REPORTS ON MATHEMATICAL LOGIC
41 (2006), 173–194

Yosuke KATOH, Tomasz KOWALSKI, Masaki UEDA

ALMOST MINIMAL VARIETIES RELATED TO FUZZY LOGIC

In memory of Willem Blok

A b s t r a c t. We present constructions producing continua of *almost minimal* subvarieties of certain varieties related to fuzzy logic. We also prove that there are only countably many almost minimal varieties of Hajek's BL-algebras — all of them rather well known. Some contrasting results on varieties satisfying the 2-potency condition $x^3 = x^2$ are also included. The uncountability results have circulated rather widely in preprint (cf. [10]); this paper is meant to emphasise a general scheme that our constructions fall under.

1 Introduction

By a *variety of logic* we mean any variety of algebras that constitutes semantics for a logical calculus. We deliberately leave it at this vague level, although in all our examples the relation between the logic and the variety

Received 22 March 2006

will be that of *algebraizability in the sense of Blok-Pigozzi* (cf. [3]). A variety is minimal, if it has only one, trivial, proper subvariety. A variety will be called *almost minimal* if it has only one nontrivial proper subvariety. Typically, minimal and almost minimal varieties are of interest as elements of the lattice of subvarieties of some larger variety. In such a context, minimal varieties are atoms, and almost minimal ones are elements of height 2 with only one subcover.

Varieties that interest us here will all be varieties of $\mathbf{FL_{ew}}$-algebras corresponding to certain generalisations of Hajek's *basic logic* (cf. [7]) that have been considered by fuzzy logicians (who are, we hasten to add, logicians working in fuzzy logic, not logicians without sharp spatio-temporal localisation). An $\mathbf{FL_{ew}}$-algebra is an algebra $\langle A; \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ of the type $\langle 2, 2, 2, 2, 0, 0\rangle$, such that the following conditions hold:

- $\langle A; \wedge, \vee, \cdot, 0, 1\rangle$ is a bounded lattice,
- $\langle A; \cdot, 1\rangle$ is a commutative monoid,
- $x \cdot y \leq z$ if and only if $y \leq x \rightarrow z$.

The operations '$\cdot$' and '$\rightarrow$' are referred to as *fusion* or *multiplication*, and *residuation*, respectively. One property of $\mathbf{FL_{ew}}$-algebras that we will use later is the following distributivity of fusion over join:

$$(\bigvee X) \cdot (\bigvee Y) = \bigvee \{x \cdot y : x \in X, y \in Y\} \tag{1}$$

which holds for any subsets X and Y of A provided that the joins on the left-hand side exist. Another is the following:

$$a \rightarrow b = \bigvee \{c \in A : c \cdot a \leq b\} \tag{2}$$

which always holds. In particular, in a finite lattice $\mathbf{L}$ with a monoid multiplication that is monotone with respect to the lattice ordering, condition (1) always holds and so (2) can be taken as a definition of residuation. This definition however involves joins over the whole universe of the lattice $\mathbf{L}$ and thus residuation will typically not be term definable.

The three defining conditions of $\mathbf{FL_{ew}}$-algebras can be expressed by equations, so they indeed define a variety, which we call $\mathsf{FL_{ew}}$. The variety $\mathsf{FL_{ew}}$ is arithmetical, has CEP and is congruence 1-regular. We will make use of the latter property rather often and without further mention, so let us also recall that it amounts to the fact that congruences are completely determined by their cosets of 1, which turn out to be lattice filters closed under monoid multiplication. Such filters will be called *congruence filters* from now on. For more on $\mathbf{FL_{ew}}$-algebras and their connection to *substructural logics* the reader may consult [11]. A wealth of results on closely related algebraic structures called *residuated lattices* is to be found in [8]. The reader should be aware that in some papers written before 2002 the name residuated lattices was used narrowly: only for what we now call $\mathbf{FL_{ew}}$-algebras. We also recall that $\mathsf{FL_{ew}}$ is the *equivalent algebraic semantics* (cf. [3]) for the logic $\mathbf{FL_{ew}}$, defined by dropping contraction from the Gentzen calculus $\mathbf{LJ}$ for intuitionistic logic.

The logic $\mathbf{FL_{ew}}$ is weak enough to contain various generalizations of Hajek's basic logic that we have already alluded to. Since all of these are algebraizable, we will speak about them only in terms of the corresponding varieties. To begin with, the variety BL of *basic logic algebras* is the subvariety of $\mathsf{FL_{ew}}$ satisfying:

- $(x \to y)x = x \wedge y$,
- $(x \to y) \vee (y \to x) = 1$.

One natural generalisation of BL is obtained by dropping the first of the above identities while retaining the second. We will refer to it as the variety of *linear* $\mathbf{FL_{ew}}$-algebras. Its logical counterpart is known as *monoidal t-norm logic* (cf. [6]). We will also consider, for each positive integer n, the following identity:

$$x^{n+1} = x^n \tag{E_n}$$

The identity E_1 is equivalent to imposing contraction on the logic, so E_n may be regarded as a weak form of contraction; its proof theoretical import being that $n+1$ instances of a formula on the left-hand side of a sequent can be contracted to n instances. Algebraically, the presence of E_n makes congruences equationally definable, which will be of use in the constructions to come.

Another subvariety of $\mathsf{FL_{ew}}$ we will consider is the variety of *involutive* $\mathbf{FL_{ew}}$-algebras, that is the subvariety of $\mathsf{FL_{ew}}$ defined by

$$\neg\neg x = x \tag{DN}$$

where $\neg x$ is defined as $x \to 0$. This is the *double negation law* in logic, and among **BL**-algebras it defines the variety MV of *Łukasiewicz algebras.* Since these are models of Łukasiewicz's many-valued logic, they are better known as **MV**-algebras. Thus, involutive $\mathbf{FL_{ew}}$-algebras can be thought of as a generalisation of **MV**-algebras, which is perhaps not very exciting in itself but it will help to pose a question at the end of the paper. We will denote the variety of involutive $\mathbf{FL_{ew}}$-algebras by $\mathsf{IFL_{ew}}$.

2 General strategy

Let $\mathbf{A}$ be an si $\mathbf{FL_{ew}}$-algebra. We say that $\mathbf{A}$ is *stiff* iff (1) $\mathbf{A}$ has exactly one proper nontrivial quotient isomorphic to the two element BA, (2) $\mathbf{A}$ has no proper subalgebras other than the two element BA.

Lemma 1. *Let $\mathbf{A}$ be a stiff $\mathbf{FL_{ew}}$-algebra, with the monolith μ. The following hold in $\mathbf{A}$:*

(i) *there is precisely one $b \in A \setminus \{0, 1\}$, with $b^2 = b$;*

(ii) *for any $a \in A$, $a \in F_\mu$ iff $a \geq b$;*

(iii) *for any $a \in A$, $a \in F_\mu$ or $\neg a \in F_\mu$;*

(iv) **Con A** *is a three-element chain.*

Moreover, if for some positive integer n, $\mathbf{A} \in \mathsf{E}_n$, all the above can be expressed by first-order formulas.

Proof. Since (i) is already a first-order formula, the claim follows by noticing that for every $c \in A \setminus \{0, 1\}$, with $c^2 = c$ has $\{x \in A : x \geq c\}$ as a congruence filter, and thus there is precisely one such, since $\mathbf{A}$ is stiff. For (ii) note that by (i) b is a definable constant; moreover, by EDPC for E_n, we can express '$a \in F_\mu$' by a first-order formula. If (iii) did not hold, then $\mathbf{A}/\mu$ would fail to be a BA. Finally, (iv) holds in $\mathbf{A}$ by definition, and is equivalent to the following: for any $a \in A$, if $a \notin F_\mu$, then the congruence generated by $(a, 1)$ is the full congruence. This, again by EDPC, can be expressed as a first-order formula. □

Lemma 2. *For any equation $\sigma \approx \tau$, we have: $\sigma \approx \tau$ is true in the two-element BA iff $\sigma \to \tau \in F_\mu$ and $\tau \to \sigma \in F_\mu$ iff $\mathbf{A} \models \phi(\sigma, \tau)$, where*

$\phi(\sigma,\tau)$ is a certain first-order formula definable from σ and τ. In particular, the fact that $\mathbf{A}/\mu$ is isomorphic to the two-element BA is expressible by a first-order formula.

Proof. The first equivalence follows from the definition of stiff $\mathbf{FL_{ew}}$-algebra and the fact that $a \to b \in F_\mu$ iff $a/\mu \leq b/\mu \in \mathbf{A}/\mu$. For the second it suffices to recall that '$a \in F_\mu$' is first-order definable. For the last statement note that $\mathbf{A}/\mu$'s being isomorphic to the two-element BA could be restated as the conjunction of '$\mathbf{A}/\mu$ has precisely two elements' and 'all equations that hold in BAs hold in $\mathbf{A}/\mu$'. The former is expressible by a first-order sentence, regardless of the type (provided we have identity in the type). The latter is further equivalent to 'all axioms of BAs hold in $\mathbf{A}/\mu$' and that, by finite axiomatisability of BAs, is a finite conjunction of first-order formulas. □

Lemma 3. *Let $\mathbf{A} \in \mathsf{E}_n$ be stiff. All the claims from Lemmas 1 and 2 hold for any ultrapower $\mathbf{A}^I/U$ of $\mathbf{A}$.*

Proof. As $\mathbf{A}$ and $\mathbf{A}^I/U$ are elementarily equivalent, $\mathbf{A}^I/U$ satisfies all the properties of $\mathbf{A}$ expressible by first-order formulas. Since $\mathbf{A} \in \mathsf{E}_n$, all the claims from Lemma 1 are such. Thus, the congruence lattice of $\mathbf{A}^I/U$ is a three-element chain, and therefore $\mathbf{A}^I/U$ is si. Let ν stand for its monolith. Then, '$a \in F_\nu$' is first-order definable on $\mathbf{A}^I/U$, which suffices to conclude that the claims from Lemma 2 also hold. □

Lemma 4. *Let $\mathbf{A}$ be a stiff algebra in E_n, for some $n \in \omega$, and let $\mathbf{C}$ be an si algebra in $V(\mathbf{A})$ nonisomorphic to the two-element BA. Then, $\mathbf{A}$ is a subalgebra of $\mathbf{C}$.*

Proof. By Jónsson Lemma, $\mathbf{C}$ belongs to $HSP_U(\mathbf{A})$. By CEP, $\mathbf{C} \in SHP_U(\mathbf{A})$. Thus, there is an ultrapower $\mathbf{A}^I/U$, with I any set of indices and U an ultrafilter on I, and a congruence ψ on $\mathbf{A}^I/U$, such that $\mathbf{C} \subseteq (\mathbf{A}^I/U)/\psi$. By Lemma 3, $\mathbf{A}^I/U$ is si, and the quotient of $\mathbf{A}^I/U$ by its monolith ν is isomorphic to the two-element BA. Thus, our congruence ψ can only be trivial. It follows that $\mathbf{C} \subseteq \mathbf{A}^I/U$. By Lemma 1 again, there is precisely one element $b \in A^I/U \setminus \{0,1\}$, with $b^2 = b$, and, moreover, any $a \in A^I/U \setminus \{0,1\}$ has either $a^n = b$ or $(\neg a)^n = b$. Since the canonical embedding of $\mathbf{A}$ into $\mathbf{A}^I/U$ sends idempotent elements to idempotent elements, we have that b generates $\mathbf{A}$ as a subalgebra of $\mathbf{A}^I/U$. Thus, $\mathbf{A} \subseteq \mathbf{C}$. □

Lemma 5. *Each stiff member of* E_n*, for any* $n \in \omega$*, generates an almost minimal variety. If* $\mathbf{A} \in \mathsf{E}_n$ *and* $\mathbf{B} \in \mathsf{E}_n$ *are nonisomorphic stiff algebras, then* $V(\mathbf{A}) \neq V(\mathbf{B})$*.*

Proof. Let $\mathbf{A}$ be a stiff algebra in E_n, and $\mathbf{C}$ be any non-Boolean si algebra in $V(\mathbf{A})$. Clearly, $V(\mathbf{A}) \supseteq V(\mathbf{C})$. By Lemma 4, $\mathbf{A} \subseteq \mathbf{C}$. Thus, $V(\mathbf{A}) = V(\mathbf{C})$, which proves the first statement.

For the second, it suffices to prove that $\mathbf{A} \notin V(\mathbf{B})$. Suppose the contrary, then by Lemma 4 we get $\mathbf{B} \subseteq \mathbf{A}$. Since $\mathbf{A}$ is stiff, it has only two subalgebras: the two-element BA and itself. Since the two-element BA is not stiff, we get that $\mathbf{B} \cong \mathbf{A}$. This contradicts the assumption. $\square$

As there are only countably many finite stiff $\mathbf{FL_{ew}}$-algebras, our purpose requires constructing infinite ones. This, however, is much less difficult than it may seem, at least for varieties with EDPC, since "stiffness", being a first-order property, carries over to ultraproducts. Assume that we work within a subvariety V of E_k, for some positive integer k. Suppose that for each $n \in \omega$, we have a finite stiff $\mathbf{FL_{ew}}$-algebra $\mathbf{A}_n$, with $|A_n| > n$. Recall from Lemma 1 that each of these has a unique idempotent element different from 0 and 1. We will reserve the symbol $\star$ for it. Let now $\mathbf{A} = \prod_{n\in\omega} \mathbf{A}_n/U$, be an ultraproduct of all the $\mathbf{A}_n$ by a free ultrafilter U on ω. Then, let $\mathbf{A}^\star$ be the subalgebra of $\mathbf{A}$ generated by $\star = \langle \star_n : n \in \omega\rangle/U$.

Lemma 6. *The algebra* $\mathbf{A}^\star$ *is an infinite stiff* $\mathbf{FL_{ew}}$*-algebra.*

Proof. By EDPC, the ultraproduct $\mathbf{A}$ is si, and its congruence lattice is a three-element chain. Thus, the same goes for $\mathbf{A}^\star$, and to show that it is stiff we only need to prove that every $a \in A^\star \setminus \{1, 0\}$ generates $\mathbf{A}^\star$. Clearly, it suffices to show that a generates $\star$. If $a \in F_\star$ (with $F_\star$ being the monolithic congruence filter), then $a^n = \star$. Otherwise, $\neg a \in F_\star$, and thus $(\neg a)^n = \star$.

Now to show that $\mathbf{A}^\star$ is infinite, note first that the elements generated by $\star$ in $\mathbf{A}$ are congruence classes of elements generated in $\prod_{n\in\omega} \mathbf{A}_n$ by $\langle \star_n : n \in \omega\rangle$. As usual we view the generation process as follows: for any $n \in \omega$, let $G^0_n = \{\star_n\}$, and $G^{k+1}_n = \{F(g) : g \in G^k_n, F - \text{a basic operation}\}$. Suppose $\mathbf{A}^\star$ is finite. Then, by properties of ultraproducts, any element x generated by $\langle \star_n : n \in \omega\rangle$ at every $k+1$-th stage of generation "later" than some fixed m, belongs to G^k_i at almost all coordinates i. This implies that $G_i^{k+1} = G_i^k$, for almost all i. Since the type is finite, $|G_i^k|$ is bounded by a finite number that does not depend on i (its very rough estimate is $\underbrace{4(4(\cdots(4}_{k-1}\cdot 3\overbrace{^{2})^{2}\ldots)^{2}}^{k-1}$, for $k \geq 2$). It follows that infinitely many $\mathbf{A}_i$ have sizes bounded by the same finite number, which contradicts the assumption that $|A_n| > n$ for each n. $\square$

It follows from Lemma 6 that one way of constructing uncountably many almost minimal varieties of $\mathbf{FL_{ew}}$-algebras can go as follows. Working within E_k, for some positive integer k, construct uncountably many families of finite stiff algebras, each family containing algebras of unbounded size. Say, for each $S \subseteq \omega$, construct a family $(\mathbf{A}_{S,n})_{n\in\omega}$, with $|\mathbf{A}_{S,n}| \geq n$, for each $n \in \omega$. Then, for each S, the algebra $\mathbf{A}_S^\star$, is stiff. The task will be completed if we manage to show that $\mathbf{A}_S^\star$ is nonisomorphic to $\mathbf{A}_T^\star$, whenever $S \neq T$.

3 Three particular constructions

We will present three constructions employing the general strategy outlined in the previous section, the first with some detail, the other two rather briefly. Let us also add that all our constructions here have a direct ancestor in [12].

3.1 The involutive case

Let S be any subset of ω. For any positive integer K, define C_K^S be the disjoint union of the sets: $A = \{a_0, \ldots, a_K\}$, $B = \{b_{-1}, b_0, \ldots, b_K\}$, $M = \{m_{-1}, m_0, \ldots, m_K\}$, $N = \{n_0, \ldots, n_K\}$, and $E = \{\mathbf{0}, \mathbf{1}\}$. C_K^S is partially ordered (see Figure 2) by the transitive reflexive closure of the relation '$\triangleright$' defined below:

- $\mathbf{1} \triangleright a_0$, $n_0 \triangleright \mathbf{0}$;
- $a_i \triangleright a_j$ iff $i < j$, $b_i \triangleright b_j$ iff $i < j$;
- $m_i \triangleright m_j$ iff $i > j$, $n_i \triangleright n_j$ iff $i > j$;
- $a_K \triangleright b_{-1}$, $m_{-1} \triangleright n_K$;

- $b_0 \triangleright m_K$, $b_K \triangleright m_0$.

Multiplication on C_K^S is defined by putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in C_K^S$;
- $a_i \cdot a_j = \begin{cases} b_{\max\{i,j\}-1}, & \text{if } i \neq j,\ \max\{i,j\} \in S, \text{ or} \\ & \quad \min\{i,j\} = 0, \\ b_{\max\{i,j\}}, & \text{if } \min\{i,j\} > 0,\ \max\{i,j\} \notin S, \text{ or} \\ & \quad i = j > 0; \end{cases}$
- $a_i \cdot b_j = b_K$;
- $a_i \cdot n_j = \begin{cases} \mathbf{0}, & \text{if } i \geq j, \\ n_0, & \text{if } i < j; \end{cases}$
- $a_i \cdot m_j = \begin{cases} n_0, & \text{if } i > j, \\ n_1, & \text{if } i = j \in \{0,1\}, \text{ or} \\ & \quad i = j > 1,\ j \notin S, \\ n_j, & \text{if } i = j > 1,\ j \in S, \text{ or} \\ & \quad 0 < i < j,\ j \notin S, \\ n_{j+1}, & \text{if } 0 = i < j < K, \text{ or} \\ & \quad 0 < i < j < K,\ j \in S, \\ m_{-1}, & \text{if } 0 = i < j = K, \text{ or} \\ & \quad 0 < i < j = K,\ K \notin S; \end{cases}$
- $b_i \cdot b_j = b_K$;
- $b_i \cdot m_j = \begin{cases} \mathbf{0}, & \text{if } i \geq j, \\ n_0, & \text{if } i < j; \end{cases}$
- $b_i \cdot n_j = \mathbf{0}$;
- $m_i \cdot m_j = \begin{cases} \mathbf{0}, & \text{if } \min\{i,j\} \leq 0, \\ n_0, & \text{otherwise}; \end{cases}$
- $m_i \cdot n_j = \mathbf{0}$;

- $n_i \cdot n_j = \mathbf{0}$.

Residuation on C_K^S is defined by putting:

- $x \to y = \bigvee\{z \in C_K^S : z \cdot x \leq y\}$.

Finally, we define $\mathbf{C}_K^S$ to be the algebra $\langle C_K^S; \vee, \wedge, \cdot, \to, \mathbf{0}, \mathbf{1}\rangle$.

Lemma 7. *The algebra $\mathbf{C}_K^S$ is a $\mathbf{FL_{ew}}$-algebra.*

Proof. We have to prove two things: that multiplication above, commutative by definition, is also associative, and that multiplication distributes over join, i.e., $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$ holds. For associativity we have to consider several cases, of which only one is not straightforward and is dealt with below:

$$(a_i \cdot a_j) \cdot m_k = \begin{cases} b_{j-1} \cdot m_k, & \text{if } i = 0, \\ b_{i-1} \cdot m_k, & \text{if } j = 0, \\ b_{j-1} \cdot m_k, & \text{if } 0 < i < j,\ j \in S, \\ b_i \cdot m_k, & \text{if } 0 < i = j, \\ b_j \cdot m_k, & \text{if } 0 < i < j,\ j \notin S, \\ b_i \cdot m_k, & \text{if } 0 < i > j,\ i \notin S; \end{cases}$$

which yields further:

$$(a_i \cdot a_j) \cdot m_k = \begin{cases} \mathbf{0}, & \text{if } j-1 \geq k,\ i=0, \text{ or} \\ & \quad i-1 \geq k,\ j=0, \text{ or} \\ & \quad j-1 \geq k,\ 0<i<j,\ j \in S, \text{ or} \\ & \quad i-1 \geq k,\ 0<j<i,\ i \in S, \text{ or} \\ & \quad i \geq k,\ 0<i=j, \text{ or} \\ & \quad j \geq k,\ 0<i<j,\ j \notin S, \text{ or} \\ & \quad i \geq k,\ 0<j<i,\ i \notin S; \\ n_0, & \text{if } j-1<k,\ i=0, \text{ or} \\ & \quad i-1<k,\ j=0, \text{ or} \\ & \quad j-1<k,\ 0<i<j,\ j \in S, \text{ or} \\ & \quad i-1<k,\ 0<j>i,\ i \in S, \text{ or} \\ & \quad i<k,\ 0<i=j, \text{ or} \\ & \quad j<k,\ 0<i<j,\ j \notin S, \text{ or} \\ & \quad i<k,\ 0<j<i,\ i \notin S. \end{cases}$$

Then, changing the bracketing, we get:

$$a_i \cdot (a_j \cdot m_k) = \begin{cases} a_i \cdot n_0, & \text{if } j>k, \\ a_i \cdot n_1 & \text{if } j=k \in \{0,1\}, \text{ or} \\ & \quad j=k>1,\ k \notin S, \\ a_i \cdot n_k, & \text{if } j=k>1,\ k \in S, \text{ or} \\ & \quad 0<j<k,\ k \notin S, \\ a_i \cdot n_{k+1}, & \text{if } 0=j<k<K, \text{ or} \\ & \quad 0<j<k<K,\ k \in S, \text{ or} \\ a_i \cdot m_{-1}, & \text{if } 0=j<k=K, \text{ or} \\ & \quad 0<j<k=K,\ K \in S. \end{cases}$$

which finally yields:

$$a_i \cdot (a_j \cdot m_k) = \begin{cases} \mathbf{0}, & \text{if } j > k, \text{ or} \\ & i \geq 1,\ j = k \in \{0,1\}, \text{ or} \\ & i \geq 1,\ j = k > 1,\ k \notin S, \text{ or} \\ & i \geq k,\ j = k > 1,\ k \in S, \text{ or} \\ & i \geq k,\ 0 < j < k,\ k \notin S, \text{ or} \\ & i \geq k+1,\ 0 = j < k < K, \text{ or} \\ & i \geq k+1,\ 0 < j < k < K,\ k \in S; \\ n_0, & \text{if } i = 0,\ j = k \in \{0,1\}, \text{ or} \\ & i = 0,\ j = k > 1,\ k \notin S, \text{ or} \\ & i < k,\ j = k > 1,\ k \in S, \text{ or} \\ & i < k,\ 0 < j < k,\ k \notin S, \text{ or} \\ & i < k+1,\ 0 = j < k < K, \text{ or} \\ & i < k+1,\ 0 < j < k < K,\ k \in S, \text{ or} \\ & 0 = j < k = K, \text{ or} \\ & 0 < j < k = K,\ K \in S. \end{cases}$$

The proof now reduces to a series of tedious case-by-case calculations confirming that $a_i \cdot (a_j \cdot m_k) = \mathbf{0}$ if and only if $(a_i \cdot a_j) \cdot m_k = \mathbf{0}$.

For distributivity of multiplication over join, we proceed in two steps. First, we show that multiplication is monotone, i.e., $x \leq y$ implies $z{\cdot}x \leq z{\cdot}y$. Out of a number of cases only two deserve attention:

The first is $z = a_i$, $x = a_j$, $y = a_k$. As $a_j \leq a_k$, we have $j \geq k$. We may assume $j > k$. Then, if $i \geq j > k$, both $a_i \cdot a_j$ and $a_i \cdot a_k$ are equal to either of b_i, b_{i-1}. The only dubious case arises when $a_i \cdot a_j = b_{i-1}$. This, however, can happen only if $i \in S$, and then $a_i \cdot a_k = b_{i-1}$ as well. If $j > i \geq k$, we get $a_i \cdot a_j \in \{b_{j-1}, b_j\}$ and $a_i \cdot a_k \in \{b_{i-1}, b_i\}$. Since $j > i$, we have $b_{j-1} \leq b_i$, which establishes the claim for all subcases. Finally, if $j > k > i$, we have $a_i \cdot a_j \in \{b_{j-1}, b_j\}$ and $a_i \cdot a_k \in \{b_{k-1}, b_k\}$. As $j > k$, the previous reasoning applies.

The second is $z = a_i$, $x = m_j$, $y = m_k$. Here we have $j \leq k$, and we may also assume $j < k$. Then, if $i \leq j$, we have $a_i \cdot m_j \in \{n_1, n_j, n_{j+1}\}$, and only the cases (1) $a_i \cdot m_j = n_j$ and (2) $a_i \cdot m_j = n_{j+1}$ are not straightforward.

Case (1) splits into two: (1a) $i = j > 1$ and $j \in S$, in which case $k > i > 1$, and thus $a_i \cdot m_k \in \{n_j, n_{j+1}, m_{-1}\}$, proving the claim; (1b) $0 < i < j$ and $j \notin S$, in which case $a_i \cdot m_k \in \{n_k, n_{k+1}, m_{-1}\}$, proving the claim as well, for $j < k$, by the assumption. Case (2) also splits into two: (2a) if $0 = i < j < K$, then $a_i \cdot m_k \in \{n_{k+1}, m_{-1}\}$, and since $j < k$, the claim is proved; (2b) if $0 < i < j < K$ and $j \in S$, then $a_i \cdot m_k \in \{n_k, n_{k+1}, m_{-1}\}$; now the assumption guarantees that $j + 1 \leq k$, therefore the claim holds here as well.

Having established monotonicity, we can approach the proof of distributivity. Consider $x(y \vee z)$. If y and z are compatible, say $y \geq z$, then $x(y \vee z) = xy = xy \vee xz$, by monotonicity. Suppose y and z are incompatible. We may take $y = b_i$ ($i \in \{1, \dots, K\}$) and $z = m_j$ ($j \in \{1, \dots, K\}$). Then, $x(y \vee z) = x(b_i \vee m_j) = xb_0$, and $xy \vee xz = xb_i \vee xm_j$. If $x = \mathbf{1}$, or $x = \mathbf{0}$, then the desired equality obviously holds. Assume $x \notin \{\mathbf{1}, \mathbf{0}\}$. Then, if $x \geq b_K$, we have: $x(y \vee z) = xb_0 = b_K$, and $xy \vee xz = xb_i \vee xm_j = b_K$, since $xm_j \leq m_0$. Thus, the equality holds here, too. If $x \leq m_K$, then two cases should be distinguished: (1) $x \leq m_0$, in which case $x(y \vee z) = xb_0 = \mathbf{0}$, and $xy \vee xz = xb_i \vee xm_j = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$; (2) $x = m_k$, with $k \in \{1, \dots, K\}$, in which case $x(y \vee z) = m_k b_0 = n_0$, and $xy \vee xz = m_k b_i \vee m_k m_j = n_0$, as well. This finishes the whole proof. □

Lemma 8. *The* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{C}_K^S$ *is stiff. Moreover, it belongs to* E_3 *and satisfies* $\neg\neg x = x$.

Proof. It is clear from the construction that $\mathbf{C}_K^S$ satisfies $x^4 = x^3$. It is also clear, that the only nontrivial and non-full congruence on $\mathbf{C}_K^S$ is the one associated with the filter $F = \{x \in C_K^S : x \geq b_K\}$. The quotient of this congruence is the two-element Boolean algebra. We also have: $\neg a_i = n_i$, $\neg b_i = m_i$, $\neg n_i = a_i$, and $\neg m_i = b_i$; we leave out the detailed calculations. This shows that $\neg\neg x = x$ holds. Moreover, for any $x \in C_K^S \setminus \{\mathbf{1}, \mathbf{0}\}$, we have either $x^3 = b_K$, or $(\neg x)^3 = b_K$. Thus, to prove stiffness and finish the whole proof, it suffices to show that $\mathbf{C}_K^S$ is generated by b_K. Take $b_K \vee \neg b_K = b_K \vee m_K = b_0$. This generates a_0, since $b_0 \to b_K = a_0$. Further, $a_0 a_0 = b_{-1}$. Then, $a_0 \to b_0 = a_1$, and $a_1^2 = b_1$. Suppose $a_0, \dots, a_n$, b_{-1}, $b_0, \dots, b_n$ have been generated. Then, $a_0 \to b_n = a_{n+1}$, and $a_{n+1}^2 = b_{n+1}$. This shows that all the elements in A and B get generated. Then the sets N and M are generated by negation, and this finishes the generation process. Observe that this process is independent from the set S, in the sense that, had S been chosen differently, the operations involved in the generation would still yield precisely the same results. □

Consider now the family $(\mathbf{C}^S_K)_{K\in\omega^+}$, where $\omega^+ = \omega\setminus\{0\}$. Take any free ultrafilter U on ω and let $\mathbf{C}_S$ stand for the ultraproduct $\prod_{K\in\omega^+}\mathbf{C}^S_K/U$. Further, let $\mathbf{C}^\star_S$ be the subalgebra of $\mathbf{C}_S$ generated by $\star$. Clearly, the assumptions of Lemma 6 apply to our construction. Thus, $\mathbf{C}^\star_S$ is an infinite stiff $\mathbf{FL_{ew}}$-algebra, and as such it generates an almost minimal variety.

Lemma 9. *For any S, T subsets of $\omega\setminus\{0,1\}$, we have: if $S\neq T$, then $\mathbf{C}^\star_S$ is not isomorphic to $\mathbf{C}^\star_T$.*

Proof. Before we embark on the proof, let us dwell for a while on what $\mathbf{C}_S$ and $\mathbf{C}_T$ look like. We will refer to certain elements of these ultraproducts by the names of the elements of factor algebras, for instance, a_i in the appropriate context, will stand for $\langle e(n) : n\in\omega^+\rangle/U$, where $e(n) = a_i(n)$, if a_i exists in C^S_n, or is arbitrary otherwise. In particular, it is helpful to think of K used in $b_K = \langle b_K(n) : n\in\omega^+\rangle/U$ as an infinitely large natural number, so that $K > n$, for any $n\in\omega$. (just like in nonstandard models for arithmetic). Notice that $\star$ is always unambiguous, being a definable constant.

Now, suppose S, T are subsets of $\omega\setminus\{0,1\}$ and $S\neq T$, yet $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$ are isomorphic. We can therefore identify the lattices underlying $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$. Now, to obtain the desired contradiction, we look at multiplication and residuation induced on these lattices.

Let i be the smallest number such that $i\in S$ but $i\notin T$; we can always assume such a number exists, if not we just swap S and T. Consider the elements a_1, a_i; note that we use these names unambiguously, because, as the generation process from the previous proof does not depend on the choice of S and T, we may use a_1 as shorthand for $((\star\vee\neg\star)\to\star)\to(\star\vee\neg\star)$, and a_i as a shorthand for something similar, only much longer. Now, consider $a_1\cdot^S a_i$. Since $0<1<i\in S$, the first clause in the definition of '$\cdot$' applies at almost all coordinates, yielding in the ultraproduct: $a_1\cdot^S a_i = b_{i-1}$. Then, for $a_1\cdot^T a_i$, we have $0<1<i\notin T$ and the second clause applies, yielding: $a_1\cdot^S a_i = b_i$. However, b_i and b_{i-1} are different in the ultraproduct, hence, $a_1\cdot^S a_i$ and $a_1\cdot^T a_i$ produce different results, contradicting the assumption that $\mathbf{C}^\star_S$ and $\mathbf{C}^\star_T$ are isomorphic. □

Theorem 1. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathsf{E}_3\cap\mathsf{IFL_{ew}}$.*

$\cdot$	1	a	0
1	1	a	0
a	a	0	0
0	0	0	0

$\cdot$	1	a	b	0
1	1	a	b	0
a	a	a	0	0
b	b	0	0	0
0	0	0	0	0

Figure 1: Algebras generating the two almost minimal subvarieties of $\mathsf{E}_2 \cap \mathsf{IFL}_{\mathsf{ew}}$.

For contrast we have the next theorem.

Theorem 2. *There are two almost minimal subvarieties of* $\mathsf{E}_2 \cap \mathsf{IFL}_{\mathsf{ew}}$.

Since the theorem above turns out to be a special case of Theorem 4, the proof will be delivered there. Multiplication tables for fusions of the two generating algebras are shown in Figure 1. Lattice operations in these algebras are determined by the linear ordering in which the elements occur on the left-hand side of the table. The algebra on the left is of course $\mathbf{Ł}_3$.

3.2 The linear case

For any S with $0 \in S \subseteq \omega$. For any positive integer K define L_K^S to be the disjoint union of the sets: $B = \{b_0, \ldots, b_{K+1}\}$, $A = \{a_0, a_1, a_2\}$, $N = \{n_0, \ldots, n_{K+1}\}$, $C = \{\mathbf{0}, \mathbf{1}\}$, and $D = \{d_s : s < K,\ s \in S\} \cup \{e\}$. L_K^S is totally ordered (see Figure 2) by the transitive reflexive closure of the relation '$\triangleright$' defined below:

- $\mathbf{1} \triangleright b_0 \triangleright b_{K+1} \triangleright a_0 \triangleright a_1 \triangleright a_2 \triangleright n_{K+1} \triangleright d_0 \triangleright e \triangleright \mathbf{0}$;
- $b_i \triangleright b_{i+1}$, $n_{i+1} \triangleright n_i$, for all $i \in \{0, \ldots, K\}$;
- $d_r \triangleright d_s$ iff $r < s$ in the natural ordering of S.

For any $i \in \omega$, let $\lfloor i \rfloor$ stand for the largest $s \in S$ with $s \leq i$. Such an s always exists, for $0 \in S$. With this notation at hand, we define multiplication on L_K^S, putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in L_K^S$;

- $b_i \cdot b_j = \begin{cases} a_0, & \text{if } \min\{i,j\} = 0, \\ a_1, & \text{if } 0 < i,\, 0 < j < K, \text{ or} \\ & \quad 0 < j,\, 0 < i < K, \text{ or} \\ & \quad i = j = K, \\ a_2 & \text{if } i \in \{K, K+1\},\, j = K+1, \text{ or} \\ & \quad j \in \{K, K+1\},\, i = K+1; \end{cases}$
- $b_i \cdot a_j = a_2$
- $b_i \cdot n_j = \begin{cases} d_{\lfloor i \rfloor}, & \text{if } j \le i, \\ e, & \text{if } j = i+1, \\ \mathbf{0}, & \text{if } j > i+1; \end{cases}$
- $x \cdot y = \mathbf{0}$, in all other cases.

Residuation is defined as previously:

- $x \to y = \bigvee\{z \in L_K^S : z \cdot x \le y\}$.

Then, let $\mathbf{L}_K^S$ be the algebra $\langle L_K^S; \vee, \wedge, \cdot, \to, \mathbf{0}, \mathbf{1}\rangle$.

Lemma 10. *The algebra $\mathbf{L}_K^S$ is a $\mathbf{FL_{ew}}$-algebra.*

We leave the proof for the reader.

Lemma 11. *The $\mathbf{FL_{ew}}$-algebra $\mathbf{L}_K^S$ is stiff. Moreover, it belongs to E_3 and satisfies $x \to y \vee y \to x = 1$.*

Proof. The only thing that is not immediately seen from the construction is that $\mathbf{L}_K^S$ has no subalgebras apart from the two-element Boolean algebra and itself. To show this it suffices to prove that the element a_2 generates $\mathbf{L}_K^S$. We have: $\neg a_2 = n_{K+1}$ and $\neg n_{K+1} = b_{K+1}$. Then, $b_{K+1} \to a_2 = b_K$. Further, $b_K \cdot n_{K+1} = e$, and $\neg b_K = n_K$, $n_K \to e = b_{K-1}$. Then, by backward induction on i, $\neg b_i = n_i$, $n_i \to e = b_{i-1}$ and thus we have generated all the elements, except a_0, a_1 and all the elements d_s, for $s \in S \cap \{0, \dots, K-1\}$. To get these, we may, for instance, employ: $b_0^2 = a_0$, $b_1^2 = a_1$, and finally $b_{K+1} \cdot n_s = d_s$, for all suitable s. □

As in the previous case, let $\mathbf{L}_S$ be the ultraproduct $\prod_{K \in \omega^+} \mathbf{L}_K^S / U$ by a free ultrafilter U on ω and $\mathbf{L}_S^\star$ its subalgebra generated by $\star$. Again, Lemma 6 applies, thus $\mathbf{L}_S^\star$ is an infinite stiff algebra.

Lemma 12. *Let S, T be subsets of ω, each containing* 0. *If $S \neq T$, then $\mathbf{L}^\star_S$ is not isomorphic to $\mathbf{L}^\star_T$.*

Proof. We adopt all the conventions from Lemmas 8 and 9. Suppose i is the smallest number with $i \in S \setminus T$. Observe that in both $\mathbf{L}^\star_S$ and $\mathbf{L}^\star_T$ we have a finite increasing sequence d_0, $d_{\lfloor 1 \rfloor}, \dots, d_{\lfloor i-1 \rfloor}$, with k distinct terms ($k \leq i$). Then, $b_i \cdot^S n_{K+1} = d_{\lfloor i \rfloor} = d_i \neq d_{\lfloor i-1 \rfloor} = b_{i-1} \cdot^S n_{K+1}$, since $i \in S$; but $b_i \cdot^T n_{K+1} = d_{\lfloor i \rfloor} = d_{\lfloor i-1 \rfloor} = b_{i-1} \cdot^T n_{K+1}$, as $i \notin T$. $\square$

Theorem 3. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathsf{E}_3 \cap \mathsf{L}$.*

Again, we contrast it with the following.

Theorem 4. *There are six almost minimal subvarieties of $\mathsf{E}_2 \cap \mathsf{L}$.*

Proof. Only a sketch. First, we observe that if an algebra in $\mathsf{E}_2 \cap \mathsf{L}$ is simple, it must have unique coatom c with $c^2 = 0$. Then. $\neg c = c$ and we conclude that $\{1, c, 0\}$ is a subuniverse. That produces the three-element Łukasiewicz algebra. Suppose that our algebra is si but not simple. Let a be the smallest element of the smallest nontrivial congruence filter. Then $a^2 = a$. We verify that the set $\{1, \neg\neg a \to a, \neg\neg a, a, \neg a, \neg(\neg\neg a \to a), (\neg\neg a \to a) \cdot \neg a, 0\}$ with the order decreasing from left to right, is closed under the operations. As a by-product, we obtain that the algebra with this universe is stiff. Next, we try to identify some of the elements; this produces the remaining five algebras. Incidentally, only one of them satisfies $\neg\neg x = x$, which proves Theorem 2. $\square$

The algebras generating these six varieties are the two from Figure 1 and the four from Figure 3. The algebra in the upper left-hand corner of the latter picture is the three-element Heyting algebra.

3.3 The distributive case

We finish off with yet one more construction, this time descending as low as E_2 in the subvarieties of R. However, as all the proofs proceed exactly as in previous cases, none will be presented.

Let S be a subset of ω with with $0 \in S$. For any positive integer K, let D^S_K be the disjoint union of the sets: $A = \{a_0, \dots, a_{K+1}\}$, $N = \{n_0, \dots, n_{K+1}\}$, $C = \{\mathbf{0}, \mathbf{1}\}$, $B = \{b_s : s < K, \ s \in S\} \cup \{c\}$. D^S_K is partially ordered (see Figure 2) by the transitive reflexive closure of the relation '$\rhd$' defined below:

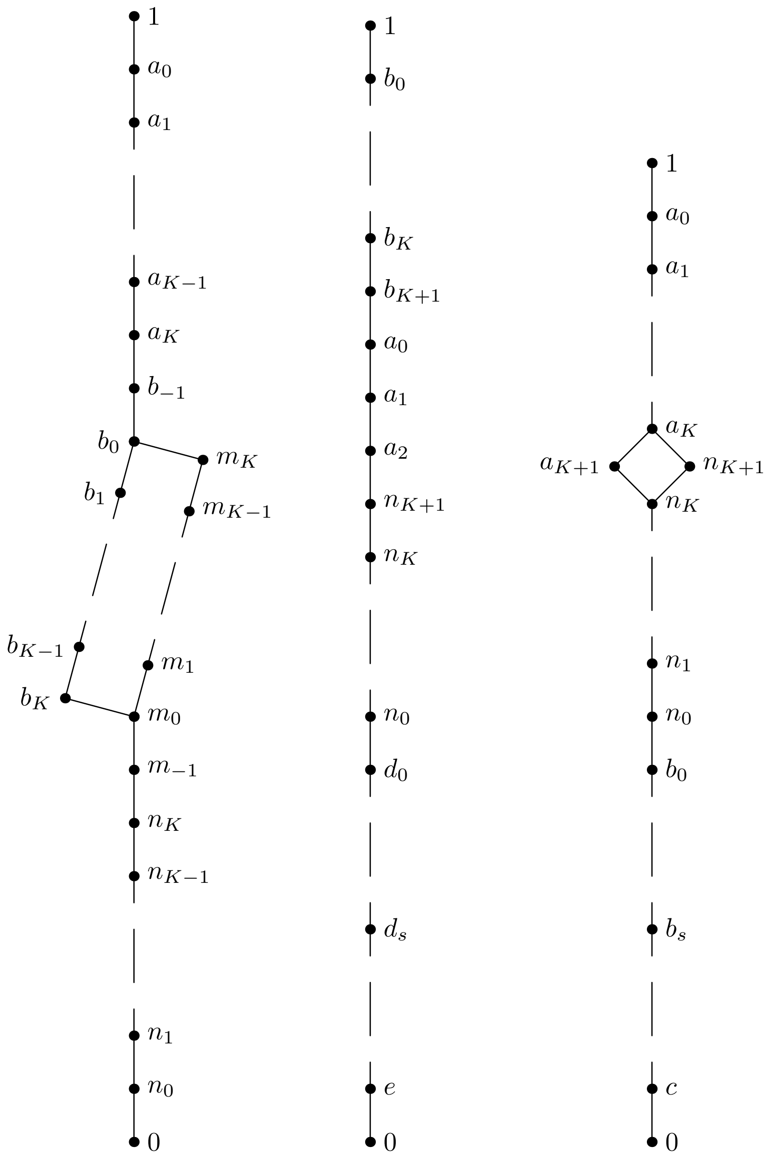

Figure 2: Algebras generating almost minimal varieties. From the left: $\mathbf{C}_K^S$, $\mathbf{L}_K^S$, and $\mathbf{D}_K^S$.

$\cdot$	1	a	0
1	1	a	0
a	a	a	0
0	0	0	0

$\cdot$	1	a	b	c	0
1	1	a	b	c	0
a	a	b	b	0	0
b	b	b	b	0	0
c	c	0	0	0	0
0	0	0	0	0	0

$\cdot$	1	a	b	c	d	e	0
1	1	a	b	c	d	e	0
a	a	c	c	c	d	0	0
b	b	c	c	c	0	0	0
c	c	c	c	c	0	0	0
d	d	e	0	0	0	0	0
e	e	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$\cdot$	1	a	b	c	d	e	f	0
1	1	a	b	c	d	e	f	0
a	a	c	c	c	f	0	0	0
b	b	c	c	c	0	0	0	0
c	c	c	c	c	0	0	0	0
d	d	f	0	0	0	0	0	0
e	e	0	0	0	0	0	0	0
f	f	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Figure 3: Four more algebras generating almost minimal subvarieties of $\mathsf{E}_2 \cap \mathsf{FL}_{\mathsf{ew}}$.

- $\mathbf{1} \triangleright a_0$, $n_0 \triangleright b_0$, $c \triangleright \mathbf{0}$;
- $x \triangleright c$, for all $x \neq \mathbf{0}$;
- $a_i \triangleright a_j$, $n_j \triangleright n_i$ iff $i < j$ $(i, j \in \{0, \dots, K+1\})$;
- $a_K \triangleright n_{K+1}$, $a_{K+1} \triangleright n_K$;
- $b_r \triangleright b_s$ iff $r < s$ in the natural ordering of S.

For any $i \in \omega$, let $\lfloor i \rfloor$ stand, as previously, for the largest $s \in S$ with $s \leq i$. Define multiplication on D^S_K, putting:

- $\mathbf{1} \cdot x = x$, $\mathbf{0} \cdot x = \mathbf{0}$, $x \cdot y = y \cdot x$, for all $x, y \in D^S_K$;
- $a_i \cdot a_j = a_{K+1}$;
- $a_i \cdot n_j = \begin{cases} b_{\lfloor i \rfloor}, & \text{if } j > i+1, \\ c, & \text{if } j = i+1, \\ \mathbf{0}, & \text{if } j \leq i; \end{cases}$
- $x \cdot y = \mathbf{0}$, in all other cases.

Lemma 13. *The algebra $\mathbf{D}^S_K$ is a stiff $\mathbf{FL_{ew}}$-algebra. Moreover, it belongs to E_2 and is distributive as a lattice.*

Lemma 14. *Let S, T be subsets of ω, each containing* 0. *If $S \neq T$, then $\mathbf{L}^\star_S$ is not isomorphic to $\mathbf{L}^\star_T$.*

Theorem 5. *There are $2^{\aleph_0}$ almost minimal subvarieties of $\mathsf{E}_2 \cap \mathsf{D}$.*

4 Minimal varieties of BL-algebras

As we have seen, even the linearity restriction on $\mathbf{FL_{ew}}$-algebras is not enough to force the number of almost minimal subvarieties below $2^{\aleph_0}$. On the other hand, we know from [9] that there are only countably many almost minimal varieties of Łukasiewicz algebras. We can do slightly better than that, namely, we can show that the same holds true about varieties of BL-algebras. The proof was developed in 2001 at Japan Advanced Institute of Science and Technologiy (JAIST) algebraic logic seminar. We begin with an easy observation, whose proof we leave to the reader. Let $\mathbf{A}$ be an si BL-algebra.

Lemma 15. *If there is an idempotent element $a \in A \setminus \{0,1\}$, then the three-element Heyting algebra $\mathbf{H}_3$ is a subalgebra of $\mathbf{A}$.*

Thus, if $\mathbf{A}$ is to generate an almost minimal variety different from $V(\mathbf{H}_3)$, it cannot contain idempotents different from 0 and 1. Suppose $\mathbf{A}$ indeed generates almost minimal variety different from $V(\mathbf{H}_3)$ and that $\mathbf{A}$ is finite. The next lemma follows from [2].

Lemma 16. *The algebra $\mathbf{A}$ is isomorphic to the Łukasiewicz algebra $\mathbf{Ł}_{p+1}$, for a prime p.*

We can now assume that $\mathbf{A}$ is infinite and has no idempotents beside 0 and 1. Let U be the filter on A that corresponds to the monolith congruence. By the assumption, U has no smallest element, and since the $\{\rightarrow, \cdot, 1\}$-reduct of $\mathbf{A}$ is a hoop (in fact, a *basic* hoop, see [1]), we can conclude, again by [2], that $\langle U; \cdot, \rightarrow, 1\rangle$ is a simple Wajsberg hoop, therefore isomorphic as a hoop to $\mathbf{C}^\infty = \langle\{a^i : i \in \omega\}; \cdot, \rightarrow, 1\rangle$, where $1 = a^0$, $a^j \cdot a^k = a^{j+k}$, and $a^j \rightarrow a^k = a^{k-j}$ if $k < j$ and $a^j \rightarrow a^k = 1$ otherwise. Thus, by linearity, we obtain that A is a disjoint union of U and a certain set N such that for each $u \in U$ and $n \in N$ we have $u > n$.

Lemma 17. *For all $u \in U$, we have $\neg u \in N$ and $\neg\neg u \in U$.*

Proof. Suppose $\neg u \in U$ for some $u \in U$. Then $u = a^j$ and $\neg u = a^k$. Therefore $0 = u \cdot \neg u = a^j \cdot a^k = a^{j+k}$, a contradiction. □

Let now M stand for $\{n \in N : n = \neg a^k \text{ for some } k \in \omega\}$.

Lemma 18. *The set $U \cap M$ is a subuniverse of $\mathbf{A}$.*

Proof. By case analysis, of which we will present the two non-obvious cases. First, closure under multiplication. Suppose $d \in U$ and $b \in M$. Then, $d = a^j$ and $b = \neg a^k$, therefore $bd = a^j \cdot \neg a^k$. If $j \geq k$, then $a^j \leq a^k$ and thus $a^j \cdot \neg a^k = 0 = \neg a^0 \in N$. If $j \geq k$, then $a^j \cdot \neg a^k = a^j \cdot \neg a^{k-j+j} = a^j(a^j \rightarrow \neg a^{k-j})$. This, by the hoop axiom, equals $a^j \wedge \neg a^{k-j}$ and that, by linearity, equals further $\neg a^{k-j}$, which belongs to M.

Then, closure under residuation. Suppose $d, b \in M$, i.e., $d = \neg a^j$ and $b = \neg a^k$. If $k \geq j$, then $\neg a^j \leq \neg a^k$ and thus $\neg a^j \rightarrow \neg a^k = 1 = a^0 \in U$. If $k < j$, then a^{j-k} is well defined and strictly smaller than 1. We get $a^{j-k} \cdot \neg a^j = a^{j-k} \cdot \neg a^{j-k+k} = a^{j-k}(a^{j-k} \rightarrow \neg a^k) = a^{j-k} \wedge \neg a^k = \neg a^k$. So, $a^{j-k} \cdot \neg a^j = \neg a^k$ and this by residuation yields $a^{j-k} \leq \neg a^j \rightarrow \neg a^k$. Therefore, $\neg a^j \rightarrow \neg a^k \in U$. □

Lemma 19. *If* $\neg a^j = \neg a^{j+1}$, *for some* $j \in \omega$, *then* $\neg a^k = 0$, *for all* $k \in \omega$.

Proof. Induction on k. For $k = 0$ the claim holds trivially, for $k = 1$ we have $\neg a = \neg a \wedge a^j = a^j(a^j \to \neg a) = a^j(a^j \to (a \to 0)) = a^j \cdot \neg a^{j+1} = a^j \cdot \neg a^j = 0$. Then, in the inductive step we obtain $\neg a^{k+1} = a \to \neg a^k$, which by inductive assumption equals further $a \to 0 = \neg a = 0$. $\square$

Lemma 20. *If* $\neg a^j \neq \neg a^k$, *for all* $j \neq k$, *then* $\neg\neg a^k = a^k$, *for all* $k \in \omega$.

Proof. Suppose the contrary, and let k be the smallest such that $\neg\neg a^k > a^k$. Since $\neg\neg a^k \in U$, by Lemma 17, we obtain that $\neg\neg a^k = a^j$ for some $j < k$. Thus, $\neg\neg a^j = a^j$, as k is the smallest number for which it fails. Therefore, $\neg\neg a^j = \neg\neg a^k$, which yields $\neg a^j = \neg\neg\neg a^j = \neg\neg\neg a^k = \neg a^k$, contradicting the assumption. $\square$

Theorem 6. *There are countably many almost minimal varieties of BL-algebras. They are: all minimal varieties of Łukasiewicz algebras, the only almost minimal variety of Heyting algebras, and the variety of product logic algebras.*

Proof. If $\mathbf{A}$ is finite, then by Lemmas 15 and 16 it is either $\mathbf{H}_3$ or $\mathbf{Ł}_{p+1}$ for some prime p. If $\mathbf{A}$ is infinite, then by Lemmas 19 and 20 the set M (defined just before Lemma 18; it contains the negations of the elements of the congruence filter U) can be either a singleton, or be bijective with U with negation being the bijection. In the former case $\mathbf{A}$ generates the variety of product logic algebras (cf. [5]), in the latter $\mathbf{A}$ is Chang's MV-algebra (cf. [4]). $\square$

We have seen that when one passes from BL to $\mathsf{FL_{ew}}$ the number of almost minimal subvarieties increases from countably to uncountably infinite. The passage from MV to $\mathsf{IFL_{ew}}$ has the same effect, and so does the passage from BL to *linear* $\mathsf{FL_{ew}}$. However, our construction of uncountable families of stiff algebras in $\mathsf{IFL_{ew}}$ produces non-linear algebras and in linear $\mathsf{FL_{ew}}$ it produces non-involutive ones. The following question, then, seems natural.

Question 1. *How many almost minimal varieties of involutive, linear* $\mathbf{FL_{ew}}$*-algebras are there?*

References

[1] P. Agliano, I.M.A. Ferreirim, F. Montagna, Basic hoops: an algebraic study of continuous t-norms, *Studia Logica*, forthcoming.

[2] W.J. Blok, I.M.A. Ferreirim, On the structure of hoops, *Algebra Universalis* 43 (1999), 233-257.

[3] W.J. Blok, D. Pigozzi, *Algebraizable Logics*, Memoirs of the American Mathematical Society, Number 396, Amer. Math. Soc., Providence, 1989.

[4] C.C. Chang, Algebraic analysis of many-valued logics, *Transactions of the American Mathematical Society* 88 (1958), 467–490.

[5] R. Cignoli, A. Torrens, An algebraic analysis of product logic, *Multiple Valued Logic* 5, 2000, 45–65.

[6] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* 123, 2001, no.3, 271–288.

[7] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic 4, Studia Logica Library, Kluwer Academic Publishers, Dordrecht 1998.

[8] P. Jipsen, C. Tsinakis, *A survey of residuated lattices*, Ordered Algebraic Structures (J. Martinez, ed.), Kluwer Academic Publishers, Dordrecht 2002, 19–56.

[9] Y. Komori, Super-Łukasiewicz propositional logics, *Nagoya Mathematical Journal* 84 (1981), 119–133.

[10] T. Kowalski, H. Ono, *Residuated lattices: an algebraic glimpse at logics without contraction*, JAIST preprint, 2000.

[11] H. Ono, Substructural logics and residuated lattices — as introduction, *Trends in Logic* vol. 20 “50 Years of Studia Logica” (V.F. Hendricks, J. Malinowski, eds), Kluwer Academic Publishers, Dordrecht 2003, 177–212.

[12] M. Ueda, *A study of classification of residuated lattices and logics without contraction rule*, Master Thesis at JAIST, 2000.

Japan Advanced Institute of Science and Technology
1-1 Asahidai, Tatsunokuchi, Ishikawa,
Japan

katoh@jaist.ac.jp, ueda@jaist.ac.jp

College of Engineering and Computer Science
Australian National University, Canberra ACT,
Australia

Tomasz.Kowalski@anu.edu.au