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SUBALGEBRAS OF A FINITE MONADIC BOOLEAN ALGEBRA

A b s t r a c t. It is well known that the number of subalgebras of a Boolean algebra with n atoms is the number of partitions of an n-element set. In this note we characterize the subalgebras of a finite monadic Boolean algebra and we determine the cardinality of the set of such subalgebras.

For a finite n-element set X, $n \geq 1$, let $N[X]$ denote the number of elements of X and let $\mathbf{p}(n)$ denote the number of all partitions of X. If B_n is a Boolean algebra with n atoms, let $\mathcal{A}(B_n)$ be the set of all atoms of B_n. It is known that there exists a bijective correspondence between the set $\mathcal{S}(B_n)$ of all subalgebras of B_n and the set of all partitions of $\mathcal{A}(B_n)$, i.e., $N[\mathcal{S}(B_n)] = \mathbf{p}(n)$. The following recursive formula for $\mathbf{p}(n)$ can be found in [5]: if we define $\mathbf{p}(0) = 1$, then

$$\mathbf{p}(n+1) = \sum_{i=0}^{n} \binom{n}{i} \mathbf{p}(i), \qquad n \geq 0.$$

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Let $(B_n, \exists)$ be a monadic Boolean algebra [2], and $K(B_n) = \{x \in B_n : \exists\ x = x\}$. Then $K(B_n)$ is a Boolean subalgebra of B_n. Furthermore if $\forall x = -\exists - x$ where $-x$ denotes the Boolean complement of x, we have that $K(B_n) = \{x \in B_n : \forall\ x = x\}$. Conversely, any Boolean subalgebra K of B_n induces a unary operator $\exists$ on B_n such that $(B_n, \exists)$ is a monadic Boolean algebra and $K = K(B_n)$. This correspondence is bijective. If $\mathcal{A}(K) = \{c_1, c_2, \ldots, c_k\}$ is the set of atoms of K and C_i is the set of atoms of B_n preceding c_i, with $N[C_i] = n_i$, for $1 \leq i \leq k$ we will denote $(B_n, \exists)$, (B_n, K) or $(B_n, n_1 + n_2 + \cdots + n_k)$, the corresponding monadic Boolean algebra.

The aim of this paper is to give a construction of every element of the set $\mathcal{S}(B_n, K)$ of all subalgebras of (B_n, K), and to determine $N[\mathcal{S}(B_n, K)]$.

Let

$$\mathcal{S}_1(B_n, K) = \{S \in \mathcal{S}(B_n, K) : S \subset K\},$$

$$\mathcal{S}_2(B_n, K) = \{S \in \mathcal{S}(B_n, K) : K \subseteq S\},$$

$$\mathcal{S}_3(B_n, K) = \{S \in \mathcal{S}(B_n, K) : S \text{ is incomparable to } K\}.$$

It is clear that

$$N[\mathcal{S}(B_n, K)] = N[\mathcal{S}_1(B_n, K)] + N[\mathcal{S}_2(B_n, K)] + N[\mathcal{S}_3(B_n, K)].$$

On the other hand,

$$\mathcal{S}_1(B_n, K) = \{S \in \mathcal{S}(B_n) : S \subset K\}$$

$$\text{and} \quad \mathcal{S}_2(B_n, K) = \{S \in \mathcal{S}(B_n) : K \subseteq S\},$$

so, if K have t atoms, $1 \leq t \leq n$,

$$N[\mathcal{S}_1(B_n, K)] = \mathbf{p}(t) - 1 \quad \text{and} \quad N[\mathcal{S}_2(B_n, K)] = \prod_{i=1}^{t} \mathbf{p}(N[C_i]).$$

H. Bass proved the following result in 1958 [1] (see also [3], [4]).

Lemma 1. *If $(B, \exists)$ is a monadic Boolean algebra such that $K = \exists B$ is finite, then for every $x \in B$, $x \neq 0$,*

$$\exists x = \bigvee\{k \in \mathcal{A}(K) : k \wedge x \neq 0\}.$$

Let (B_n, K) be a monadic Boolean algebra, $\mathcal{A}(K) = \{c_1, c_2, \ldots, c_t\}$, $1 \leq t \leq n$, and S a Boolean subalgebra of (B_n, K), where $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_h\}$. For every s_i, $1 \leq i \leq h$, let

$$T_i = \{c_j : c_j \wedge s_i \neq 0\}.$$

Theorem 1. *Let S be a Boolean subalgebra of B_n. Then S is a monadic subalgebra of (B_n, K) if and only if the family of different sets T_i, $1 \leq i \leq h$, is a partition of $\mathcal{A}(K)$.*

Proof. Suppose that the different sets T_i form a partition of $\mathcal{A}(K)$. Observe that if $s_k \wedge c_r \neq 0$ for some $c_r \in T_i$, then $s_k \wedge c_j \neq 0$ for every $c_j \in T_i$. Indeed, if for some $c_j \in T_i$, $s_k \wedge c_j = 0$, then $c_j \notin T_k$, that is, $T_k \neq T_i$. On the other hand, $c_r \in T_i \cap T_k$, with $T_i \neq T_k$, which is a contradiction.

Besides, $\bigvee\limits_{c_j \in T_i} c_j = \bigvee \{s_k : s_k \wedge c_j \neq 0 \text{ for every } c_j \in T_i\}$. Indeed, if $x \in \mathcal{A}(B_n)$ and $x \leq \bigvee\limits_{c_j \in T_i} c_j$, then $x \leq c_r$ for some $c_r \in T_i$, that is, $c_r \wedge s_i \neq 0$. On the other hand, $x \leq s_k$, for some k, $1 \leq k \leq h$. Thus $x \leq s_k \wedge c_r$ and consequently, $s_k \wedge c_r \neq 0$, with $c_r \in T_i$. Then, by the previous remark, $s_k \wedge c_j \neq 0$ for every $c_j \in T_i$. So $x \leq \bigvee \{s_k : s_k \wedge c_j \neq 0 \text{ for every } c_j \in T_i\}$.

Suppose now that $x \leq \bigvee \{s_k : s_k \wedge c_j \neq 0 \text{ for every } c_j \in T_i\}$. Then $x \leq s_k$ for some s_k such that $s_k \wedge c_j \neq 0$ for every $c_j \in T_i$. Hence $c_j \in T_k$, and consequently, $c_j \in T_k \cap T_i$. So $T_k \cap T_i \neq \emptyset$ and then $T_k = T_i$. In addition, $x \leq c_r$ for some c_r, and then $x \leq c_r \wedge s_k$, that is, $c_r \wedge s_k \neq 0$. So $c_r \in T_k = T_i$, that is, $c_r \in T_i$. Therefore, $x \leq \bigvee\limits_{c_j \in T_i} c_j$.

Let us see that $\exists s_i \in S$ for every atom s_i of S. Indeed, from Lemma 1,

$$\exists s_i = \bigvee_{c_j \in T_i} c_j = \bigvee \{s_k : s_k \wedge c_j \neq 0 \text{ for every } c_j \in T_i\} \in S.$$

Conversely, suppose that S is a monadic subalgebra of (B_n, K) and let us prove that the different sets T_i form a partition of $\mathcal{A}(K)$. Observe that $T_i \neq \emptyset$, being that for every i there exists j such that $s_i \wedge c_j \neq 0$. Suppose that $T_i \neq T_j$. From Lemma 1, $\exists s_i = \bigvee\limits_{c_k \in T_i} c_k$ and $\exists s_j = \bigvee\limits_{c_k \in T_j} c_k$. As S is a monadic subalgebra and s_i, s_j are distinct atoms of S, then $\exists s_i$ and $\exists s_j$ are atoms of the subalgebra of constants of S, and in addition, they are distinct since $T_i \neq T_j$. So $\exists s_i \wedge \exists s_j = 0$, and consequently, $T_i \cap T_j = \emptyset$. $\square$

Examples

1. Using Theorem 1 we are going to determine the number of subalgebras of the monadic Boolean algebra $(B_n, \underbrace{1+1+\cdots+1}_{k}+(n-k))$, where $0 \leq k < n$.

   Let S be a subalgebra of (B_n, K), with $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$. We know that there exists i, $1 \leq i \leq t$, such that $s_i \wedge c_{k+1} \neq 0$. Let $I = \{i : 1 \leq i \leq t, s_i \wedge c_{k+1} \neq 0\}$.

   If $N[I] = 1$, say $I = \{i\}$, then $c_{k+1} \leq s_i$. The number of Boolean subalgebras of B_n verifying this condition equals to the number of partitions of the set $\{c_1, c_2, \ldots, c_{k+1}\}$, i.e., $\mathbf{p}(k+1)$, and clearly all these subalgebras are monadic.

   If $N[I] > 1$, it follows from Theorem 1 that S is monadic if and only if $\bigvee_{i \in I} s_i = c_{k+1}$, where $s_i \neq c_{k+1}$ for all $i \in I$. So, in this case, there are $\mathbf{p}(k) \cdot (\mathbf{p}(n-k) - 1)$ monadic subalgebras.

   Then, if $1 \leq k < n$,

   $$N[\mathcal{S}(B_n, \underbrace{1+1+\cdots+1}_{k}+(n-k))] = \mathbf{p}(k+1) + \mathbf{p}(k).(\mathbf{p}(n-k) - 1).$$

   If $k = 0$ then it is clear that $N[S(B_n, n)] = \mathbf{p}(n)$.

   We know that

   $$N[\mathcal{S}_1(B_n, \underbrace{1+1+\cdots+1}_{k}+(n-k))] = \mathbf{p}(k+1) - 1,$$

   and

   $$N[\mathcal{S}_2(B_n, \underbrace{1+1+\cdots+1}_{k}+(n-k))] = \mathbf{p}(n-k),$$

   so

   $$\begin{aligned} & N[\mathcal{S}_3(B_n, \underbrace{1+1+\cdots+1}_{k}+(n-k))] \\ &= \mathbf{p}(k+1) + \mathbf{p}(k).(\mathbf{p}(n-k) - 1) - (\mathbf{p}(k+1) - 1 + \mathbf{p}(n-k)) \\ &= (\mathbf{p}(k) - 1) \cdot (\mathbf{p}(n-k) - 1). \end{aligned}$$

2. Let K be a Boolean subalgebra of B_n, $n \geq 4$, such that K is isomorphic to B_2 and $\mathcal{A}(K) = \{c_1, c_2\}$. From Theorem 1, if $N[C_1] = 1$ and $N[C_2] = n - 1$ there are no monadic subalgebras incomparable to K. Suppose now $N[C_1] = j > 1$ and $N[C_2] = n - j > 1$. Then there exist $\dfrac{(2^j - 2) \cdot (2^{n-j} - 2)}{2}$ monadic subalgebras isomorphic to B_2 and incomparable to K.

3. If $n \geq 4$ and K is a Boolean subalgebra of B_n isomorphic to B_{n-1} then there exist exactly $\binom{n-2}{2}$ monadic subalgebras isomorphic to K and incomparable to K.

Our next objective is to obtain a formula to determine the number of subalgebras of a finite monadic Boolean algebra.

It is known that the number of partitions of an n-element set X into t classes is the Stirling number of second kind

$$S(n,t) = \frac{\sum\limits_{i=0}^{t-1} (-1)^i \binom{t}{i} (t-i)^n}{t!}.$$

Theorem 2. *The number of subalgebras of* (B_n, K) *is*

$$\sum_{\mathcal{P}} \prod_{j=1}^{k} \left(\sum_{h=1}^{m_j} S(n_{j1}, h) \cdot \ldots \cdot S(n_{jl_j}, h) \cdot (h!)^{l_j - 1} \right),$$

where $\mathcal{P} = \{\{c_{11}, c_{12}, \ldots, c_{1l_1}\}, \ldots, \{c_{k1}, c_{k2}, \ldots, c_{kl_k}\}\}$ *ranges over the set of partitions of* $\mathcal{A}(K)$, $n_{ji} = N[C_{ji}]$, $m_j = min\{N[C_{ji}] : i = 1, \ldots, l_j\}$.

Proof. Let us count the subalgebras of (B_n, K) associated to a given partition

$$\mathcal{P} = \{\{c_{11}, c_{12}, \ldots, c_{1l_1}\}, \ldots, \{c_{k1}, c_{k2}, \ldots, c_{kl_k}\}\}.$$

If S is the subalgebra associated to $\mathcal{P}$, let $\mathcal{A}(S) = \{s_1, s_2, \ldots, s_t\}$.
The sets T_i of Theorem 1 verify:

$$\bigvee_{j \in T_i} c_j = \bigvee \{s_k : s_k \wedge c_j \neq 0, \ \text{ for every } \ c_j \in T_i\}.$$

Consider the class $C = \{c_{j1}, c_{j2}, \ldots, c_{jl_j}\}$ of $\mathcal{P}$.

If $\{s_\alpha : s_\alpha \wedge c_{ji} \neq 0 \text{ for every } c_{ji} \in C\} = \{s_{\alpha 1}, s_{\alpha 2}, \ldots, s_{\alpha h}\}$, then $C_{ji} \cap S_{\alpha t} \neq \emptyset$ for $1 \leq i \leq l_j$ and $1 \leq t \leq h$, and $C_{ji} \cap S_\alpha = \emptyset$ for any other S_α. So every C_{ji} is partitioned into h classes, where $1 \leq h \leq m_j = min\{N[C_{ji}] : i = 1, \ldots, l_j\}$. This can be done in

$$\sum_{h=1}^{m_j} S(n_{j1}, h) \cdot \ldots \cdot S(n_{jlj}, h) \cdot (h!)^{l_j - 1}$$

different ways.

So the number of subalgebras associated to the partition $\mathcal{P}$ is

$$\prod_{j=1}^{k} \left(\sum_{h=1}^{m_j} S(n_{j1}, h) \cdot \ldots \cdot S(n_{jl_j}, h) \cdot (h!)^{l_j - 1} \right).$$

Finally, the number of subalgebras of (B_n, K) is

$$\sum_{\mathcal{P}} \prod_{j=1}^{k} \left(\sum_{h=1}^{m_j} S(n_{j1}, h) \cdot \ldots \cdot S(n_{jl_j}, h) \cdot (h!)^{l_j - 1} \right),$$

where $\mathcal{P}$ ranges over the set of all partitions of $\mathcal{A}(K)$. $\square$

By an application of the previous results, we give the following table.

NUMBER OF MONADIC SUBALGEBRAS

Boolean Algebra	Partition of the atoms of B_i	$S \subset K$	$K \subseteq S$	Incom-parable	Number
B_1	1	0	1	0	1
B_2	2	0	2	0	2
	$1+1$	1	1	0	2
B_3	3	0	5	0	5
	$1+2$	1	2	0	3
	$1+1+1$	4	1	0	5
B_4	4	0	15	0	15
	$1+3$	1	5	0	6
	$2+2$	1	4	2	7
	$1+1+2$	4	2	1	7
	$1+1+1+1$	14	1	0	15
B_5	5	0	52	0	52
	$1+4$	1	15	0	16
	$2+3$	1	10	6	17
	$1+1+3$	4	5	4	13
	$1+2+2$	4	4	4	12
	$1+1+1+2$	14	2	4	20
	$1+1+1+1+1$	51	1	0	52
B_6	6	0	203	0	203
	$1+5$	1	52	0	53
	$2+4$	1	30	14	45
	$3+3$	1	25	24	50
	$1+1+4$	4	15	14	33
	$1+2+3$	4	10	11	25
	$2+2+2$	4	8	19	31
	$1+1+1+3$	14	5	16	35
	$1+1+2+2$	14	4	13	31
	$1+1+1+1+2$	51	2	14	67
	$1+1+1+1+1+1$	202	1	0	203

References

[1] H. Bass, *Finite monadic algebras*, Proc. Amer. Math. Soc. **9** (1958), pp. 258–268.

[2] P.R. Halmos, Algebraic Logic, Chelsea Pub. Co. New York 1962.

[3] A. Masse, Anneaux monadiques libres sur un ensemble fini, (I) et (II), Seminaire de Logique Algèbrique , Faculté des Sciences, Université de Lyon, (1970).

[4] L. Monteiro, M. Abad, S. Savini, J. Sewald, *Notes on free monadic Boolean algebras*, Order **16** (1999), pp. 277–289.

[5] O. Ore, *Theory of equivalence relations*, Duke Math. Journal **9** (1942), pp. 573–627.

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