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REMARKS ON SPLITTINGS IN THE VARIETY OF RESIDUATED LATTICES

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1. Introduction

A *residuated lattice* is an algebra $\mathbf{A} = \langle A; \vee, \wedge, \cdot, \rightarrow, 0, 1\rangle$, such that:

(1) $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded lattice with the greatest element 1 and smallest 0;

(2) $\langle A; \cdot, 1\rangle$ is a commutative monoid;

(3) $\mathbf{A}$ satisfies: $x \cdot y \leq z$ iff $x \leq y \rightarrow z$.

The class $\mathcal{R}$ of residuated lattices is a variety. It is arithmetical, has CEP, and is genereated by its finite members (cf. [4], also [3]). It is also congruence 1-regular, i.e., for any congruence θ, the coset of 1 determines θ uniquely. Cosets of 1 are called congruence filters. A finite subdirectly irreducible (si) residuated lattice $\mathbf{A}$ always has a unique coatom c. By finiteness, $c^{n+1} = c^n$, for some positive integer n. We will denote such a

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c^n by $\star$. The congruence filter corresponding to the monolith of $\mathbf{A}$ is $\{a \in A : a \geq \star\}$. For more details on residuated lattices and their connections with *logics without contraction* see [7]. For a broader perspective onto the latter, see [5] and [6].

For a given (quasi)variety $\mathcal{W}$, let $L^q(\mathcal{W})$, $L^v(\mathcal{W})$, stand, respectively, for the lattice of subquasivarieties and subvarieties of $\mathcal{W}$.

A pair $(\mathcal{Q}_1, \mathcal{Q}_2)$ of sub(quasi)varieties of a given (quasi)variety $\mathcal{W}$ is said to *split* $L^v(\mathcal{W})$ $(L^q(\mathcal{W}))$ iff $\mathcal{Q}_1 \not\subseteq \mathcal{Q}_2$ and for any $\mathcal{S}$, sub(quasi)variety of $\mathcal{W}$, either $\mathcal{Q}_1 \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{Q}_2$. In other words, $\mathcal{Q}_2$ is the largest sub(quasi)variety of $\mathcal{W}$ not containing $\mathcal{Q}_1$.

If $(\mathcal{V}_1, \mathcal{V}_2)$ is a *splitting pair* of subvarieties of a variety $\mathcal{V}$, i.e., when $(\mathcal{V}_1, \mathcal{V}_2)$ splits $L^v(\mathcal{V})$ of subvarieties of $\mathcal{V}$, then $\mathcal{V}_1$ is generated by a si algebra, called *splitting algebra*. It follows from general algebraic results that every finite si algebra in $\mathcal{R}$ splits $L^q(\mathcal{R})$. However, we will show that only one such algebra is splitting in $\mathcal{V}$, i.e., splits $L^v(\mathcal{R})$.

If $\mathcal{V}$ is congruence-distributive and generated by its finite members, then every splitting algebra in $\mathcal{V}$ is finite and uniquely determined by the splitting pair (cf. [2]). Thus, the only candidates for splitting algebras are finite subdirect irreducibles.

Fact 1. *The two-element boolean algebra* $\mathbf{2}$ *splits* $L^v(\mathcal{R})$.

We will now slightly modify the technique introduced by Jankov in [1], to suit our purposes.

Let $\mathbf{A}$ be a finite si residuated lattice. Fix a set X of $|A|$ distinct variables, and index them by the elements of A, so that x_a, x_b be distinct iff $a \neq b$. Let $\neg$ be the term operation defined as $\neg z = z \to 0$, and $\diamond \in \{\vee, \wedge, \cdot, \to\}$. The *diagram* of $\mathbf{A}$ is defined, as usual, by $\Delta_{\mathbf{A}} = \bigwedge\{x_{\neg a} \leftrightarrow \neg x_a : a \in A\} \wedge \bigwedge\{x_{a \diamond b} \leftrightarrow x_a \diamond x_b : a, b \in A\}$, Then, the *Jankov term of order* n for $\mathbf{A}$ is defined as $Y^{(n)}_{\mathbf{A}} = \Delta^n_{\mathbf{A}} \to x_\star$, where $\star \in A$ is the smallest member of the monolithic filter on $\mathbf{A}$.

By a *valuation* on an algebra $\mathbf{B}$ we mean any homomomorphism from the absolutely free algebra of the appropriate type into $\mathbf{B}$.

Lemma 1. *Let* $\mathbf{B} \in \mathcal{R}$*. Then,* $\mathbf{A} \subseteq \mathbf{B}$ *iff there is a valuation* v*, such that* $\mathbf{B} \models_v \Delta_{\mathbf{A}} = 1 \ \& \ x_\star \neq 1$.

We know already that the two-element boolean algebra splits $L^v(\mathcal{R})$. Our aim is to show no other algebra has this property. The technique we use will be guided by the following lemma, which characterises non-splitting algebras in $\mathcal{R}$.

Lemma 2. *The following are equivalent:*

(i) $\mathbf{A}$ *is not a splitting algebra in* $\mathcal{R}$,

(ii) $(\forall i \in \omega)(\exists \mathbf{B} \in \mathcal{R}) : \mathbf{A} \notin \mathcal{V}(\mathbf{B})$ *and* $\mathbf{B} \not\models Y^{(i)}_{\mathbf{A}} = 1$.

2. Expansions of residuated lattices

This section is entirely devoted to presenting a construction that embeds a given finite si residuated lattice $\mathbf{A}$ into another one, called an *expansion* of $\mathbf{A}$, in a certain special way that will prove useful later on.

We begin the construction by fixing a finite si residuated lattice $\mathbf{A}$, with the coatom c. Now, take the set $A_0 = \{a \in A : ca < a\}$, and let D be any set disjoint from A, with $|D| = |A_0|$. Thus, by means of any bijection, we can index the elements of D by the elements of A_0, getting $D = \{d_a : a \in A_0\}$. Let $B = A \cup D$. We will proceed to define a relation and an operation on B.

Definition 1. For $x, y \in B$, we put $x \leq y$ if either:

- $x, y \in A$ and $x \leq^{\mathbf{A}} y$; or
- $x = d_a \in D, y \in A$ and $a \leq^{\mathbf{A}} y$; or
- $x \in A, y = d_a \in D$ and $x \leq^{\mathbf{A}} ca$; or
- $x = d_a, y = d_b \in D$ and $a \leq^{\mathbf{A}} b$.

Notice that we have $ca < d_a < a$ whenever $ca < a$.

Then, we pass on to define a binary operation '$\cdot$' on B. To avoid overloaded notation, we will abbreviate $x \cdot^{\mathbf{A}} y$ everywhere by xy. Thereby, we commit ourselves to never abbreviating the new operation $x \cdot y$, within the present section.

Definition 2. For any $x, y \in B$, let:

$$x \cdot y = y \cdot x = \begin{cases} xy, & \text{if } \ x, y \in A; \\ d_{ay}, & \text{if } \ x = d_a \in D, \ y \in A, \ cay < ay; \\ ay, & \text{if } \ x = d_a \in D, \ y \in A, \ cay = ay; \\ cab, & \text{if } \ x = d_a, y = d_b \in D \end{cases}.$$

The structure $\mathbf{B} = \langle B; \cdot, 1, 0, \leq \rangle$ defined above turns out to be almost a residuated lattice. Namely, we have:

Fact 2. *The structure* $\mathbf{B}$ *is a partially ordered, bounded, commutative, integral monoid. Moreover, '$\cdot$' is monotonic, i.e., if $x \leq y$, then $z \cdot x \leq z \cdot y$, for any $x, y, z \in B$.*

To state the next observation, it will be convenient to view $\mathbf{B}$ as a partial algebra $\langle B; \wedge^?, \vee^?, \rightarrow^?, \cdot, 0, 1 \rangle$, where $\wedge^?, \vee^?, \rightarrow^?$ coincide, respectively, with the meet, join, and residuation, whenever they exist, and are undefined otherwise.

Fact 3. $\mathbf{A}$ *is a subalgebra of* $\mathbf{B}$.

So far, we have been dealing with two 'sorts' of elements: members of A, and members of D. To get rid of this tiresome division, let's write d instead of d_1 (notice that the element d_1 indeed exists, i.e., is in D, for $c1 = c < 1$), and state:

Fact 4. *For any $x, y \in A$, the following hold:*

(i) *if $cx < x$, then $d \cdot x = d_x$, otherwise $d \cdot x = x$;*

(ii) $d \cdot x \cdot d \cdot y = cxy$;

(iii) $d \cdot x \leq d \cdot y$ *iff* $d \cdot x \leq y$ *iff* $x \leq y$;

Despite Fact 3 above, we cannot expect $\mathbf{B}$ to be a fully fledged (i.e., not partial) residuated lattice. Indeed, simple examples show that $\mathbf{B}$ might be neither residuated nor a lattice. To deal with this unwelcome situation, we will resort to a completion technique, reminiscent of what has been used in [5] or [6], yet quite substantially different.

Definition 3. A subset X of B is closed, if the following four conditions are satisfied:

- $0^{\mathbf{A}} \in X$;
- $\forall x, y \in B : x \in X$ and $y \leq x$ imply $y \in X$;
- $\forall x, y \in A : x \in X$ and $y \in X$ imply $x \vee y \in X$;
- $\forall x, y \in A : d \cdot x \in X$ and $d \cdot y \in X$ imply $d \cdot (x \vee y) \in X$.

Since, as it is easy to verify, the intersection of any family of closed sets in the above sense is itself closed, we can define $C : \wp(B) \longrightarrow \wp(B)$ to be a map sending each $X \subseteq B$ to the smallest closed subset of B containing X. As usual we denote it by $C(X)$ and call the closure of X. To justify this terminology, we have the following:

Fact 5. *The map C defined above is a closure operation on B.*

For a closed $X \subseteq B$, define $\hat{x}$ to be $\bigvee\{x \in A : x \in X\}$, and $\dot{x}$ to be $\bigvee\{x \in A : d \cdot x \in X\}$. Since, by Fact 3, joins of elements of A exist, and are again in A, these are legitimate definitions.

Fact 6. *If $X \subseteq B$ is closed, then $X = (\hat{x}] \cup (d \cdot \dot{x}]$. Moreover, if $a \in A$, then $(a]$ is closed.*

For $X, Y \subseteq B$, we define $X \Rightarrow Y$ to be the set $\{z \in B \mid \forall x \in X : z \cdot x \in Y\}$, and $Y \circ X$ to be $\{x \cdot y \mid x \in X, y \in Y\}$.

Fact 7. *Let X, Y be closed subsets of B, and let $Q = C(X \circ Y)$. The following hold:*

(i) $\hat{q} = c\dot{x}\dot{y} \vee \hat{x}\hat{y}$, $\dot{q} = \dot{x}\hat{y} \vee \hat{x}\dot{y}$;

(ii) $C(X \circ Y) = (c\dot{x}\dot{y} \vee \hat{x}\hat{y}] \cup (d \cdot (\dot{x}\hat{y} \vee \hat{x}\dot{y})]$;

(iii) *$X \Rightarrow Y$ is closed.*

Definition 4. *Let $\mathbf{C}$ be the algebra $\langle C; \wedge, \vee, \cdot, \rightarrow, 1, 0\rangle$, with the universe C being the set of all closed subsets of B, and the operations defined as follows:*

- $X \wedge Y = X \cap Y$,
- $X \vee Y = C(X \cup Y)$,

- $X \cdot Y = C(X \circ Y)$,
- $X \to Y = X \Rightarrow Y$,
- $1^{\mathbf{C}} = B$, $0^{\mathbf{C}} = \{0^{\mathbf{A}}\}$.

We will refer to the algebra **C** defined above, as the *expansion* of **A**.

Fact 8. *The expansion* **C** *of* **A** *is a residuated lattice, and* **A** *is a subalgebra of* **C**.

We will sum up the properties of **C** in the lemma below. First, however, yet another definition. Let **A** be a finite si residuated lattice with the monolithic congruence filter μ.

Definition 5. *The filter μ is of depth n iff n is the smallest natural number for which $c^n = c^{n+1} = \star$, where c is the coatom of* **A**.

Lemma 3. *Let* **A** *be a finite si residuated lattice with the monolithic congruence filter μ of depth n, and* **C** *be its expansion. Then, the following hold:*

(i) $\mathbf{A} \subseteq \mathbf{C}$,

(ii) **C** *is si,*

(iii) ν, *the monolithic congruence filter of* **C**, *is of depth* $2n$,

(iv) $\mu = \nu|_{\mathbf{A}}$.

It is easy to observe that the construction presented here can be iterated. The next lemma is a consequence of this observation.

Lemma 4. *Let* **A** *be a finite si residuated lattice with the monolith μ of depth n, and k be any natural number. Then, there is an si residuated lattice* **B** *with the monolith ν, such that:*

(i) $\mathbf{A} \subseteq \mathbf{B}$,

(ii) ν *is of depth greater than k,*

(iii) $\mu = \nu|_{\mathbf{A}}$.

3. Lack of splittings

To make use of Lemma 2 we will need another construction. Let **A**, **B** be finite si residuated lattices. We reserve the letter c to stand for the coatom of **A**, and q for the coatom of **B**.

Then, we proceed to define $\mathbf{A} \odot \mathbf{B} = \langle ((A \setminus \{1\}) \times (B \setminus \{1\})) \cup \{\langle 1, 1\rangle\}; \wedge, \vee, \cdot, \to, \langle 1, 1\rangle, \langle 0, 0\rangle\rangle$.

We will write a_i, instead of $\langle a, i\rangle$, and 1_1, 0_0 we will further abbreviate to 1, 0, whenever it will not cause confusion. In other words, we view the elements of $A \odot B$ as elements of A indexed by elements of B.

The operations on $A \odot B$ are defined by:

$$\begin{aligned} a_i \wedge b_j &=_{\mathrm{df}} (a \wedge b)_{i \wedge j}, \\ a_i \vee b_j &=_{\mathrm{df}} (a \vee b)_{i \vee j}, \\ a_i \cdot b_j &=_{\mathrm{df}} (a \cdot b)_{i \cdot j}, \\ a_i \to b_j &=_{\mathrm{df}} \begin{cases} (a \to b)_{i \to j}, & \text{if } \ a \not\leq b, \ i \not\leq j; \\ (a \to b)_q, & \text{if } \ a \not\leq b, \ i \leq j; \\ 1_1, & \text{if } \ a \leq b, \ i \leq j; \\ c_{i \to j}, & \text{if } \ a \leq b, \ i \not\leq j. \end{cases} \end{aligned}$$

Fact 9. $\mathbf{A} \odot \mathbf{B}$ *is an si residuated lattice.*

Now, let **A**, μ, **B**, $n \leq k \in \omega$ be as in Lemma 4, and let $m \geq k$ be the depth of ν. $\mathbf{B} \odot \mathbf{Ł}_{p+1}$, where $\mathbf{Ł}_{p+1}$ is the simple Łukasiewicz algebra with $p + 1$ elements, for the first prime number p greater or equal to $|B|$. As previously, let c, q stand for the unique coatoms of **A**, $\mathbf{Ł}_{p+1}$, respectively.

Fact 10. $\mathbf{A} \notin \mathcal{V}(\mathbf{B} \odot \mathbf{Ł}_{p+1})$. *Moreover, there is a valuation v such that $\mathbf{B} \odot \mathbf{Ł}_{p+1} \models_v \Delta_{\mathbf{A}} = c_q$, and $\mathbf{B} \odot \mathbf{Ł}_{p+1} \not\models_v Y^{(k)}_{\mathbf{A}} = 1$, for any $k < p$.*

We are now ready to state our main result.

Theorem 1. *The only algebra that splits $L^v(\mathcal{R})$ is the two-element boolean algebra* **2**.

Proof. That **2** splits $L^v(\mathcal{R})$ follows by Fact 2. Let **A** be a finite si residuated lattice different from **2**. Take any $k \in \omega$. Let B be the algebra whose existence is guaranteed by Lemma 4. Then, by Fact 10, $\mathbf{B} \odot \mathbf{Ł}_{p+1}$ falsifies $Y_{\mathbf{A}}^{(k)} = 1$, and $\mathbf{A} \notin \mathcal{V}(\mathbf{B} \odot \mathbf{Ł}_{p+1})$. Together, these constitute precisely the condition (ii) of Lemma 2, by which the conclusion follows.

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