrm{Kcdn}}$ is axiomatized by adding to B_{Km} the weak contraposition and double negation axioms. Thirdly, the logic $\mathrm{B}_{\mathrm{Kcdnr}}$ is the result of adding to $\mathrm{B}_{\mathrm{Kcdn}}$ the reductio axioms. Finally, the logic B_{Kj} is axiomatized by adding the EFQ ("E falso quodlibet") axioms to $\mathrm{B}_{\mathrm{Kcdnr}}$.

In [6], it is shown how to strengthen these logics with some strong positive axioms. Ternary relational semantics are defined for these extensions.

The aim of this paper is to provide logics definitionally equivalent to $\mathrm{B}_{\mathrm{Kcdn}}$, $\mathrm{B}_{\mathrm{Kcdnr}}$ and B_{Kj} in which negation is introduced as a primitive unary connective $\neg$ (negation). Let us briefly explain the importance of these results. Let S_{+} be a positive logic, S_{F} , the result of introducing a negation via a falsity constant F and $\mathrm{S}_{\neg}$, the result of introducing a negation with a negation connective. If S_{+} is a relatively strong logic, it is not difficult to find a logic $\mathrm{S}_{\neg'}$ definitionally equivalent to S_{F} and a logic $\mathrm{S}_{F'}$ definitionally equivalent to $\mathrm{S}_{\neg}$. Thus, for example, let J_{+} be positive intuitionistic logic. Intuitionistic negation can be introduced in J_{+} by adding a falsity constant F to the positive language together with the definition

$$\text{(c).}\ \neg A \leftrightarrow (A \rightarrow F)$$

and by adding to J_{+} the axiom

$$\text{(d).}\ F \rightarrow A$$

Alternatively, negation can be introduced by adding to the positive language a (negation) connective and supplementing J_{+} with the following axioms

$$\text{(e). } (A \to \neg B) \to (B \to \neg A)$$
$$\text{(f). } \neg A \to (A \to B)$$

Once F is defined in the latter language, it is not difficult to prove that the two alternatives are definitionally equivalent. So far, so good. But if either the positive language or the negation introduced is weaker, the task of finding definitionally equivalent logics (with the constant and the connective) is, if not impossible, a harder one. Thus, for example, which extension, if any, of $\mathrm{B_{K+}}$ with a negation connective (and without F) is equivalent to the logic $\mathrm{B_{Km}}$ mentioned above?

Although $\mathrm{B_{K+}}$ is not a particularly strong logic, we shall show how to build up logics definitionally equivalent to $\mathrm{B_{Kcdn}}$, $\mathrm{B_{Kcdnr}}$ and $\mathrm{B_{Kj}}$ by introducing negation as a primitive unary connective.

The structure of the paper is as follows. In §2, the logic $\mathrm{B_{Kcdn,\neg}}$ definitionally equivalent to $\mathrm{B_{Kcdn}}$ is introduced. In §3, 4, semantics for $\mathrm{B_{Kcdn,\neg}}$ is provided, and soundness and completeness are proved. In §5, the logic $\mathrm{B_{Kcdnr,\neg}}$ definitionally equivalent to $\mathrm{B_{Kcdnr}}$ is defined and semantics for it together with soundness and completeness are given. In §6, the logic $\mathrm{B_{Kj,\neg}}$ definitionally equivalent to $\mathrm{B_{Kj}}$ is introduced together with a semantics and sketches of soundness and completeness based on [6]. Finally, in §7, the definitional equivalence is proved.

2. $\mathbf{B_{K+}}$ with weak contraposition and weak double negation. The logic $\mathbf{B_{Kcdn,\neg}}$

As pointed out above, $\mathrm{B_{K+}}$ is the result of adding the rule

$$\text{K: } \vdash A \Rightarrow \vdash B \to A$$

to Routley and Meyer's basic positive logic $\mathrm{B_+}$. On the other hand, $\mathrm{B_{K+}}$ models are defined, similarly, as $\mathrm{B_+}$ models except that the set O of designated points is deleted and consequently, validity is defined in respect of all points in K (cf. [6], [7]). Next, the logic $\mathrm{B_{Kcdn,\neg}}$ is defined.

The unary connective $\neg$ (negation) is added to the positive language. Then, the logic $\mathrm{B_{Kcdn,\neg}}$ can be axiomatized by adding to $\mathrm{B_{K+}}$ the following axioms

$$\text{A1. } \neg B \to [(A \to B) \to \neg A]$$
$$\text{A2. } (A \to \neg B) \to (B \to \neg A)$$

The logic $B_{Kcdn,\neg}$ can intuitively be described as the logic B_{K+} plus the weak constructive contraposition and double negation axioms. We note the following theorems and rules of inference of $B_{Kcdn,\neg}$ (a proof for each one of them is sketched to their right):

T1. $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$	A2
T2. $A \to \neg\neg A$	T1
T3. $(A \to B) \to (\neg B \to \neg A)$	A2, T2
T4. $B \to [(A \to \neg B) \to \neg A]$	A1, T2
T5. $A \to [(A \to \neg B) \to \neg B]$	A2, T4
T6. $\neg A \to [A \to \neg(A \to A)]$	A1, A2
T7. $[B \to \neg(A \to A)] \to \neg B$	T4
T8. $\vdash A \Rightarrow \vdash \neg A \to \neg B$	K, T3
T9. $\neg A \to (A \to \neg B)$	T6, T8
T10. $(\neg A \vee \neg B) \to \neg(A \wedge B)$	T3
T11. $(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$	T1, T3

Note that A2 and T3 are the weak contraposition axioms; A1 and T4, the weak "permuted" contraposition axioms, and T2, introduction of double negation. In addition, the following theorems are useful in establishing the definitional equivalence between B_{Kcdn} and $B_{Kcdn,\neg}$: given T7 and T9, it is proved

$$\text{T12. } \neg A \leftrightarrow [A \to \neg(B \to B)] \qquad \text{T7, T9}$$

and by using T12,

T13. $(A \to B) \to \{[B \to \neg\,(A \to A)] \to [A \to \neg\,(A \to A)]\}$	T3, T12
T14. $[B \to \neg\,(A \to A)] \to \{(A \to B) \to [A \to \neg\,(A \to A)]\}$	A1, T12
T15. $A \to \{[A \to \neg\,(A \to A)] \to \neg(A \to A)\}$	T5
T16. $\{A \to [B \to \neg\,(A \to A)]\} \to \{B \to [A \to \neg\,(A \to A)]\}$	A2, T12
T17. $B \to \{[A \to [B \to \neg\,(A \to A)]] \to [A \to \neg\,(A \to A)]\}$	T4, T12

Next, we note the following

Proposition 2.1. *$B_{Kcdn,\neg}$ is well axiomatized in respect of B_{K+}. That is, given B_{K+}, A1 and A2 are mutually independent.*

Proof. By MaGIC, the matrix generator developed by J. Slaney (see [8]). □

Alternative axiomatizations of $\mathrm{B_{Kcdn,\neg}}$ with A1 and A2 and T1-T4 can be proposed (cf. [6]). We end this section with the following remark:

Remark 2.2. The logic $\mathrm{B_{Kcdn,\neg}}$ is deductively equivalent to the logic $\mathrm{B_{Kc2}}$ defined in [5].

3. Semantics for $\mathbf{B_{Kcdn,\neg}}$

A *$B_{\mathrm{Kcdn},\neg}$ model* is a quadruple $\langle K, S, R, \vDash\rangle$ where S is a non-empty subset of K, and K, R and $\vDash$ are defined, in a similar way, as in $\mathrm{B_{K+}}$ models except for the addition of the following clause and postulates:

(i). $a \vDash \neg A$ iff for all b, $c \in K$, $(Rabc \ \&\ c \in S) \Rightarrow b \nvDash A$

P1. $(R^2abcd \ \&\ d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Rbcx \ \&\ Raxy)$

P2. $(R^2abcd \ \&\ d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Racx \ \&\ Rbxy)$

A is $\mathrm{B_{Kcdn,\neg}}$ *valid* ($\vDash_{\mathrm{B_{Kcdn,\neg}}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

The intuitive meaning of clause (i) is briefly discussed in the following section, Remark 4.7.

Before proving soundness, we note two useful (and meaningful) lemmas:

Lemma 3.1. $(a \leq b \ \&\ a \vDash A) \Rightarrow b \vDash A$

Proof. As in the case of $\mathrm{B_{K+}}$ (or $\mathrm{B_+}$), induction on the length of A. □

Lemma 3.2. *$\vDash A \to B$ iff for all $a \in K$ in all $B_{Kcdn,\neg}$ models, $a \vDash A \Rightarrow a \vDash B$.*

Proof. Similarly, as in the case of $\mathrm{B_{K+}}$ (or $\mathrm{B_+}$). □

Now, we prove

Theorem 3.3 (soundness of $\mathrm{B_{Kcdn,\neg}}$)**.** *If $\vdash_{B_{Kcdn,\neg}} A$, then $\vDash_{B_{Kcdn,\neg}} A$.*

Proof. Given the soundness of B_{K+} (cf. [6]), it remains to prove that A1 and A2 are valid.

A1 is $B_{\mathrm{Kcdn},\neg}$ valid: Suppose, for reductio, that A1 is not $B_{\mathrm{Kcdn},\neg}$ valid. By Lemma 3.2, $a \vDash \neg B$, $a \nvDash (A \to B) \to \neg A$ for wff A, B and $a \in K$ in some model. Then, $b \vDash A \to B$, $c \nvDash \neg A$ for b, $c \in K$ such that $Rabc$. By clause (i), $d \vDash A$ for $d \in K$, $e \in S$ such that $Rcde$. Then, R^2abde whence, by P1, $Rbdz$ and $Razu$ for $z \in K$, $u \in S$. So, $z \vDash B$. On the other hand, by $a \vDash \neg B$ and clause (i), $(Raxy \;\&\; y \in S) \Rightarrow x \nvDash A$ for all $x \in K$ and $y \in S$. Consequently, $z \nvDash A$, a contradiction. The proof of the validity of A2 is similar and is left to the reader. □

We end this section with the following proposition:

Proposition 3.4. *Let $\langle K, S, R, \vDash\rangle$ be a $B_{Kcdn,\neg}$ model. Let $\neg A$ be a theorem. Then, A is false in every $a \in S$.*

Proof. Suppose $\neg A$ is a theorem. Then, $\neg A$ is valid by the soundness theorem. Let $a \in S$. As for every $a \in K$, there is some $x \in K$ such that $Rxaa$, clearly $x \vDash \neg A$, and so, $a \nvDash A$ by clause (i). □

So, note that the argument of a negative formula that is a theorem is false at each point of S in each model.

In the next section, the completeness of $B_{\mathrm{Kcdn},\neg}$ in respect of the semantics defined in this one is proved.

4. Completeness of $\mathbf{B_{Kcdn,\neg}}$

We begin by defining the concept of a theory.

Definition 4.1. Let L be a language and S be a logic defined on L. An S-theory is a set of formulas of L closed under adjunction and provable entailment. That is a is an S-theory if whenever A, $B \in A$ then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of S and $A \in a$, then $B \in a$.

Next, we recall the two senses of weak consistency introduced in [5]. These notions will be used in the completeness proof.

Definition 4.2. Let a be an S-theory. Then, a is w1-inconsistent (weakly inconsistent in a first sense) iff $\neg A \in a$, A being a theorem of S (a is w1-consistent iff it is not w1-inconsistent).

Definition 4.3. Let a be an S-theory. Then, a is w2-inconsistent (weakly inconsistent in a second sense) iff $A \in a$, $\neg A$ being a theorem of S (a is w2-consistent iff it is not w2-inconsistent).

In other words, a theory is w1-inconsistent iff it contains the negation of a theorem; and it is w2-inconsistent iff it contains the argument of a negation formula that is a theorem.

Now, let $B_{+,\neg}$ be any negation extension of B_+ in which the rule introduction of double negation (dn)

$$\text{dn.} \ \vdash A \Rightarrow \vdash \neg\neg A$$

holds. It is proved:

Proposition 4.4. *Let a be a $B_{+,\neg}$ theory. Then, if a is w1-inconsistent, then a is w2-inconsistent.*

Proof. Suppose $\neg A \in a$, A being a theorem. By dn, $\neg\neg A$ is also a theorem, and thus, a contains the argument of a negation formula that is a theorem. □

Next, let $B_{+,\neg}$ be any negation extension of B_+ in which the principle of introduction of double negation T2 holds. It is proved:

Proposition 4.5. *Let a be a $B_{+,\neg}$ theory. Then, if a is w2-inconsistent, then a is w1-inconsistent.*

Proof. Suppose $A \in a$, $\neg A$ being a theorem. By T2, $\neg\neg A \in a$, and thus, a contains the negation of a theorem. □

As a corollary of Propositions 4.1 and 4.2, we have:

Proposition 4.6. *Let a be a $B_{Kcdn,\neg}$ theory. Then, a is w1-consistent iff a is w2-consistent.*

Therefore, in $\mathrm{B}_{\mathrm{Kcdn},\neg}$, w1-consistency and w2-consistency are equivalent.

As we shall see, in this logic, consistency has to be understood in one of these two senses. But we remark that w1-consistency and w2-consistency are not, in general, equivalent. In particular, they are not equivalent in theories whose underlying logic is a constructive entailment logic: in these logics, consistency has to be understood as w2-consistency (cf. the logic B_{Km} in [6] or the logics defined in [4]).

On the other hand, we note the following remark on the intuitive meaning of clause (i):

Remark 4.7. Clause (i) is an adaptation of the negation clause characteristic of minimal intuitionistic logic in binary relational semantics. The intuitionistic clause reads:

$$a \vDash \neg A \text{ iff } (Rab \ \ \& \ \ b \in S) \Rightarrow b \nvDash A$$

That is, a formula of the form $\neg A$ is true at point a iff A is false in all consistent points accessible from a. "Consistent" is here understood in the minimal intuitionistic way. So, in the ternary relational semantics, the (minimal) intuitionistic clause would be translated as clause (i). That is, a formula of the form $\neg A$ is true at point a iff A is false at all points b such that $Rabc$ for all consistent points c. "Consistent" is here understood as w-consistent in either of the two senses of this concept.

Before introducing the canonical model, some definitions are needed. A theory is *null* iff it does not contain any wff; it is *regular* iff it contains all theorems. Finally, a theory a is *prime* iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$. Now, let K^T be the set of all $B_{Kcdn,\neg}$ theories. The relation R^T is defined as follows: for all a, b, $c \in K^T$, and wff A, B, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$. Next, let K^C be the set of all prime non-null $B_{Kcdn,\neg}$ theories, S^C be the set of all prime w2-consistent non-null $B_{Kcdn,\neg}$ theories, and R^C be the restriction of R^T to K^C. Finally, $\vDash^C A$ iff $A \in a$. Then, the $B_{Kcdn,\neg}$ *canonical model* is the quadruple $\langle K^C, S^C, R^C, \vDash^C \rangle$. We note the following:

Remark 4.8. Given Proposition 4.6, S^C could, of course, have been defined as the set of all prime w1-consistent theories.

Next, we record a couple of lemmas on non-null theories, and a series of lemmas which are an easy restriction of the corresponding B_+ lemmas to the case of non-null theories (cf. [6]). By the K rule, it is proved:

Lemma 4.9. *Let $a \in K^T$. Then, a is regular iff a is non-null.*

By Lemma 4.9, we have:

Lemma 4.10. *Let a, b be non-null theories. The set $x = \{B \mid \exists A[A \to B \in a \ \& \ A \in b]\}$ is a non-null theory such that $R^T abx$.*

Then, we have,

Lemma 4.11. *Let A be a wff, a a non-null element in K^T and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4.12. *Let a be a non-null element in K^T, $b \in K^T$ and c a prime member in K^C such that $R^T abc$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4.13. *Let $a \in K^T$, b a non-null element in K^T and c a prime member in K^C such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Lemma 4.14. *$a \leq^C b$ iff $a \subseteq b$.*

An important corollary of Lemma 4.11 is:

Lemma 4.15. *If $\nvdash_{B_{Kcdb,\neg}} A$, then there is some $x \in K^C$ such that $A \notin x$.*

Now, we prove that $\mathrm{B_{Kcdn,\neg}}$ is w2-consistent and the primeness lemma.

Proposition 4.16. *The logic $B_{Kcdn,\neg}$ is w2-consistent.*

Proof. Let $\neg A$ be a theorem. By Proposition 3.4, A is not $\mathrm{B_{Kcdn,\neg}}$ valid. So, A is not derivable in $\mathrm{B_{Kcdn,\neg}}$ by the soundness theorem. Consequently, $\mathrm{B_{Kcdn,\neg}}$ is w2-consistent (so, it is, of course, w1-consistent by Proposition 4.5). □

Next, we prove the primeness lemma.

Lemma 4.17 (Primeness lemma)**.** *If a is a w2-consistent $B_{Kcdn,\neg}$ theory, then there is a prime w2-consistent $B_{Kcdn,\neg}$ theory x (i.e., a member of S^C) such that $a \subseteq x$.*

This lemma is immediate from Proposition 4.6 and Proposition 4.18 that follows.

Let $\mathrm{B_{+,\neg}}$ be any negation extension of $\mathrm{B_+}$ in which the De Morgan law

$$\text{dm1. } (\neg A \vee \neg B) \to \neg(A \wedge B)$$

holds. It is proved:

Proposition 4.18. *Let a be a w1-consistent $B_{+,\neg}$ theory. Then, there is some prime w1-consistent $B_{+,\neg}$ theory x such that $a \subseteq x$.*

Proof. Define from a a maximal w1-consistent theory x such that $a \subseteq x$. If x is not prime, then there are wff A, B such that $A \vee B \in x$, $A \notin x$, $B \notin x$. Define the set $[x, A] = \{C \mid \exists D[D \in x \ \ \& \ \ \vdash_{\mathrm{B}_{+,\neg}} (A \wedge D) \to C]\}$. Define $[x, B]$ similarly. It is not difficult to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. By the maximality of x, they are w1-inconsistent. That is, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems C, D. By definitions, we have $\vdash_{\mathrm{B}_{+,\neg}} (A \wedge E) \to \neg C$, $\vdash_{\mathrm{B}_{+,\neg}} (B \wedge E') \to \neg C$ for some E, $E' \in x$. By basic theorems of B_{+}, $\vdash [(A \vee B) \wedge (E \wedge E')] \to (\neg C \vee \neg D)$. So, $\neg C \vee \neg D \in x$, and by dm1, $\neg(C \wedge D) \in x$. But, by adjunction, $\vdash_{\mathrm{B}_{+,\neg}} C \wedge D$. Therefore, if x is not prime, it is w1-inconsistent, which is impossible. $\square$

Consequently, in any logic including B_{+} plus dm1, w1-consistent theories can be extended to prime w1-consistent theories.

Finally, we prove a lemma that will be useful for proving the canonical adequacy of postulate P1.

Lemma 4.19. *Let a, b, c be non-null elements in K^T, and d be a w2-consistent element in K^T such that $R^{T2}abcd$. Then, there are $x \in K^T$ and w2-consistent y in K^T such that $R^T bcx$ and $R^T axy$.*

Proof. Assume the hypothesis of the lemma. Then, $R^T abz$ and $R^T zcd$ for some $z \in K^T$. Define the non-null theories $x = \{B \mid \exists A[A \to B \in b \ \ \& \ \ A \in c]\}$, $y = \{B \mid \exists A[A \to B \in a \ \ \& \ \ A \in x]\}$ such that $R^T bcx$ and $R^T axy$ (cf. Lemma 4.10). We prove that y is w2-consistent. Suppose it is not. Then, $A \in y$, $\neg A$ being a theorem. So, $B \to A \in a$, $C \to B \in b$ for some wff B and $C \in c$. By A1, $(B \to A) \to \neg B$ is a theorem. So, $\neg B \in a$. Again, by A1, $(C \to B) \to \neg C \in a$. Therefore, $\neg C \in z$ ($R^T abz$).

Now, by T6, $\neg C \to [C \to \neg(C \to C)]$. So, $\neg(C \to C) \in d$ ($R^T zcd$), contradicting the w2-consistency of d (note that $\neg\neg(C \to C)$ is a theorem by T2). $\square$

Next, we prove the completeness theorem:

Theorem 4.20 (Completeness of $\mathrm{B}_{\mathrm{Kcdn},\neg}$). *If $\vDash_{B_{Kcdn,\neg}} A$, then $\vdash_{B_{Kcdn,\neg}} A$.*

Proof. Given the completeness of B_{K+}, we have to prove:

1. The set S^C is not empty:

The logic $B_{Kcdn,\neg}$ is w2-consistent by proposition 4.16. Then, by the primeness lemma (Lemma 4.17), there is some $x \in S^C$ such that $B_{Kcdn,\neg} \subseteq x$.

2. Postulates P1, P2 hold in the $B_{Kcdn,\neg}$ canonical model:

We prove that P1 holds. The proof for P2 is similar and is left to reader.

P1 reads canonically as follows: P1^C. $(R^{C2}abcd \ \& \ d \in S^C) \Rightarrow (\exists x \in K^C)(\exists y \in S^C)(R^C bcx \ \& \ R^C axy)$. So, suppose $R^{C2}abcd$, $d \in S^C$. By Lemma 4.19, there are $z \in K^T$ and w2-consistent u in K^T such that $R^T bcz$ and $R^T azu$. By the primeness lemma (Lemma 4.17), there is some $y \in S^C$ such that $u \subseteq y$. Clearly, $R^T azy$. By Lemma 4.13, there is $x \in K^C$ such that $z \subseteq x$ and $R^C axy$. Clearly, $R^C bcx$. Therefore, we have $x \in K^C$, $y \in S^C$ such that $R^C bcx$ and $R^C axy$, as it was required.

3. $\vDash^C$ is a valuation relation satisfying condition (i):

Suppose $a \vDash \neg A$, $R^C abc$, $c \in S^C$, and for reductio, $A \in b$. Then, by T6, $\neg(A \to A) \in c$, contradicting the w2-consistency of c.

Now, suppose $a \nvDash \neg A$. Define the non-null theories $z = \{C \mid \vdash_{B_{Kcdn,\neg}} A \to C\}$, $u = \{C \mid \exists D[D \to C \in a \ \ \& \ \ D \in z]\}$ such that $R^T azu$ (cf. Lemma 4.10). Clearly, $A \in z$. Suppose now that u is not w2-consistent. i.e., $B \in u$, $\neg B$ being a theorem. Then, for some wff C, $C \to B \in a$ and $\vdash_{B_{Kcdn,\neg}} A \to C$. So, $A \to B \in a$. Then, by A1, $\neg A \in a$, contradicting the hypothesis. Now, by the primeness lemma (Lemma 4.17) and Lemma 4.13, u and z are extended to prime w2-consistent y and prime x, respectively, such that $R^C axy$ and $A \in x$ ($A \in z$), as it was required. □

5. $B_{Kcdn,\neg}$ with the reductio axioms: the logic $B_{Kcdnr,\neg}$

The logic $B_{Kcdnr,\neg}$ can be axiomatized by adding to $B_{Kcdn,\neg}$ the axiom:

$$\text{A3. } (A \to B) \to [(A \to \neg B) \to \neg A]$$

The logic $B_{Kcdn,\neg}$ can intuitively be described as the logic B_{K+} plus the weak contraposition, weak double negation and weak reductio axioms. In addition to T1-T17, we note the following theorems of $\mathrm{B}_{\mathrm{Kcdnr},\neg}$:

T18.	$(\vdash A \to B \ \& \vdash A \to \neg B) \Rightarrow \vdash \neg A$	A3
T19.	$\vdash A \to B \Rightarrow \vdash (A \to \neg B) \to \neg A$	A3
T20.	$(A \to \neg B) \to [(A \to B) \to \neg A]$	T2, A3
T21.	$\vdash A \to \neg B \Rightarrow \vdash (A \to B) \to \neg A$	T20
T22.	$(A \to \neg A) \to \neg A$	T19
T23.	$\neg(A \wedge \neg A)$	T18
T24.	$A \to \neg(A \to \neg A)$	A2, T22
T25.	$(A \to \neg B) \to \neg(A \wedge B)$	T19
T26.	$(A \to B) \to \neg(A \wedge \neg B)$	T21
T27.	$(A \wedge B) \to \neg(A \to \neg B)$	A2, T25
T28.	$(A \wedge \neg B) \to \neg(A \to B)$	A2, T26
T29.	$(A \wedge \neg A) \to \neg B$	T9, T23

Note that A3 and T20 are the reductio axioms; T18, T19 and T21, the reductio rules; T22 (or T24 in a contrapositional expression) is special reductio; T23 is the principle of non-contradiction, and T29, a restricted version of the ECQ ("E contradictione quodlibet") axiom. Finally, T25-T28 are part of the classical interdefinition between the conditional and the conjunction.

As in the case of $\mathrm{B}_{\mathrm{Kcdn},\neg}$, the following theorems, which are derivable by using T12, are useful in establishing definitional equivalence.

T30. $[A \wedge [A \to \neg(A \to A)]] \to \neg(A \to A)$ T29, T12

T31. $(\vdash A \to B \ \& \vdash A \to [B \to \neg(A \to A)]) \Rightarrow \vdash A \to \neg(A \to A)$ T18, T12

T32. $[A \to [A \to \neg(A \to A)]] \to [A \to \neg(A \to A)]$ T22, T12

T33. $A \to \{[A \to [A \to \neg(A \to A)]] \to \neg(A \to A)\}$ T15, T32

T34. $\vdash A \to B \Rightarrow \vdash [A \to [B \to \neg(A \to A)]] \to [A \to \neg(A \to A)]$ T19, T12

T35. $\vdash [A \to [B \to \neg(A \to A)]] \Rightarrow \vdash (A \to B) \to [A \to \neg(A \to A)]$ T21, T12

T36. $[A \to [B \to \neg(A \to A)]] \to [(A \wedge B) \to \neg(A \to A)]$ T25, T12

T37. $(A \to B) \to [[A \wedge [B \to \neg(A \to A)]] \to \neg(A \to A)]$ T26, T12

T38. $(A \wedge B) \to \{[A \to [B \to \neg(A \to A)]] \to \neg(A \to A)\}$ T27, T12

T39. $[[A \wedge [B \to \neg(A \to A)]] \to [(A \to B) \to \neg(A \to A)]$ T28, T12

T40. $(A \to B) \to \{[A \to [B \to \neg(A \to A)]] \to [A \to \neg(A \to A)]\}$ A3, T12

T41. $\{A \to [B \to \neg(A \to A)]\} \to \{(A \to B) \to [A \to \neg(A \to A)]\}$ T20, T12

We end this general syntactical description of $B_{\mathrm{Kcdnr},\neg}$ with the following:

Proposition 5.1. *(a) Axiom A1 is not independent (the proof is left to the reader). So, A2 and A3 do axiomatize $B_{Kcdnr,\neg}$, and given B_{K+}, they are mutually independent (proof by MaGIC). (b) Given $B_{Kcdn,\neg}$, A3 can be substituted by any one of T18-T29 (cf. [6]). (c) The logic $B_{Kcdnr,\neg}$ is deductively equivalent to the logic B_{Kc5} defined in [3].*

We now define the semantics for $B_{\mathrm{Kcdnr},\neg}$.

A $B_{\mathrm{Kcdnr},\neg}$ *model* is defined, similarly, as $B_{\mathrm{Kcdn},\neg}$ models except for the addition of the following postulate

P3. $(R^2abcd \ \& \ d \in S) \Rightarrow (\exists x, y \in K)(\exists z \in S)(Racx \ \& \ Rbcy \ \& \ Ryxz)$

A is $B_{\mathrm{Kcdnr},\neg}$ *valid* ($\vDash_{B_{\mathrm{Kcdnr},\neg}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

Given the soundness of $B_{\mathrm{Kcdn},\neg}$, it is clear that to prove the soundness of $B_{\mathrm{Kcdnr},\neg}$, we just have to prove that A3 is $B_{\mathrm{Kcdnr},\neg}$ valid.

Theorem 5.2 (Soundness of $B_{\mathrm{Kcdnr},\neg}$). *If $\vdash_{B_{Kcdnr,\neg}} A$, then $\vDash_{B_{Kcdnr,\neg}} A$.*

A3 is $B_{\mathrm{Kcdnr},\neg}$ valid:

Proof. Suppose, for reductio, $a \vDash A \to B$, $a \nvDash (A \to \neg B) \to \neg A$ for wff A, B and $a \in K$ in some model. Then, $b \vDash A \to \neg B$, $c \nvDash \neg A$ for b, $c \in K$ such that $Rabc$; by clause (i), $d \vDash A$ for $d \in K$, $e \in S$ such that $Rcde$. By P3, $Radz$, $Rbdu$, $Ruzw$ for u, $z \in K$ and $w \in S$. So, $z \vDash B$ ($Radz$) and $u \vDash \neg B$ ($Rbdu$). Now, by clause (i), $(Ruxy \ \& \ y \in S) \Rightarrow x \nvDash A$

for all $x \in K$ and $y \in S$, and by $Ruzw$, $w \in S$ and clause (i), $z \nvDash B$, a contradiction. $\Box$

On the other hand, given the completeness of $\mathrm{B_{Kcdn,\neg}}$, it is obvious that to prove completeness for $\mathrm{B_{Kcdnr,\neg}}$, we just have to prove that P3 holds canonically.

Theorem 5.3 (Completeness of $\mathrm{B_{Kcdnr,\neg}}$). *If* $\vDash_{B_{Kcdnr,\neg}} A$, *then* $\vdash_{B_{Kcdnr,\neg}} A$.

In order to prove that P3 holds canonically, let us first introduce the following definition (cf. Definitions 4.2, 4.3):

Definition 5.4. Let a be an S-theory. Then, a is n-inconsistent (negation inconsistent) iff for some wff A, $A \wedge \neg A \in a$ (a is n-consistent —negation consistent— iff a is not n-inconsistent).

Notice that Definitions 4.2, 4.3 and 5.4 all involve negation. They can be viewed as different ramifications of the notion of (in)consistency with respect to negation in classical logic. Definition 6.2 below introduces yet another concept of (in)consistency, which is an adaptation of absolute (in)consistency to theories of non-classical systems (cf. e.g. [2], §17). We have the three propositions that follow:

Proposition 5.5. *Let* $B_{K+,\neg}$ *be any negation extension of* B_{K+}, *and let* a *be any* $B_{K+,\neg}$ *theory. Then, (a) if* a *is w1-inconsistent, then* a *is n-inconsistent, and (b) if* a *is w2-inconsistent, then* a *is n-inconsistent.*

Proof. (a) Let $\neg A \in a$, A being a theorem. By the K rule, $\neg A \to A$ is also a theorem. So, $A \in a$, and consequently, $A \wedge \neg A \in a$. (b) The proof is similar to case (a). $\Box$

Proposition 5.6. *Let* $B_{K+,\neg}$ *be any negation extension of* B_{K+} *in which the rule*

$$r.\ \vdash B \Rightarrow \vdash (A \wedge \neg A) \to \neg B$$

holds, and let a *be any* $B_{K+,\neg}$ *theory. Then,* a *is n-consistent iff* a *is w1-consistent.*

Proof. By proposition 5.5 and the rule r. $\Box$

Proposition 5.7. *Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which the principle of non-contradiction T23 holds, and let a be any $B_{K+,\neg}$ theory. Then, a is n-consistent iff a is w2-consistent.*

Proof. By Proposition 5.5 and T23. □

A corollary of Propositions 5.6 and 5.7 is:

Proposition 5.8. *Let a be a $B_{Kcdnr,\neg}$ theory. Then, a is n-consistent iff a is w1-consistent iff a is w2-consistent.*

Finally, and, similarly, as in the case of w1-consistency (cf. Proposition 4.18), a proposition on the extension of n-consistent theories to prime n-consistent theories is provable. Let $\mathrm{B_{K+,\neg}}$ be any negation extension of $\mathrm{B_{K+}}$ in which T11 and T23 hold. We leave to the reader the proof of the following proposition (cf. Proposition 4.18).

Proposition 5.9. *Let a be an n-consistent $B_{K+,\neg}$ theory. Then, there is some prime n-consistent $B_{K+,\neg}$ theory x such that $a \subseteq x$.*

Therefore, note that given Proposition 5.8, consistency in $\mathrm{B_{Kcdnr,\neg}}$ can be understood (at our convenience) either as n-consistency or else in any one of the two senses of w-consistency.

Next, we prove a lemma from which the canonical validity of P3 follows immediately.

The $\mathrm{B_{Kcdnr,\neg}}$ *canonical model* is defined, similarly, as the $\mathrm{B_{Kcdn,\neg}}$ canonical model, its items being now referred, of course, to $\mathrm{B_{Kcdnr,\neg}}$ theories (note that S^C is the set of all prime consistent theories, where consistency can equivalently be understood in any one of the three senses discussed so far). Then, it is proved:

Lemma 5.10. *Let a, b, c be non-null elements in K^T and d be an n-consistent element in K^T such that $R^{T2}abcd$. Then, there are x, $y \in K^T$ and n-consistent z in K^T such that R^Tacx, R^Tbcy and R^Tyxz.*

Proof. Assume the hypothesis of the lemma. Then, $Rabu$ and $Rucd$ for some $u \in K^T$. Define the non-null theories $x = \{B \mid \exists A[A \to B \in a \ \ \& \ \ A \in c]\}$, $y = \{B \mid \exists A[A \to B \in b \ \ \& \ \ A \in c]\}$, $z = \{B \mid \exists A[A \to B \in y \ \ \& \ \ A \in x]\}$ such that R^Tacx, R^Tbcy and R^Tyxz. Suppose that z is n-inconsistent. Then, $\neg A \in z$ for some theorem A. Then, $C \to (B \to \neg A) \in$

b, $D \to B \in a$ for some wff B and C, $D \in c$. By T4, $(B \to \neg A) \to \neg B$ is a theorem. So, $[C \to (B \to \neg A)] \to (C \to \neg B)$ is also a theorem. Now, $C \to \neg B \in b$ and by *Rabu*, $C \to \neg B \in u$. Then, $\neg B \in d$ (*Rucd*, $C \in c$). On the other hand, as $D \to D \in u$ (cf. Lemma 4.9), $D \in d$ (*Rucd*, $D \in c$). Therefore, $D \wedge \neg B \in d$. Next, we prove that $\neg(D \wedge \neg B) \in d$, contradicting the n-consistency of d. As $D \to B \in a$, $\neg(D \wedge \neg B) \in a$ by T26. Now, let $E \in b$. By T9, $(D \wedge \neg B) \to \neg E \in a$, whence $E \to \neg(D \wedge \neg B) \in a$ by A2. Then, $\neg(D \wedge \neg B) \in u$ (*Rabu*). By a similar argument and *Rucd*, $\neg(D \wedge \neg B) \in d$. $\square$

Finally, we remark that the canonical validity of P3 follows from Lemma 5.10 in a similar way to which P1 follows from Lemma 4.19.

6. $\mathbf{B_{K+}}$ with intuitionistic negation: the logic $\mathbf{B_{Kj,\neg}}$

The logic $\mathrm{B_{Kj,\neg}}$ can be axiomatized by adding to $\mathrm{B_{Kcdnr,\neg}}$ the axiom

$$\text{A4. } \neg A \to (A \to B)$$

We note that, in addition to T1-T41, the following are provable in $\mathrm{B_{Kj,\neg}}$:

T42. $A \to (\neg A \to B)$	A4, T2
T43. $(A \wedge \neg A) \to B$	A4, T23
T44. $\neg(A \to A) \to B$	T42

The logic $\mathrm{B_{Kj,\neg}}$ can intuitively be described as the logic $\mathrm{B_{K+}}$ plus intuitionistic negation: that is, $\mathrm{B_{K+}}$ plus the weak contraposition, double negation, reductio, ECQ ("E contradictione quodlibet") and EFQ ("E falso quodlibet") axioms (the ECQ axiom is T43; the EFQ axioms are A4 and T42).

Next, we define the semantics.

A $\mathrm{B_{Kj,\neg}}$ *model* is a triple $\langle K, R, \vDash\rangle$ where K, R and $\vDash$ are defined, similarly, as in a $\mathrm{B_{K+}}$ model except that the following postulates and clause are added:

P4. $R^2abcd \Rightarrow (\exists x, y \in K)(Rbcx \ \& \ Raxy)$

P5. $R^2abcd \Rightarrow (\exists x, y \in K)(Racx \ \& \ Rbxy)$

P6. $R^2abcd \Rightarrow (\exists x, y, z \in K)(Racx \ \& \ Rbcy \ \& \ Ryxz)$

(i′). $a \vDash \neg A$ iff for all b, $c \in K$, $Rabc \Rightarrow b \nvDash A$

A is $\mathrm{B_{Kj,\neg}}$ *valid* ($\vDash_{\mathrm{B_{Kj,\neg}}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

It is clear that a $\mathrm{B_{Kj,\neg}}$ model is like a $\mathrm{B_{Kcdn,\neg}}$ model save for the omission of all references to the set S: now $S = K$. Therefore, the soundness of $\mathrm{B_{Kj,\neg}}$ can be proved, similarly, as that of $\mathrm{B_{Kcdnr,\neg}}$ save for this difference: given that in $\mathrm{B_{Kj,\neg}}$ models $S = K$, now A4 is valid.

A4 is $B_{\mathrm{Kj},\neg}$ valid:

Proof. Suppose $a \vDash \neg A$, $a \nvDash A \to B$ for $a \in K$ in some model. Then, $b \vDash A$, $c \nvDash B$ for b, $c \in K$ such that $Rabc$. By clause (i'), $b \nvDash A$, a contradiction (note that in $\mathrm{B_{Kcdnr,\neg}}$, if $c \notin S$, A4 is not $\mathrm{B_{Kcdnr,\neg}}$ valid). □

Thus, we can state the soundness of $\mathrm{B_{Kj,\neg}}$.

Theorem 6.1 (Soundness of $\mathrm{B_{Kj,\neg}}$)**.** *If $\vdash_{B_{Kj,\neg}} A$, then $\vDash_{B_{Kj,\neg}} A$.*

Next, we proceed into proving completeness. We set the following definition (cf. Definitions 4.2, 4.3, 5.4):

Definition 6.2. Let a be an S-theory. Then, a is a-inconsistent (absolutely inconsistent) iff a contains every wff (a is a-consistent —consistent in an absolute sense— iff it is not a-inconsistent).

It is proved:

Proposition 6.3. *Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which A4 holds, and let a be any $B_{K+,\neg}$ theory. Then, a is a-inconsistent iff a is w1-inconsistent.*

Proof. (a) If a is a-inconsistent, then a is obviously w1-inconsistent. (b) Suppose a is w1-inconsistent. Then, $\neg A \in a$, A being a theorem. By the K rule, $\neg A \to A$ is a theorem. So, $A \in a$, whence $A \wedge \neg A \in a$. Then, by T43, for arbitrary wff B, $B \in a$. □

And from this proposition and Proposition 5.8, we have:

Proposition 6.4. *Let a be a $B_{Kj,\neg}$ theory. Then, a is a-inconsistent iff a is n-inconsistent iff a is w1-inconsistent iff a is w2-inconsistent.*

Next, we have the following proposition on the extension of a-consistent theories to prime a-consistent theories (the proof is left to the reader).

Proposition 6.5. *Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which T43 holds, and let a be an a-consistent $B_{K+,\neg}$ theory. Then, there is some prime a-consistent $B_{K+,\neg}$ theory x such that $a \subseteq x$.*

Therefore, note that given Proposition 6.4, consistency in $\mathrm{B_{Kj,\neg}}$ can be understood (at our convenience) either as a-consistency, or as n-consistency or even as any one of the two senses of w-consistency.

But let us return to the completeness proof.

The $\mathrm{B_{Kj,\neg}}$ *canonical model* is the triple $\langle K^C, R^C, \vDash^C\rangle$ where R^C and $\vDash^C$ are defined as in the $\mathrm{B_{Kcdnr,\neg}}$ canonical model, but K^C is now the set of all prime consistent non-null $\mathrm{B_{Kj,\neg}}$ theories, consistency being understood in any of the senses in Proposition 6.4. Now, it is clear that in order to prove the completeness of $\mathrm{B_{Kj,\neg}}$, most of the lemmas proved so far must be modified. We have to prove that all theories defined in these lemmas are consistent. As shown in [6], it can easily be done with the aid of Proposition 6.4. The reader is referred to [6] for details. We end this section by stating the completeness of $\mathrm{B_{Kj,\neg}}$:

Theorem 6.6 (Completeness of $\mathrm{B_{Kj,\neg}}$)**.** *If $\vDash_{B_{Kj,\neg}} A$, then $\vdash_{B_{Kj,\neg}} A$.*

7. The definitional equivalence between $\mathbf{B_{Kcdn}}$ and $\mathbf{B_{Kcdn,\neg}}$ and their respective extensions

Firstly, we recall the logics $\mathrm{B_{Kcdn}}$, $\mathrm{B_{Kcdnr}}$ and $\mathrm{B_{Kj}}$ defined in [6]. We add the propositional falsity constant F to the positive language. Consider then the following axioms and rule of inference.

a1. $(A \to B) \to [(B \to F) \to (A \to F)]$

a2. $(B \to F) \to [(A \to B) \to (B \to F)]$

a3. $A \to [(A \to F) \to F]$

a4. $[A \to (B \to F)] \to [B \to (A \to F)]$

a5. $B \to [[A \to (B \to F)] \to (A \to F)]$

a6. $\vdash [A \to (B \to F)] \Rightarrow \vdash [B \to (A \to F)]$

The logic $\mathrm{B_{Kcdn}}$ is equivalently axiomatized by adding to $\mathrm{B_{K+}}$ any of the six following group of axioms:

a: a2, a4
b: a4, a5
c: a1, a2, a3
d: a1, a2, a6
e: a1, a3, a5
f: a1, a5, a6
Consider now the following axioms and rules of inference:

a7. $[A \wedge (A \to F)] \to F$
a8. $(\vdash A \to B \ \& \ \vdash A \to (B \to F)) \Rightarrow \vdash A \to F$
a9. $[A \to (A \to F)] \to (A \to F)$
a10. $A \to [[A \to (A \to F)] \to F]$
a11. $\vdash A \to B \Rightarrow \vdash [[A \to (B \to F)] \to (A \to F)]$
a12. $\vdash A \to (B \to F) \Rightarrow \vdash (A \to B) \to (A \to F)$
a13. $[A \to (B \to F)] \to [(A \wedge B) \to F]$
a14. $(A \to B) \to [[A \wedge (B \to F)] \to F]$
a15. $(A \wedge B) \to [[A \to (B \to F)] \to F]$
a16. $[(A \wedge B) \to F)] \to [(A \to B) \to F]$
a17. $[A \to (B \to F)] \to [(A \to B) \to (A \to F)]$
a18. $(A \to B) \to [[A \to (B \to F)] \to (A \to F)]$

The logic $\mathrm{B_{Kcdnr}}$ is equivalently axiomatized by adding to $\mathrm{B_{Kcdn}}$ any one of the axioms a8-a18.

Finally, $\mathrm{B_{Kj}}$ is the result of adding to $\mathrm{B_{Kcdnr}}$ the axiom

a19. $F \to A$

Now, negation is introduced in these logics via the definition

D¬. $\neg A \leftrightarrow (A \to F)$

and F is introduced in $\mathrm{B_{Kcdn,\neg}}$, $\mathrm{B_{Kcdnr,\neg}}$ and $\mathrm{B_{Kj,\neg}}$ with the definition

DF. $F \leftrightarrow \neg(A \to A)$

That is, F replaces any wff of the form $\neg(A \to A)$. Note that for any formulas A, B, $\neg(A \to A)$ and $\neg(B \to B)$ are equivalent by T8, and so, the defining formula does not depend on the choice of A.

We shall now discuss the notion of definitional equivalence summarily. We shall understand this notion as "definitional equivalence via translations", which can briefly be defined as follows. Let L1 and L2 be two logics in different languages, t1 a set of terms absent in L2, and t2, a set of terms absent in L1. Then, L1 and L2 are definitional equivalent iff there are definitions of t1 in terms of L2 (Dt1) and definitions of t2 in terms of L1 (Dt2) such that L1 $\cup$ {Dt2} = L2 $\cup$ {Dt1} (x $\cup$ y is the deductive closure of the union of x and y, and definitions are expressed as a set of suitable biconditionals). It is important to note that it is not sufficient to prove L1 $\subseteq$ L2 $\cup$ {Dt1} and L2 $\subseteq$ L1 $\cup$ {Dt2} . It additionally has to be shown that Dt2 is provable in L2 $\cup$ {Dt1} and that Dt1 is provable in L1 $\cup$ {Dt2} (cf. [1]). Then, we have:

Proposition 7.1. *$D\neg$ is provable in $B_{Kcdn,\neg} \cup \{DF\}$*

Proof. By T6 and DF, $\neg A \to (A \to F)$. By T7 and DF, $(A \to F) \to \neg A$. So, $\neg A \leftrightarrow (A \to F)$. $\square$

Note that an immediate corollary is:

Proposition 7.2. *(a) $D\neg$ is provable in $B_{Kcdnr,\neg} \cup \{DF\}$. (b) $D\neg$ is provable in $B_{Kj,\neg} \cup \{DF\}$*

Next, it is proved:

Proposition 7.3. *DF is provable in $B_{Kcdn} \cup \{D\neg\}$*

Proof. (a) By the K rule, $(A \to A) \to (F \to F)$. So, $F \to [(A \to A) \to F]$ by a4, i.e., $F \to \neg(A \to A)$ by D$\neg$. (b) By a3, $[(A \to A) \to F] \to F$, i.e., $\neg(A \to A) \to F$ by D$\neg$. Therefore, $F \leftrightarrow \neg(A \to A)$. $\square$

Again, an immediate corollary is :

Proposition 7.4. *(a) DF is provable in $B_{Kcdnr} \cup \{D\neg\}$. (b) DF is provable in $B_{Kj} \cup \{D\neg\}$.*

On the other hand, we prove:

Proposition 7.5. *(a) $B_{Kcdn,\neg} \subseteq B_{Kcdn} \cup \{D\neg\}$. (b) $B_{Kcdn} \subseteq B_{Kcdn,\neg} \cup \{DF\}$.*

Proof. (a) A1 and A2 are a2 and a4 by D¬. (b) a1-a5 are T13-17, respectively (a6 is immediate by T16). □

Proposition 7.6. *(a)* $B_{Kcdnr,\neg} \subseteq B_{Kcdnr} \cup \{D\neg\}$.
(b) $B_{Kcdnr} \subseteq B_{Kcdnr,\neg} \cup \{DF\}$.

Proof. (a) A3 is a18 by D¬. (b) a7-a18 are T30-T41, respectively. Then, Proposition 7.6 follows from Proposition 7.5. □

Proposition 7.7. *(a)* $B_{Kj,\neg} \subseteq B_{Kj} \cup \{D\neg\}$. *(b)* $B_{Kj} \subseteq B_{Kj,\neg} \cup \{DF\}$.

Proof. (a) By a19, $(A \to F) \to (A \to B)$, i.e., A4 by D¬. (b) T44 is a19 by DF. Then, Proposition 7.7 follows from Proposition 7.6. □

Therefore, given Propositions 7.1 and 7.3, $\mathrm{B_{Kcdn}}$ and $\mathrm{B_{Kcdn,\neg}}$ are definitionally equivalent by Proposition 7.5; given Propositions 7.2 and 7.4, $\mathrm{B_{Kcdnr}}$ and $\mathrm{B_{Kcdnr,\neg}}$ are definitionally equivalent by Proposition 7.6, and finally, given Propositions 7.2 and 7.4, $\mathrm{B_{Kj}}$ and $\mathrm{B_{Kj,\neg}}$ are definitionally equivalent by Proposition 7.7.

We end this paper with two remarks:

1. In $\mathrm{B_{Kj,\neg}}$, F could have been introduced by the alternative definition

   $$\mathrm{D}F'.\ F \leftrightarrow (A \wedge \neg A)$$

   because for any wff A, B, $A \wedge \neg A$ and $B \wedge \neg B$ are equivalent by T43.

2. As it was noted in the introduction, a series of extensions of $\mathrm{B_{Kcdn}}$, $\mathrm{B_{Kcdnr}}$ and $\mathrm{B_{Kj}}$ with some strong positive axioms are defined in [6]. Given that $\mathrm{B_{Kcdn,\neg}}$, $\mathrm{B_{Kcdnr,\neg}}$ and $\mathrm{B_{Kj,\neg}}$ are definitionally equivalent to $\mathrm{B_{Kcdn}}$, $\mathrm{B_{Kcdnr}}$ and $\mathrm{B_{Kj}}$, respectively, definitionally equivalent extensions of the former with the same positive axioms are easily defined. The details are left to the reader.

Notes

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