Bracket Operations in the Homotopy Theory of a 2-category

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Summary. - *We study properties of a class of secondary operations, similar to the classical Toda bracket, that are defined in the context of a 2-category with zeros. Specialised to the 2-category of based spaces, maps and tracks, the applications include new formulae for matrix Toda brackets and for a new operation that we call the box bracket. Sample computations for the brackets are given in the homotopy groups of spheres.*

As a category, the homotopy category of pointed topological spaces has a very striking feature: it permits definition of secondary operations. Indeed the secondary composition operations discovered by H. Toda, the Toda brackets, have played a fundamental role in the ongoing computation of the homotopy groups of spheres and are largely responsible for the flavour of the subject. Values of secondary operations generally present themselves as cosets of a subgroup (or subgroups) of a *track group* of form $\pi(\Sigma X, Y)$ or, equivalently up to adjoint isomorphism, $\pi(X, \Omega Y)$. As pointed out by Barratt [1] and Rutter [16], a *track* is, properly considered, a relative homotopy class of homotopies.

By using nullity sets of 2-morphisms it was shown in [6] that a generalised notion of track is available in an arbitrary 2-category with zeros. If $f : A \to B$ and $f \simeq o$ (o denotes the zero morphism) then

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the track group from f to f may be identified with the group $\mathcal{A}(o : A \to B)$ of all invertible self 2-morphisms of the zero 1-morphism from A to B. (See also §1.) In [6], this notion was used to define abstract analogues of the classical Toda bracket and also matrix Toda brackets, establishing the relevant indeterminacies. In the present paper we study systematically a somewhat more general secondary operation which we call the **box bracket**. Namely, we consider a homotopy commutative diagram

$$
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B \\
{\scriptstyle h}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle b} \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\ } & X
\end{array}
$$

in which the composites $g \circ w$ and $a \circ u$ are assumed to be null homotopic. Then the box bracket

$$
\boxbox\left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B \\
{\scriptstyle h}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle b} \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\ } & X
\end{array}
\right) \quad \subset \quad \mathcal{A}(o : W \to X)
$$

is defined (see Definition 2.1 below). Of course in this situation the composite $a \circ f \circ w$ must be null homotopic; however the Toda bracket $\{a, f, w\}$ is not necessarily defined, because it need not be true that the composites $f \circ w$ and $a \circ f$ are null homotopic (cf Remark 3.5 below).

One very basic situation where box brackets naturally arise is the following. Suppose given 1-morphisms

$$
W \xrightarrow{\ w\ } C \xrightarrow{\ f\ } A \xrightarrow{\ a\ } X \xrightarrow{\ s\ } Y
$$

satisfying $a \circ f \simeq o$ and $s \circ a \simeq o$. Then the Toda bracket $\{s, a, f\}$ is defined and we may consider $\{s, a, f\} \circ w$ (equivalently $\{s, a, f\} \circ Ew$ in the topological case). If in fact $f \circ w \simeq o$ then we have

$$
\{s, a, f\} \circ w = -s \circ \{a, f, w\}
$$

by a widely used lemma of Toda [17, Proposition 1.4]. Otherwise the formula fails and one of our main themes here is to investigate how

the failure may be measured in terms of matrix Toda brackets and box brackets. Corollary 4.5 is a good example of what we have in mind. This suggests a potential computational role for box brackets.

An outline of the paper is as follows. In §1 as preliminary we consider in the category of homotopy pair maps of a 2-category with zeros an operation called the basic box operation. Its indeterminacy is identified as a double coset. The basic box operation includes as a special case the homotopy pair bracket set studied in [4]. Somewhat in passing and as in [4] we observe that an exact sequence of Mayer-Vietoris type continues to hold. Then the box bracket operation is defined in §2 and we compute its indeterminacy in Proposition 2.4. Proposition 2.5 characterizes the image of the box bracket under one of the maps in the above mentioned exact sequence. We devote §3 to various specific situations where box brackets are expressible in terms of (classical) Toda brackets and matrix Toda brackets. Under appropriate restrictions we exhibit the relationship of the box bracket to a toral construction of Rutter [16].

In §4 and §5 we explore analogues for box brackets of several properties of Toda brackets. Generalizations of both horizontal and vertical type are treated. One of the principal results is Theorem 4.4. We draw attention especially to Corollary 4.6. This result is an equality for matrix Toda brackets which evidently has escaped previous notice. Some computations based on this corollary are given in [11]. Another key result is Theorem 5.5 where a 3×3 equality for box brackets is obtained.

In the remainder of the paper we specialize to the 2-category $\mathcal{T}op_*$ of based topological spaces, maps and track classes of homotopies. In this topological setting further properties of the classical Toda bracket can be generalized to the box bracket. In particular an "extension-coextension" definition is available. In §7 we present some examples where the box bracket is relevant. It is well to observe that results in the abstract theory do not always transfer neatly to the topological case and so comparison with the topological case is not redundant. For, while there is a natural identification of $\mathcal{A}(o : W \to X)$ with $\pi(\Sigma W, X)$ (see Proposition 6.1), certain subgroups of $\mathcal{A}(o : W \to X)$ are not easily described in $\pi(\Sigma W, X)$. For instance this is the situation even with regard to the indeterminacy

of the box bracket (see Theorem 6.4).

This article is devoted to presenting the elementary theory (basic definitions and fundamental properties) of box brackets in a very general categorical setting. Our primary focus has not been to make specific computations. Box brackets (in the topological case) can be detected by Hopf invariants and (for elements of the homotopy groups of spheres) evaluated by methods similar to those used for Toda brackets. Further detailed computations are postponed to subsequent papers. Also it seems likely that in an appropriately restricted setting there is some connection between our work and that of Baues [2] on universal Toda brackets. However we have not undertaken such a comparison here.

1. Certain 2-categories; null homotopic 1-morphisms

In this section we present the setting in which we will work and formulate the basic box operation. We freely use notations and concepts introduced in [6].

If $\mathcal{C}$ is a 2-category then the lax morphism category of $\mathcal{C}$, denoted m$\mathcal{C}$, can be formed. Its objects are the 1-morphisms of $\mathcal{C}$. For given 1-morphisms $h : W \to U$ and $f : C \to A$ in $\mathcal{C}$ a 1-morphism in m$\mathcal{C}$ is defined by a square

$$
\begin{array}{ccc}
W & \xrightarrow{w} & C \\
h \downarrow & \overset{F}{\Longrightarrow} & \downarrow f \\
U & \xrightarrow{u} & A
\end{array}
\tag{1}
$$

with specified 2-morphism $F : u \circ h \Rightarrow f \circ w$. The 2-morphisms in m$\mathcal{C}$ are modifications (in the categorical sense; cf [8], [9] or [10, p. 554]) of these squares; the exact definition is not specifically recalled but may be inferred from the definition of null homotopic 1-morphism given below. In [9] the op-lax dual of m$\mathcal{C}$ is denoted HPM($\mathcal{C}$) and the associated homotopy category HPC($\mathcal{C}$). However here we work directly with m$\mathcal{C}$ and let Hm$\mathcal{C}$ denote the associated homotopy category. Also we use $\pi(h, f)$ to denote the morphism set in Hm$\mathcal{C}$ whose elements by definition are "homotopy" classes of 1-morphisms in m$\mathcal{C}$;

these are denoted

$$\left[\begin{array}{ccc} W & \xrightarrow{\ w\ } & C \\ h \downarrow & \overset{F}{\Longrightarrow} & \downarrow f \\ U & \xrightarrow{\ u\ } & A \end{array}\right]$$

for want of a better notation.

For $\mathcal{C} = \mathcal{T}op_*$, the 2-category of based topological spaces (cf Section 6 below), m$\mathcal{T}op_*$ and Hm$\mathcal{T}op_*$ correspond to the category of homotopy pair maps and the category of homotopy pair classes as introduced by the first author [3] and much utilized thereafter. For (1) in $\mathcal{T}op_*$ (thus here F is a track class of homotopies) the homotopy class has been referred to as the coherence class of the square.

If the 2-category $\mathcal{C}$ has zeros then also m$\mathcal{C}$ is a 2-category with zeros. Note that the unique zero map in $\hom_{\mathrm{m}\mathcal{C}}(h, f)$ corresponds to the square

$$\begin{array}{ccc} W & \xrightarrow{\ o\ } & C \\ h \downarrow & \overset{1_o}{\Longrightarrow} & \downarrow f \\ U & \xrightarrow{\ o\ } & A \end{array}$$

To say that (1) is a **null homotopic** 1-morphism in m$\mathcal{C}$ means that there exist invertible 2-morphisms $H : o \Rightarrow u$ and $K : o \Rightarrow w$ such that

$$\begin{array}{ccc} W & \xrightarrow{\ w\ } & C \\ h \downarrow & \overset{F}{\Longrightarrow} & \downarrow f \\ U & \xrightarrow{\ u\ } & A \\ & \underset{\Uparrow H}{\overset{o}{\frown}} & \end{array} \quad = \quad \begin{array}{ccc} & \overset{w}{\Uparrow K} & \\ W & \xrightarrow{\ o\ } & C \\ h \downarrow & \overset{1_o}{\Longrightarrow} & \downarrow f \\ U & \xrightarrow{\ o\ } & A \end{array}$$

in $\mathcal{C}$. That is, $F + Hh = fK + 1_o$ or equivalently $-fK + F + Hh = 1_o$. Recalling the relation $\langle conj \rangle$ of conjugation of 2-morphisms [6, Definition 1.2], we see that if (1) is null homotopic then $F \langle conj \rangle 1_o$. In particular this implies (see [6, Propositions 2.2, 2.4 and 2.5]) that F itself must be invertible with $w \simeq o$ and $u \simeq o$. Moreover the nullity set N(F) of F (see [6, Definition 2.1]) must coincide with the group $\mathcal{A}(o : W \to A)$ of all invertible self 2-morphisms of the zero 1-morphism $o : W \to A$.

In the next definition we consider a notion weaker than requiring (1) to be null homotopic in m$\mathcal{C}$; namely we drop the requirement that the composite 2-morphism $-fK + F + Hh$ be equal to 1_o. However we do maintain that F, H and K be invertible; this is done in order to obtain a subset of $\mathcal{A}(o : W \rightarrow A)$ rather than just a subset of N(F).

DEFINITION 1.1. *Let $\mathcal{C}$ be a 2-category with zeros and (1) a square of 1-morphisms with a fixed invertible 2-morphism $F : u \circ h \Rightarrow f \circ w : W \rightarrow A$. Assume that $w \simeq o$ and $u \simeq o$. Then the* **basic box operation**

$$\mathcal{B}(F) \subset \mathcal{A}(o : W \rightarrow A)$$

is defined. It corresponds to the set of all 2-morphisms of the form

$$-fK + F + Hh : o \Rightarrow o : W \rightarrow A$$

for all possible homotopies $H : o \Rightarrow u$ and $K : o \Rightarrow w$. Recall that a **homotopy** *is an invertible 2-morphism and notice that there is a "direction" associated to $\mathcal{B}(F)$. Actually our choice of the notation $\mathcal{B}(F)$ is somewhat deficient in that it does not reflect that we are regarding (1) as a 1-morphism in m$\mathcal{C}$. Moreover it is well to emphasize that the basic box operation is defined for an individual F. In fact the subset obtained by considering simultaneously all possible F's, that is,*

$$\{-fK + F + Hh \mid \textit{for all possible homotopies } H : o \Rightarrow u,$$
$$K : o \Rightarrow w, F : u \circ h \Rightarrow f \circ w\},$$

is readily seen to equal the whole of $\mathcal{A}(o : W \rightarrow A)$ itself and so is not of interest.

DEFINITION 1.2. *We say that $\mathcal{B}(F)$ is* **trivial** *if the identity 2-morphism 1_o belongs to $\mathcal{B}(F)$. We note that (1) is null homotopic as a 1-morphism of m$\mathcal{C}$ if and only if $\mathcal{B}(F)$ is trivial.*

If $\mathcal{C}$ has zeros and $h : W \rightarrow U$ and $f : C \rightarrow A$ are arbitrary 1-morphisms then there is defined a function

$$\nabla_{(h,f)} : \mathcal{A}(o : W \rightarrow A) \rightarrow \pi(h, f)$$

given by

$$
W \; \overset{o}{\underset{o}{\Uparrow \xi}} \; A \qquad \mapsto \qquad
\begin{bmatrix}
W \xrightarrow{\;o\;} C \\
h \downarrow \quad \overset{\xi}{\Rightarrow} \quad \downarrow f \\
U \xrightarrow{\;o\;} A
\end{bmatrix}
$$

where "[ ]" here denotes the "homotopy" class of a 1-morphism in $m\mathcal{C}$. There is also a function $\delta_{(h,f)} : \pi(h,f) \to \pi(U,A) \times \pi(W,C)$ given by

$$
\begin{bmatrix}
W \xrightarrow{\;w\;} C \\
h \downarrow \quad \overset{F}{\Rightarrow} \quad \downarrow f \\
U \xrightarrow{\;u\;} A
\end{bmatrix}
\qquad \mapsto \qquad ([u],[w]).
$$

The subscript notation (h,f) is used for these functions in order to show dependence on h and f. (The notation of [4] for $\delta_{(h,f)}$ in the topological case was (c,d).) Then the sequence

$$
\mathcal{A}(o : U \to A) \oplus \mathcal{A}(o : W \to C) \overset{\binom{\cdot\, h}{f\, \cdot}}{\longrightarrow} \mathcal{A}(o : W \to A)
$$
$$
\overset{\nabla_{(h,f)}}{\longrightarrow} \pi(h,f) \overset{\delta_{(h,f)}}{\longrightarrow} \pi(U,A) \times \pi(W,C) \quad (2)
$$

is an exact sequence as pointed sets. We omit the proof as it just follows the lines of [4, §2]. Moreover by the exactness we may state the equality

$$
\bigcup \{ \nabla_{(h,f)}(\mathcal{B}(F)) \mid \text{all such } F \text{ in } (1) \} = \delta^{-1}_{(h,f)}(o,o).
$$

PROPOSITION 1.3. *In Definition 1.1 denote by* $\theta = -fK + F + Hh$ *a fixed but arbitrary element of* $\mathcal{B}(F)$. *Then*

$$
\mathcal{B}(F) = f \circ \mathcal{A}(o : W \to C) + \theta + \mathcal{A}(o : U \to A) \circ h
$$

as subsets of $\mathcal{A}(o : W \to A)$. *That is,* $\mathcal{B}(F)$ *is a double coset of the subgroups* $f \circ \mathcal{A}(o : W \to C)$ *and* $\mathcal{A}(o : U \to A) \circ h$. *And consequently, whenever* $\mathcal{A}(o : W \to A)$ *is abelian,* $\mathcal{B}(F)$ *is a coset of the subgroup*

$$
f \circ \mathcal{A}(o : W \to C) + \mathcal{A}(o : U \to A) \circ h.
$$

Proof. Let the double coset

$$f \circ \mathcal{A}(o : W \to C) + \theta + \mathcal{A}(o : U \to A) \circ h$$

be denoted by $\mathcal{S}$. An arbitrary element of $\mathcal{B}(F)$ is of the form $-fK' + F + H'h$ for some homotopies $H' : o \Rightarrow u$ and $K' : o \Rightarrow w$. Now we may write

$$-fK' + F + H'h = -fK' + fK - fK + F + Hh - Hh + H'h$$
$$= f(-K' + K) + \theta + (-H + H')h.$$

Noting that $-K' + K \in \mathcal{A}(o : W \to C)$ and $-H + H' \in \mathcal{A}(o : U \to A)$ we conclude that this last expression is an element of $\mathcal{S}$. This proves that $\mathcal{B}(F) \subset \mathcal{S}$.

To establish the reverse inclusion let ζ be an element of $\mathcal{S}$. It is of the form $\zeta = fL + \theta + Ph$ for some homotopies $L : o \Rightarrow o : W \to C$ and $P : o \Rightarrow o : U \to A$. Then

$$\zeta = fL + \theta + Ph$$
$$= fL - fK + F + Hh + Ph$$
$$= -f(K - L) + F + (H + P)h.$$

This last expression represents an element of $\mathcal{B}(F)$. Hence $\zeta \in \mathcal{B}(F)$ as claimed. $\qquad \square$

EXAMPLE 1.4. *Consider a diagram*

with fixed homotopies F and G. Also suppose that $a \circ u \simeq o$ and $g \circ w \simeq o$. Then we set $\{G, F\} = \mathcal{B}(Gw + aF)$. By Proposition 1.3 $\{G, F\}$ is a double coset of the subgroups $b \circ \mathcal{A}(o : W \to B)$ and $\mathcal{A}(o : U \to X) \circ h$. If the diagram given is regarded as the composite

of two 1-morphism in mC then this composite is null homotopic in mC if and only if $\{G, F\}$ is trivial (cf Definition 1.2). In the topological case $\{G, F\}$ is the operation considered by Hardie-Kamps [4, (1.5)], where it is called the **homotopy pair bracket set**. *In this latter case the operation is to be regarded as a subset of $\pi(\Sigma W, X)$ and is a double coset of the subgroups $b \circ \pi(\Sigma W, B)$ and $\pi(\Sigma U, X) \circ \Sigma h$.*

EXAMPLE 1.5. *Consider the diagram (regarded as a square)*

$$
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{f} & A \\
{\scriptstyle w}\downarrow & & {\scriptstyle 1_{a\circ f\circ w}}\Downarrow & & \downarrow{\scriptstyle a} \\
C & \xrightarrow{f} & A & \xrightarrow{a} & X
\end{array}
$$

in which it is assumed that $f \circ w \simeq 0$ and $a \circ f \simeq 0$, and where $1_{a\circ f\circ w}$ denotes the identity 2-morphism on $a \circ f \circ w$. Then the basic box operation is defined and clearly we have $\mathcal{B}(1_{a\circ f\circ w}) = \{a, f, w\}$ where the notation $\{a, f, w\}$ denotes the classical Toda bracket as used in [6, Definition 8.1]. We caution that this usage corresponds to what would be denoted $-\{a, f, w\}$ in [4, (1.3)]. (Also see Proposition 3.1 below.)

2. The box bracket

In this section $\mathcal{C}$ is a 2-category with zeros and for reference we fix the following diagram in $\mathcal{C}$.

$$
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}
\tag{3}
$$

DEFINITION 2.1. *In diagram (3) assume that $u \circ h \simeq f \circ w$, $a \circ f \simeq b \circ g$, $g \circ w \simeq o$ and $a \circ u \simeq o$. Then the* **box bracket**

$$
\Box\left(
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}
\right) \subset \mathcal{A}(o : W \to X)
$$

is defined. It corresponds to the set of all composite 2-*morphisms of the form*

$$-bK + Gw + aF + Hh : o \Rightarrow o : W \to X$$

for all possible homotopies $H : o \Rightarrow a \circ u$, $F : u \circ h \Rightarrow f \circ w$, $G : a \circ f \Rightarrow b \circ g$ *and* $K : o \Rightarrow g \circ w$ *as indicated in the following diagram.*

(4)

In relation to the operation in Example 1.4 it is clear that the equality

$$\boxdot \left(\begin{array}{c} W \xrightarrow{w} C \xrightarrow{g} B \\ h \downarrow \quad\quad \downarrow f \quad\quad \downarrow b \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \bigcup \{ \{G, F\} \mid \text{all possible homotopies} \\ G : a \circ f \Rightarrow b \circ g, F : u \circ h \Rightarrow f \circ w \}$$

is valid.

In the next proposition we compute the indeterminacy of the box bracket operation.

PROPOSITION 2.2. *In Definition 2.1 let* $\theta = -bK + Gw + aF + Hh$ *denote a fixed but arbitrary element of the box bracket. Then*

$$\boxdot \left(\begin{array}{c} W \xrightarrow{w} C \xrightarrow{g} B \\ h \downarrow \quad\quad \downarrow f \quad\quad \downarrow b \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \begin{array}{l} b \circ \mathcal{A}(o : W \to B) + [\mathcal{A}(b \circ g) \circ w]^{bK} \\ + \theta + [a \circ \mathcal{A}(u \circ h)]^{Hh} \\ + \mathcal{A}(o : U \to X) \circ h \end{array}$$

as subsets of $\mathcal{A}(o : W \to X)$. *Alternatively the equalities*

$$[\mathcal{A}(b \circ g) \circ w]^{bK} = [\mathcal{A}(a \circ f) \circ w]^{-Gw + bK}$$

$$[a \circ \mathcal{A}(u \circ h)]^{Hh} = [a \circ \mathcal{A}(f \circ w)]^{aF + Hh}$$

may be used. Furthermore, if the composites $a \circ u \circ h$ and $b \circ g \circ w$ are assumed to be admissible (in the sense of [6, Definition 6.3]) then the subsets

$$b \circ \mathcal{A}(o : W \to B) + [\mathcal{A}(b \circ g) \circ w]^{bK}$$

$$[a \circ \mathcal{A}(u \circ h)]^{Hh} + \mathcal{A}(o : U \to X) \circ h$$

are subgroups of $\mathcal{A}(o : W \to X)$ and consequently the box bracket is a double coset of these subgroups.

Proof. First we note that the equalities

$$[b \circ \mathcal{A}(g \circ w) + \mathcal{A}(b \circ g) \circ w]^{bK} = b \circ \mathcal{A}(o : W \to B) + [\mathcal{A}(b \circ g) \circ w]^{bK}$$

$$[a \circ \mathcal{A}(u \circ h) + \mathcal{A}(a \circ u) \circ h]^{Hh} = [a \circ \mathcal{A}(u \circ h)]^{Hh} + \mathcal{A}(o : U \to X) \circ h$$

are valid (cf [6, §6]). Now if $b \circ g \circ w$ is admissible then $b \circ \mathcal{A}(g \circ w) + \mathcal{A}(b \circ g) \circ w$ is a subgroup and hence so is $b \circ \mathcal{A}(o : W \to B) + [\mathcal{A}(b \circ g) \circ w]^{bK}$. For a similar reason $[a \circ \mathcal{A}(u \circ h)]^{Hh} + \mathcal{A}(o : U \to X) \circ h$ is a subgroup if $a \circ u \circ h$ is admissible. These observations establish the last part of the proposition.

Next, to verify the description of the box bracket which is claimed in the proposition, set

$$\mathcal{R} = \Box\Box \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\quad f\downarrow \quad\quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right)$$

and

$$\mathcal{T} = b \circ \mathcal{A}(o : W \to B) + [\mathcal{A}(b \circ g) \circ w]^{bK} + \theta$$
$$+ [a \circ \mathcal{A}(u \circ h)]^{Hh} + \mathcal{A}(o : U \to X) \circ h.$$

By the above equations it will be sufficient to show that $\mathcal{R} = \mathcal{T}$.

Now an arbitrary element of $\mathcal{R}$ is of the form

$$-bK' + G'w + aF' + H'h$$

for some homotopies $H' : o \Rightarrow au$, $F' : uh \Rightarrow fw$, $G' : af \Rightarrow bg$, and $K' : o \Rightarrow gw$. We write

$$- bK' + G'w + aF' + H'h$$
$$\begin{aligned}
= \;\; & -bK' + bK - bK + G'w - Gw + bK - bK + Gw \\
& +aF + Hh - Hh - aF + aF' + Hh - Hh + H'h \\
= \;\; & b(-K' + K) - bK + (G' - G)w + bK + \theta - Hh \\
& +a(-F + F') + Hh + (-H + H')h.
\end{aligned}$$

By observing that $(-K' + K) \in \mathcal{A}(o : W \to B)$, $(G' - G) \in \mathcal{A}(b \circ g)$, $(-F + F') \in \mathcal{A}(u \circ h)$ and $(-H + H') \in \mathcal{A}(o : U \to X)$, it is seen that this last expression is an element of $\mathcal{T}$. This proves that $\mathcal{R} \subset \mathcal{T}$.

To establish the reverse inclusion let ζ be an element of $\mathcal{T}$. It is of the form $\zeta = bN + (Pw)^{bK} + \theta + (aL)^{Hh} + Mh$ for homotopies $N : o \Rightarrow o : W \to B$, $P : b \circ g \Rightarrow b \circ g$, $L : u \circ h \Rightarrow u \circ h$ and $M : o \Rightarrow o : U \to X$. Then

$$\begin{aligned}
\zeta &= bN + (Pw)^{bK} + \theta + (aL)^{Hh} + Mh \\
&= bN - bK + Pw + bK + \theta - Hh + aL + Hh + Mh \\
&= -b(K - N) + (P + G)w + a(F + L) + (H + M)h
\end{aligned}$$

where $K - N : o \Rightarrow g \circ w$, $P + G : a \circ f \Rightarrow b \circ g$, $F + L : u \circ h \Rightarrow f \circ w$ and $H + M : o \Rightarrow a \circ u$. Hence $\zeta \in \mathcal{R}$ as claimed. $\square$

With reference to the exact sequence (2) we have the following proposition. The proof is straightforward and thus omitted. (Compare [4, Proposition 2.4].)

PROPOSITION 2.3. *Suppose $\mathcal{C}$ is a 2-category with zeros. Let the diagram*

$$\begin{array}{ccccc}
W & \xrightarrow{\;w\;} & C & \xrightarrow{\;g\;} & B \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\
U & \xrightarrow{\;u\;} & A & \xrightarrow{\;a\;} & X
\end{array}$$

be homotopy commutative with $g \circ w \simeq o$ and $a \circ u \simeq o$. Then

$$\nabla_{(h,b)} \left(\; \square \left(\begin{array}{ccccc}
W & \xrightarrow{\;w\;} & C & \xrightarrow{\;g\;} & B \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\
U & \xrightarrow{\;u\;} & A & \xrightarrow{\;a\;} & X
\end{array} \right) \right) = \delta^{-1}_{(f,b)}([a],[g]) \circ \delta^{-1}_{(h,f)}([u],[w])$$

in $\pi(h, b)$ where the "$\circ$" on the right hand side denotes composition in $\mathrm{Hm}\mathcal{C}$.

The definition of the box bracket admits a relative form which we now describe. Consider a diagram

$$\mathcal{D} \xrightarrow{\ t\ } \mathcal{C} \xleftarrow{\ s\ } \mathcal{E}$$

of 2-categories and 2-functors. Fix homotopy commutative squares

$$
\begin{array}{ccc}
W & \xrightarrow{\ w\ } & C' \\
\ \downarrow{\scriptstyle h} & & \ \downarrow{\scriptstyle f'} \\
U & \xrightarrow{\ u\ } & A'
\end{array}
\qquad\qquad
\begin{array}{ccc}
C'' & \xrightarrow{\ g\ } & B \\
\ \downarrow{\scriptstyle f''} & & \ \downarrow{\scriptstyle b} \\
A'' & \xrightarrow{\ a\ } & X
\end{array}
$$

in $\mathcal{D}$ and $\mathcal{E}$ respectively and suppose that

$$(tC' \ \xrightarrow{\ tf'\ }\ tA') = (C \ \xrightarrow{\ f\ }\ A) = (sC'' \ \xrightarrow{\ sf''\ }\ sA'')$$

as 1-morphisms in $\mathcal{C}$. Also assume that $sg \circ tw \simeq o$ and $sa \circ tu \simeq o$. Then by the box bracket

$$
\Box_{t,\,s}\left(
\begin{array}{ccccc}
tW & \xrightarrow{\ tw\ } & C & \xrightarrow{\ sg\ } & sB \\
\ \downarrow{\scriptstyle th} & & \ \downarrow{\scriptstyle f} & & \ \downarrow{\scriptstyle sb} \\
tU & \xrightarrow[\ tu\ ]{} & A & \xrightarrow[\ sa\ ]{} & sX
\end{array}
\right) \ \subset \ \mathcal{A}(o : tW \to sX) \qquad (5)
$$

we shall mean the set of all composite 2-morphisms in $\mathcal{C}$ of the form

$$-(sb)K + (sG)(tw) + (sa)(tF) + H(th) : o \Rightarrow o : tW \to sX$$

for all possible homotopies $F : u \circ h \Rightarrow f' \circ w$ in $\mathcal{D}$, $G : a \circ f'' \Rightarrow b \circ g$ in $\mathcal{E}$ and $H : o \Rightarrow sa \circ tu$, $K : o \Rightarrow sg \circ tw$ in $\mathcal{C}$.

We leave it to the reader to establish the following relative version of Proposition 2.2.

PROPOSITION 2.4. *Let* $\theta = -(sb)K + (sG)(tw) + (sa)(tF) + H(th)$ *denote a fixed but arbitrary element of the box bracket* (5). *Then*

$$\boxdot_{t,s}\left(\begin{array}{ccc} tW & \xrightarrow{tw} & C \xrightarrow{sg} sB \\ th\downarrow & & \downarrow f \quad\quad \downarrow sb \\ tU & \xrightarrow{tu} & A \xrightarrow{sa} sX \end{array}\right) = \begin{array}{l} sb \circ \mathcal{A}(o : tW \to sB) \\ + [s\mathcal{A}(b \circ g) \circ tw]^{(sb)K} \\ +\theta + [sa \circ t\mathcal{A}(u \circ h)]^{H(th)} \\ +\mathcal{A}(o : tU \to sX) \circ th \end{array}$$

as subsets of $\mathcal{A}(o : tW \to sX)$.

3. Some special box brackets

In this section we consider some specific situations where box brackets reduce to, or are expressible in terms of, classical and matrix Toda brackets. In particular it is proven that if one of the diagonal composites in the box bracket is null homotopic then the box bracket is the sum of a classical Toda bracket and a matrix Toda bracket (Proposition 3.3, (1) and (2)) and that if both diagonal composites are null homotopic then there is a decomposition in terms of three classical Toda brackets (Proposition 3.3(3)).

PROPOSITION 3.1. *For given 1-morphisms* $W \xrightarrow{w} C \xrightarrow{f} A \xrightarrow{a} X$ *in* $\mathcal{C}$ *suppose that* $f \circ w \simeq o$ *and* $a \circ f \simeq o$. *Then the relations*

$$-\{a, f, w\} = \boxdot\left(\begin{array}{ccc} W & \xrightarrow{w} & C \xrightarrow{\quad} * \\ \downarrow & & \downarrow f \quad\quad \downarrow \\ * & \xrightarrow{\quad} & A \xrightarrow{a} X \end{array}\right) \subset -\boxdot\left(\begin{array}{ccc} W & \xrightarrow{w} & C \xrightarrow{f} A \\ w\downarrow & & \downarrow f \quad\quad \downarrow a \\ C & \xrightarrow{f} & A \xrightarrow{a} X \end{array}\right)$$

are valid.

Proof. Recall [6, Definition 8.1], that we take

$$\{a, f, w\} = \{-aK + Lw \mid \text{all homotopies } K : o \Rightarrow f \circ w, L : o \Rightarrow a \circ f\}$$

as the definition of the classical Toda bracket $\{a, f, w\}$, while for the specific case at hand the definition of the box bracket simplifies to

$$\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \longrightarrow * \\ \big\downarrow \qquad \big\downarrow f \qquad \big\downarrow \\ * \longrightarrow A \xrightarrow{\ a\ } X \end{array} \right) = \begin{array}{l} \{Gw + aF \mid \text{all homotopies} \\ \quad F : o \Rightarrow f \circ w, G : a \circ f \Rightarrow o\}. \end{array}$$

But $-(-aK + Lw) = -Lw + aK$ and $Gw + aF = -(-aF + (-G)w)$. It follows that the equality claimed in the proposition holds. Also note that we may write

$$Gw + aF = -(-aF - Gw) = -(-aF + (1_{a \circ f})w + a(1_{f \circ w}) + (-G)w).$$

Hence the inclusion of box brackets claimed in the proposition holds as well. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

PROPOSITION 3.2. *As subsets of $\mathcal{A}(o : W \to X)$, the following equalities hold.*

(1) *If $b \simeq o$ then* $\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\big\downarrow \qquad \big\downarrow f \qquad \big\downarrow b \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) = \left\{ a, \begin{array}{c} f \\ u \end{array}, \begin{array}{c} w \\ h \end{array} \right\}.$

(2) *If $g \simeq o$ then*

$$\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\big\downarrow \qquad \big\downarrow f \qquad \big\downarrow b \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) = b \circ \mathcal{A}(o : W \to B) + \left\{ a, \begin{array}{c} f \\ u \end{array}, \begin{array}{c} w \\ h \end{array} \right\}.$$

(3) *If $h \simeq o$ then* $\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\big\downarrow \qquad \big\downarrow f \qquad \big\downarrow b \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) = \left\{ \begin{array}{c} b \\ a \end{array}, \begin{array}{c} g \\ f \end{array}, w \right\}.$

(4) *If $u \simeq o$ then*

$$\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\big\downarrow \qquad \big\downarrow f \qquad \big\downarrow b \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) = \left\{ \begin{array}{c} b \\ a \end{array}, \begin{array}{c} g \\ f \end{array}, w \right\} + \mathcal{A}(o : U \to X) \circ h.$$

Proof. We restrict ourselves to proving part (2) only; the other three parts have proofs which are quite similar and so are omitted.

Let $L : g \Rightarrow o$ be a homotopy. If the homotopies H, F, G and K are as in Definition 2.1 then we may write:

$$
\begin{aligned}
-bK + Gw + aF + Hh &= -bK - bLw + bLw + Gw + aF + Hh \\
&= b(-K - Lw) + (-(-G - bL)w + aF + Hh)
\end{aligned}
$$

Now the first term here is an arbitrary element of

$$
\boxempty \left(
\begin{array}{ccc}
W & \xrightarrow{w} C \xrightarrow{g} B \\
h \downarrow & \quad \downarrow f \quad \downarrow b \\
U & \xrightarrow{u} A \xrightarrow{a} X
\end{array}
\right)
$$

while the last term represents an element of

$$
b \circ \mathcal{A}(o : W \to B) \; + \; \left\{ a, \; \begin{array}{c} f \\ u \end{array}, \; \begin{array}{c} w \\ h \end{array} \right\}.
$$

Conversely given $\xi : o \Rightarrow o : W \to B$ and a diagram

$$
\begin{array}{c}
W \xrightarrow{w} C \\
h \Big\downarrow \overset{F}{\Rightarrow} f \Big\downarrow \overset{-J}{\Rightarrow} o \\
U \xrightarrow{u} A \\
\overset{H}{\Rightarrow} \quad a \searrow \\
o \qquad X
\end{array}
$$

representing an element of $\left\{ a, \; \begin{array}{c} f \\ u \end{array}, \; \begin{array}{c} w \\ h \end{array} \right\}$ (see [6, Definition 5.1]) we may write

$$
\begin{aligned}
b\xi + (-Jw + aF + Hh) &= b\xi + bLw - bLw - Jw + aF + Hh \\
&= -b(-Lw - \xi) + (-bL - J)w + aF + Hh
\end{aligned}
$$

since the relations $o \circ w = o : W \to B$, $b \circ o = o : C \to X$ and $-L + L = 1_o : o \Rightarrow o : C \to B$ are valid. Then the last expression given is recognizable as an element of $\boxplus \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h \downarrow \quad f \downarrow \quad b \downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right)$.

Consequently the equality stated in part (2) is valid. $\qquad\square$

PROPOSITION 3.3. *In the homotopy commutative diagram of 1-morphisms in $\mathcal{C}$*

$$\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\quad f\downarrow \quad\quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array}$$

suppose that $g \circ w \simeq o$ and $a \circ u \simeq o$.

(1) If $b \circ g \simeq o$ then

$$\boxplus \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad f\downarrow \quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \{b, g, w\} + \left\{ a, \begin{array}{c} f \\ u \end{array}, \begin{array}{c} w \\ h \end{array} \right\}$$

as subsets of $\mathcal{A}(o : W \to X)$.

(2) If $u \circ h \simeq o$ then

$$\boxplus \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad f\downarrow \quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \left\{ \begin{array}{c} b \\ a \end{array}, \begin{array}{c} g \\ f \end{array}, w \right\} + \{a, u, h\}$$

as subsets of $\mathcal{A}(o : W \to X)$.

(3) If $b \circ g \simeq o$ and $u \circ h \simeq o$ then

$$\boxplus \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad f\downarrow \quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \{b, g, w\} - \{a, f, w\} + \{a, u, h\}$$

as subsets of $\mathcal{A}(o : W \to X)$. *Moreover, if* $\mathcal{A}(o : W \to X)$ *is abelian then this subset is a coset of the subgroup*

$$b \circ \mathcal{A}(o : W \to B) + \mathcal{A}(o : C \to X) \circ w +$$
$$a \circ \mathcal{A}(o : W \to A) + \mathcal{A}(o : U \to X) \circ h.$$

Proof. We give the proof of part (1). The similar proofs of parts (2) and (3) are left to the reader.

Let $L : o \Rightarrow b \circ g$ be an assumed homotopy. Then with reference to Definition 2.1 we may write

$$-bK + Gw + aF + Hh = -bK + Lw - Lw + Gw + aF + Hh$$
$$= (-bK + Lw) + ((-L + G)w + aF + Hh)$$

so that the inclusion

$$\square\!\square \left(\begin{array}{c} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad \downarrow f \quad \downarrow b \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) \subset \{b, g, w\} + \left\{ a, \begin{array}{c} f \\ u \end{array}, \begin{array}{c} w \\ h \end{array} \right\}$$

follows at once.

An arbitrary element of $\{b, g, w\} + \left\{ a, \begin{array}{c} f \\ u \end{array}, \begin{array}{c} w \\ h \end{array} \right\}$ on the other hand has the form $(-bK + Lw) + (Jw + aF + Hh)$ for homotopies $K : o \Rightarrow g \circ w$, $L : o \Rightarrow b \circ g$, $J : a \circ f \Rightarrow o$, $F : u \circ h \Rightarrow f \circ w$ and $H : o \Rightarrow a \circ u$. This composite 2-morphism may be re-expressed in the form $-bK + (L + J)w + aF + Hh$ which shows that it also represents an

element of $\square\!\square \left(\begin{array}{c} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad \downarrow f \quad \downarrow b \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right)$. Hence the reverse inclusion

holds and therefore the equation in part (1) is established. $\square$

PROPOSITION 3.4. *Let*

be a homotopy commutative diagram in which $g \circ w = 0$ and $\overline{a} \circ \overline{u} = 0$. Then

$$\square\square\left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\downarrow \qquad 1_C\downarrow \qquad b\downarrow \\ U \xrightarrow{\ \overline{u}\ } C \xrightarrow{\ \overline{a}\ } X \end{array} \right) \ = \ \{b, g \circ \overline{u}, h\}$$

as subsets of $\mathcal{A}(o : W \to X)$.

Proof. Since the central vertical arrow in the box bracket is the identity, the Toda bracket $\{b, g \circ \overline{u}, h\}$ may be formulated. Moreover it follows directly from the definitions that the box bracket is a subset of $\{b, g \circ \overline{u}, h\}$. To show the reverse inclusion, fix homotopies $F : \overline{u} \circ h \Rightarrow w$ and $G : \overline{a} \Rightarrow b \circ g$. Now an arbitrary element of $\{b, g \circ \overline{u}, h\}$ is of the form $-bK + Lh$ for homotopies $L : o \Rightarrow b \circ g \circ \overline{u}$ and $K : o \Rightarrow g \circ \overline{u} \circ h$. Note that we may write

$$-bK + Lh = -bK - (b \circ g)F + (b \circ g)F + G(\overline{u} \circ h) - G(\overline{u} \circ h) + Lh$$
$$= -b(gF + K) + Gw + \overline{a}F + (-G\overline{u} + L)h$$

with homotopies $gF + K : o \Rightarrow g \circ w$ and $-G\overline{u} + L : o \Rightarrow \overline{a} \circ \overline{u}$. In this latter form we see that $-bK + Lh$ is also an element of the box bracket. Thus the reverse inclusion also holds. $\square$

REMARK 3.5. *Let notation be as in Proposition 3.4. It may be remarked that whenever $\{b, g \circ \overline{u}, h\}$ is nontrivial then the box bracket considered in Proposition 3.4 is nontrivial and has the property that the Toda bracket $\{\overline{a}, 1_C, w\}$ which is constructed from the central composition of 1-morphisms in the box bracket is undefined. For an*

explicit occurrence of this let $\mathcal{C} = \mathcal{T}op_*$ (see §6 below) and observe that in the homotopy groups of spheres we have

$$\square\square \left(\begin{array}{ccc} S^{11} & \xrightarrow{2\nu_8} S^8 & \xrightarrow{4\iota_8} S^8 \\ \nu_8 \downarrow & \iota_8 \downarrow & \downarrow \nu_5 \\ S^8 & \xrightarrow{2\iota_8} S^8 & \xrightarrow{\eta_5^3} S^5 \end{array} \right) \ = \ \{\nu_5, 8\iota_8, \nu_8\} \ = \ \sigma''' \ \neq \ 0$$

in $\pi_{12}^7 = \mathbb{Z}/2 = \{\sigma'''\}$ by Lemma 5.13 of [17]. Further note that this is an example with zero indeterminacy.

We conclude this section by exploring the relationship of the box bracket to a toral construction considered by Rutter[16]. But first a general remark. Let $\mathcal{C}$ be an arbitrary 2-category. Associated to any homotopy commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{w} C & \xrightarrow{g} B \\ h \downarrow & \downarrow f & \downarrow b \\ U & \xrightarrow{u} A & \xrightarrow{a} X \end{array}$$

in $\mathcal{C}$ is a subset of $\hom_{\mathcal{C}}(a \circ u \circ h, b \circ g \circ w)$ given by the union

$$\bigcup \{ \ Gw + aF \ \big| \ \text{all possible homotopies}$$
$$G : a \circ f \Rightarrow b \circ g, F : u \circ h \Rightarrow f \circ w\}.$$

If $\theta = Gw + aF$ is a fixed element then this union is readily identifiable as

$$\mathcal{A}(b \circ g) \circ w + \theta + a \circ \mathcal{A}(u \circ h).$$

Obviously this latter usually has no meaning as a double coset.

Following Rutter [16, §1], we may consider a toral construction

$$< k; g, h; f > \subset \mathcal{A}(k \circ g \circ f)$$

for 1-morphisms

$$X \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C \xrightarrow{k} D$$

defined whenever $g \circ f \simeq h \circ f$ and $k \circ g \simeq k \circ h$. This is just the above mentioned union construction for

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
B & \xrightarrow{\ g\ } & C & \xrightarrow{\ k\ } & D
\end{array}
$$

but here fortuitously in the corresponding description

$$\mathcal{A}(k \circ g) \circ f + \theta + k \circ \mathcal{A}(g \circ f)$$

the terms $\mathcal{A}(k \circ g) \circ f$ and $k \circ \mathcal{A}(g \circ f)$ are both subgroups of $\mathcal{A}(k \circ g \circ f)$ so that a double coset is obtained.

Now further let $\mathcal{C}$ have zeros and assume that $g \circ f \simeq o$ and $k \circ g \simeq o$. Then the box bracket

$$
\boxbox \left(
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
B & \xrightarrow{\ g\ } & C & \xrightarrow{\ k\ } & D
\end{array}
\right) \subset \mathcal{A}(o : X \to D)
$$

will also be defined. By Proposition 3.3(3) this box bracket equals

$$\{k, g, f\} - \{k, h, f\} + \{k, g, f\}.$$

Recall [6, Proposition 6.4] that if $L : o \Rightarrow k \circ g \circ f$ is a homotopy then there is an isomorphism

$$(\ )^L : \mathcal{A}(k \circ g \circ f) \to \mathcal{A}(o : X \to D), \quad \xi^L = -L + \xi + L,$$

which is independent of L if $\mathcal{A}(k \circ g \circ f)$ is abelian.

The following proposition is readily established.

PROPOSITION 3.6. *Assume that* $\mathcal{A}(k \circ g \circ f)$ *is abelian. If* $L : o \Rightarrow k \circ g \circ f$ *is* **any** *homotopy then*

$$
\boxbox \left(
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle k} \\
B & \xrightarrow{\ g\ } & C & \xrightarrow{\ k\ } & D
\end{array}
\right)
\begin{array}{l}
= \ (< k; g, h; f >)^L + \{k, g, f\} \\
= \ \{k, g, f\} + (< k; g, h; f >)^L
\end{array}
$$

as subsets of $\mathcal{A}(o : X \to D)$.

4. Horizontal lemmas; invariance under homotopy

In this section we establish several "horizontal" properties of the box bracket. Most of these are analogues of known properties of the classical topological Toda bracket as developed in [17, Chapter 1]. We also show that a box bracket depends only on the homotopy classes of the 1-morphisms used to define it (Theorem 4.2). In the first result we examine what happens when an arbitrary square is added either to the right or to the left of a box bracket.

PROPOSITION 4.1. *Let*

$$
\begin{array}{ccccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B & \xrightarrow{r} & D \\
\downarrow h & & \downarrow f & & \downarrow b & & \downarrow d \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X & \xrightarrow{s} & Y
\end{array}
$$

be a fixed homotopy commutative diagram of 1*-morphisms in* $\mathcal{C}$.

(1) *If* $r \circ g \simeq o$ *and* $s \circ a \simeq o$ *then*

$$
\square\square\left(\begin{array}{ccccc}
C & \xrightarrow{g} & B & \xrightarrow{r} & D \\
\downarrow f & & \downarrow b & & \downarrow d \\
A & \xrightarrow{a} & X & \xrightarrow{s} & Y
\end{array}\right) \circ w \subset \square\square\left(\begin{array}{ccccc}
W & \xrightarrow{g \circ w} & B & \xrightarrow{r} & D \\
\downarrow h & & \downarrow b & & \downarrow d \\
U & \xrightarrow{a \circ u} & X & \xrightarrow{s} & Y
\end{array}\right)
$$

with equality if both w *and* u *are homotopy equivalences in* $\mathcal{C}$.

(2) *If* $g \circ w \simeq o$ *and* $a \circ u \simeq o$ *then*

$$
s \circ \square\square\left(\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
\downarrow h & & \downarrow f & & \downarrow b \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}\right) \subset \square\square\left(\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{r \circ g} & D \\
\downarrow h & & \downarrow f & & \downarrow d \\
U & \xrightarrow{u} & X & \xrightarrow{s \circ a} & Y
\end{array}\right)
$$

with equality if both r *and* s *are homotopy equivalences in* $\mathcal{C}$.

Proof. The proofs of inclusion are readily given and similar to [4, Proposition 3.1, (ii) and (iii)]. We omit them and will only prove the last assertion of part (2).

Hence assume that both r and s are homotopy equivalences in $\mathcal{C}$. In this case we claim that the reverse inclusion holds. Let $L : s \circ b \Rightarrow d \circ r$ be a homotopy. Now we may regard the square

$$
\begin{array}{ccc}
B & \xrightarrow{\;r\;} & D \\
{\scriptstyle b}\big\downarrow & \overset{L}{\Rightarrow} & \big\downarrow{\scriptstyle d} \\
X & \xrightarrow{\;s\;} & Y
\end{array}
$$

as a 1-morphism in the 2-category m$\mathcal{C}$. By general results on homotopy equivalences (see [9]) this square must be a homotopy equivalence in m$\mathcal{C}$, because by hypothesis r and s are homotopy equivalences in $\mathcal{C}$ and the 2-morphism L is invertible. Let

$$
\begin{array}{ccc}
D & \xrightarrow{\;\overline{r}\;} & B \\
{\scriptstyle d}\big\downarrow & \overset{M}{\Rightarrow} & \big\downarrow{\scriptstyle b} \\
Y & \xrightarrow{\;\overline{s}\;} & X
\end{array}
$$

be a homotopy inverse. It follows that there exist homotopies N, P, Q and R so that the following equalities of 2-morphisms in $\mathcal{C}$ are valid.

Let the diagram

define an arbitrary element $\widetilde{\theta}$ of $\square\square$ $\left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{r \circ g} D \\ h \downarrow \quad\quad f \downarrow \quad\quad d \downarrow \\ U \xrightarrow{u} A \xrightarrow{s \circ a} Y \end{array}\right)$. By using the above homotopy equivalences we may also write $\widetilde{\theta}$ as the following composite 2-morphism.

$$
\begin{array}{c}
\\
\xuparrow{} -\widetilde{K} \qquad \Uparrow P \\
W \xrightarrow{w} C \xrightarrow{r \circ g} D \xrightarrow{\overline{r}} B \xrightarrow{r} D \\
h \downarrow \;\overset{F}{\Rightarrow}\; f \downarrow \;\overset{\widetilde{G}}{\Rightarrow}\; d \downarrow \;\overset{M}{\Rightarrow}\; b \downarrow \;\overset{L}{\Rightarrow}\; d \downarrow \\
U \xrightarrow{u} X \xrightarrow{s \circ a} Y \xrightarrow{\overline{s}} X \xrightarrow{s} Y \\
\Uparrow \widetilde{H} \qquad \Uparrow N
\end{array}
$$

This in turn may be simplified to the composite 2-morphism

$$
\begin{array}{c}
\Uparrow -\widetilde{K} \\
W \xrightarrow{w} C \xrightarrow{r \circ g} D \xrightarrow{\overline{r}} B \\
h \downarrow \;\overset{F}{\Rightarrow}\; f \downarrow \;\overset{\widetilde{G}}{\Rightarrow}\; d \downarrow \;\overset{M}{\Rightarrow}\; b \downarrow \\
U \xrightarrow{u} X \xrightarrow{s \circ a} Y \xrightarrow{\overline{s}} X \xrightarrow{s} Y \\
\Uparrow \widetilde{H}
\end{array}
$$

since the equalities

$$ dPo = 1_o = L\overline{r}o : o \Rightarrow o : W \to Y \qquad No = 1_o : o \Rightarrow o : U \to Y $$

are valid. And finally this latter may be rewritten as the following composite 2-morphism (by using the equalities $Q - Q = 1_{\overline{s} \circ s}$ and

$-R + R = 1_{\overline{r} \circ r}$).
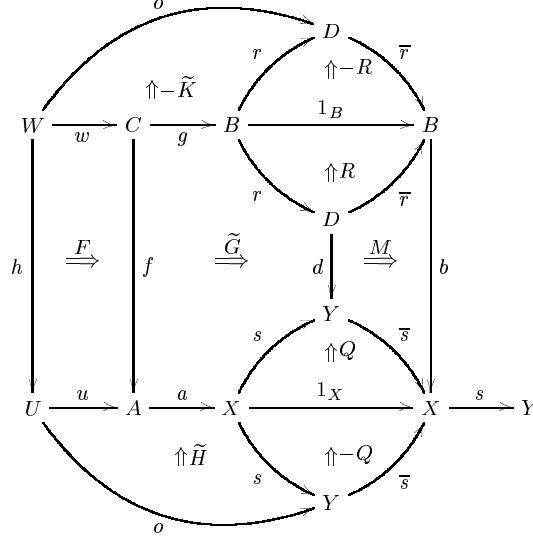
This last representation evidently displays the original element $\widetilde{\theta}$ in

the form of an element of $s \circ \boxed{} \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\quad f\downarrow \quad\quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right)$ and verifies

our claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

THEOREM 4.2. *("Invariance under homotopy") A box bracket de-
pends only on the homotopy classes of the 1-morphisms in the dia-
gram which defines it. More precisely, suppose that the box bracket
of diagram* (3) *is defined and that* $w \simeq \overline{w}$, $h \simeq \overline{h}$, $f \simeq \overline{f}$, $u \simeq \overline{u}$,
$g \simeq \overline{g}$, $a \simeq \overline{a}$ *and* $b \simeq \overline{b}$. *Then*

$$\boxed{} \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\quad f\downarrow \quad\quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \boxed{} \left(\begin{array}{ccc} W \xrightarrow{\overline{w}} C \xrightarrow{\overline{g}} B \\ \overline{h}\downarrow \quad\quad \overline{f}\downarrow \quad\quad \overline{b}\downarrow \\ U \xrightarrow{\overline{u}} A \xrightarrow{\overline{a}} X \end{array} \right) .$$

Proof. Let $\mathcal{R}$ denote the box bracket on the left above. First we
show the dependence of $\mathcal{R}$ only on the homotopy class of w. Set

$$\overline{\mathcal{R}} = \Box\Box \left(\begin{array}{ccc} W \xrightarrow{\overline{w}} C \xrightarrow{g} B \\ {\scriptstyle h}\downarrow \quad\quad {\scriptstyle f}\downarrow \quad\quad {\scriptstyle b}\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right). \text{ Let } -bK + Gw + aF + Hh \text{ be an}$$

arbitrary element of $\mathcal{R}$ and let $J : w \Rightarrow \overline{w}$ be an assumed homotopy. Since $-J + J = 1_w$ we have:

$$\begin{aligned} -bK + Gw + aF + Hh &= -bK + b \circ g(-J + J) + Gw + aF + Hh \\ &= -b(gJ + K) + G\overline{w} + a(fJ + F) + Hh \end{aligned}$$

This implies that $\mathcal{R} \subset \overline{\mathcal{R}}$. In a similar way we can show that $\overline{\mathcal{R}} \subset \mathcal{R}$. Thus $\mathcal{R}$ depends only on the homotopy class of w.

That $\mathcal{R}$ depends only on the homotopy classes of g, f, u, and a can be demonstrated in much the same way. However the dependence of $\mathcal{R}$ on the homotopy classes of h and b requires a different approach—namely, application of Proposition 4.1.

If $h \simeq \overline{h}$ then we may form a homotopy commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{1_W} & W & \xrightarrow{w} & C & \xrightarrow{g} & B \\ {\scriptstyle \overline{h}}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow \\ U & \xrightarrow{1_U} & U & \xrightarrow{u} & A & \xrightarrow{a} & X \end{array}$$

in which 1_W and 1_U are manifestly homotopy equivalences. Therefore by Proposition 4.1(1)

$$\Box\Box \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ {\scriptstyle h}\downarrow \quad {\scriptstyle f}\downarrow \quad {\scriptstyle b}\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) \circ 1_W = \Box\Box \left(\begin{array}{ccc} W \xrightarrow{w \circ 1_W} C \xrightarrow{g} B \\ {\scriptstyle \overline{h}}\downarrow \quad {\scriptstyle f}\downarrow \quad {\scriptstyle b}\downarrow \\ U \xrightarrow{u \circ 1_U} A \xrightarrow{a} X \end{array} \right)$$

but this just reduces to the equality

$$\Box\Box \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ {\scriptstyle h}\downarrow \quad {\scriptstyle f}\downarrow \quad {\scriptstyle b}\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) = \Box\Box \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ {\scriptstyle \overline{h}}\downarrow \quad {\scriptstyle f}\downarrow \quad {\scriptstyle b}\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right).$$

Hence $\mathcal{R}$ depends only on the homotopy class of h. In a similar way $\mathcal{R}$ depends only on the homotopy class of b. $\qquad\square$

PROPOSITION 4.3.

(1) *In the homotopy commutative diagram*

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g'\ } & B' & \xrightarrow{\ g\ } & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & & & {\scriptstyle b}\downarrow \\
U & \xrightarrow{\ u\ } U' & \xrightarrow{\ u'\ } & A & & \xrightarrow{\ a\ } & X
\end{array}
$$

suppose that $g' \circ w \simeq o$ *and* $a \circ u' \simeq o$. *Then the inclusion*

$$
\boxminus \left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g'\ } & B' \\
{\scriptstyle u\circ h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b\circ g}\downarrow \\
U' & \xrightarrow{\ u'\ } & A & \xrightarrow{\ a\ } & X
\end{array}
\right)
\subset
\boxminus \left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\circ g'\ } & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow \\
U & \xrightarrow{\ u'\circ u\ } & A & \xrightarrow{\ a\ } & X
\end{array}
\right)
$$

is valid.

(2) *In the homotopy commutative diagram*

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ w\ } W' & \xrightarrow{\ w'\ } & C & \xrightarrow{\quad g\quad} & & B \\
{\scriptstyle h}\downarrow & & & {\scriptstyle f}\downarrow & & & {\scriptstyle b}\downarrow \\
U & & \xrightarrow{\quad u\quad} & A & \xrightarrow{\ a'\ } A' & \xrightarrow{\ a\ } & X
\end{array}
$$

suppose that $g \circ w' \circ w \simeq o$ *and* $a \circ a' \circ u \simeq o$. *Then the inclusion*

$$
\boxminus \left(
\begin{array}{ccccc}
W & \xrightarrow{\ w'\circ w\ } & C & \xrightarrow{\ g\ } & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\circ a'\ } & X
\end{array}
\right)
\subset
\boxminus \left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & W' & \xrightarrow{\ g\circ w'\ } & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle a'\circ f\circ w'}\downarrow & & {\scriptstyle b}\downarrow \\
U & \xrightarrow{\ a'\circ u\ } & A' & \xrightarrow{\ a\ } & X
\end{array}
\right)
$$

is valid.

Proof. The proof, being straightforward, is omitted.   □

THEOREM 4.4. *In the homotopy commutative diagram*

$$
\begin{array}{ccccccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B & \xrightarrow{\ r\ } & D \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle d}\downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\ } & X & \xrightarrow{\ s\ } & Y
\end{array}
$$

suppose that all horizontal pair composites are null homotopic. Then:

(1) *The relation*

$$
s \ \circ \ \boxplus \left(
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
\downarrow{h} & & \downarrow{f} & & \downarrow{b} \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}
\right) \ \subset
$$

$$
d \circ \{r, g, w\} \ + \ \boxplus \left(
\begin{array}{ccccc}
C & \xrightarrow{g} & B & \xrightarrow{r} & D \\
\downarrow{f} & & \downarrow{b} & & \downarrow{d} \\
A & \xrightarrow{a} & X & \xrightarrow{s} & Y
\end{array}
\right) \circ w \ - \ \{s, a, u\} \circ h
$$

is valid, with equality whenever the equalities $\mathcal{A}(s \circ a) \circ u \circ h = \{1_{s \circ a \circ u \circ h}\}$ *and* $d \circ \mathcal{A}(r \circ g) \circ w = \{1_{d \circ r \circ g \circ w}\}$ *are valid.*

(2) *The relation*

$$
\boxplus \left(
\begin{array}{ccccc}
C & \xrightarrow{g} & B & \xrightarrow{r} & D \\
\downarrow{f} & & \downarrow{b} & & \downarrow{d} \\
A & \xrightarrow{a} & X & \xrightarrow{s} & Y
\end{array}
\right) \circ \ w \ \subset
$$

$$
-d \circ \{r, g, w\} \ + \ s \circ \ \boxplus \left(
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
\downarrow{h} & & \downarrow{f} & & \downarrow{b} \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}
\right) \ + \ \{s, a, u\} \circ \ h
$$

is valid, with equality whenever the equalities $d \circ r \circ \mathcal{A}(g \circ w) = \{1_{d \circ r \circ g \circ w}\}$ *and* $s \circ \mathcal{A}(a \circ u) \circ h = \{1_{s \circ a \circ u \circ h}\}$ *are valid.*

Proof. We establish the first part only; the similar proof of the second part is omitted. Set $\mathcal{L} = s \circ \boxplus \left(
\begin{array}{ccccc}
W & \xrightarrow{w} & C & \xrightarrow{g} & B \\
\downarrow{h} & & \downarrow{f} & & \downarrow{b} \\
U & \xrightarrow{u} & A & \xrightarrow{a} & X
\end{array}
\right)$ and

$$
\mathcal{R} = d \circ \{r, g, w\} \ + \ \boxplus \left(
\begin{array}{ccccc}
C & \xrightarrow{g} & B & \xrightarrow{r} & D \\
\downarrow{f} & & \downarrow{b} & & \downarrow{d} \\
A & \xrightarrow{a} & X & \xrightarrow{s} & Y
\end{array}
\right) \circ w \ - \ \{s, a, u\} \circ h.
$$

We start with the following diagram of composite 2-morphisms
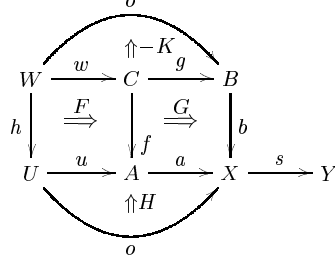

which represents an arbitrary element of $\mathcal{L}$. By hypothesis there exist homotopies $J : s \circ b \Rightarrow d \circ r$, $M : o \Rightarrow s \circ a$ and $N : o \Rightarrow r \circ g$. Then we may rewrite this diagram as
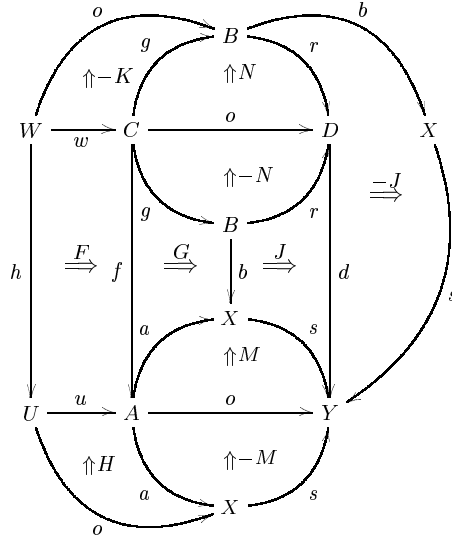
using the relations $N - N = 1_{r \circ g}$, $-J + J = 1_{s \circ b}$ and $M - M = 1_{s \circ a}$. The composite 2-morphism in this last diagram represents the element $-Jo + d(-rK + Nw) + (-dN + Jg + sG + Mf)w + oF - (-sH + Mu)h$ which reduces to $d(-rK + Nw) + (-dN + Jg + sG + Mf)w - (-sH + Mu)h$ since $-Jo = 1_o : o \Rightarrow o : W \to Y$ and $Fo = 1_o : o \Rightarrow o : W \to Y$. Note that this latter expression is in a form that allows it to be recognized as an element of $\mathcal{R}$. Thus $\mathcal{L} \subset \mathcal{R}$.

Next assume additionally that the equalities $\mathcal{A}(s \circ a) \circ u \circ h = \{1_{s \circ a \circ u \circ h}\}$ and $d \circ \mathcal{A}(r \circ g) \circ w = \{1_{d \circ r \circ g \circ w}\}$ hold. Select a homotopy $F : u \circ h \Rightarrow f \circ w$. Then we may use the fact that $oF = 1_o : o \Rightarrow o : W \to Y$ to represent an arbitrary element of $\mathcal{R}$ as a composite 2-morphism of the following form.
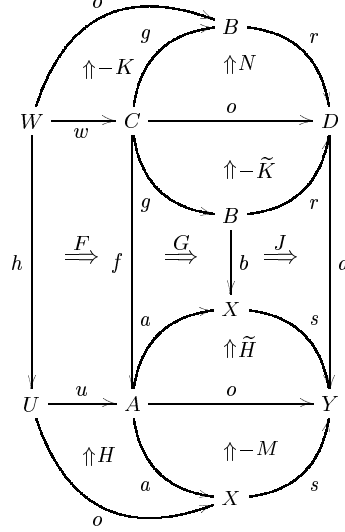
Observe that $(\widetilde{H} - M)(u \circ h) = 1_{s \circ a \circ u \circ h}$ since $\mathcal{A}(s \circ a) \circ u \circ h = \{1_{s \circ a \circ u \circ h}\}$ and that we may write

$$
\begin{aligned}
- (d \circ r)K &+ d(N - \widetilde{K})w + J(g \circ w) \\
&= -(d \circ r)K + d(N - \widetilde{K})w + (d \circ r)K + Jo - (s \circ b)K \\
&= -(d \circ r)K + 1_{d \circ r \circ g \circ w} + (d \circ r)K + 1_o - (s \circ b)K \\
&= -(s \circ b)K
\end{aligned}
$$

using $d \circ \mathcal{A}(r \circ g) \circ w = \{1_{d \circ r \circ g \circ w}\}$. It follows that the composite 2-morphism above takes the form of an element of $\mathcal{L}$. Thus $\mathcal{R} \subset \mathcal{L}$ in this case and the proof of part (1) is complete. $\square$

Theorem 4.4 has some useful corollaries which we now state. The first corollary shows that a box bracket frequently decomposes under pre-composition or post-composition with a selected 1-morphism.

COROLLARY 4.5.

(1) *In the homotopy commutative diagram*

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B & \longrightarrow & * \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\ } & X & \xrightarrow{\ s\ } & Y
\end{array}
$$

suppose that $g \circ w \simeq o$, $a \circ u \simeq o$ *and* $s \circ a \simeq o$. *Then the inclusion*

$$
s \circ \square\!\square \left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B \\
{\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ a\ } & X
\end{array}
\right)
\subset
\left\{ s,\ \begin{array}{c} b \\ a \end{array},\ \begin{array}{c} g \\ f \end{array} \right\} \circ w - \{s, a, u\} \circ h
$$

is valid and moreover if $\mathcal{A}(s \circ a) \circ u \circ h = \{1_{s \circ a \circ u \circ h}\}$ *then equality holds.*

(2) *In the homotopy commutative diagram*

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B & \xrightarrow{\ r\ } & D \\
\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle d}\downarrow \\
* & \longrightarrow & A & \xrightarrow{\ a\ } & X & \xrightarrow{\ s\ } & Y
\end{array}
$$

suppose that $g \circ w \simeq o$, $r \circ g \simeq o$ *and* $s \circ a \simeq o$. *Then the inclusion*

$$
\square\!\square \left(
\begin{array}{ccccc}
C & \xrightarrow{\ g\ } & B & \xrightarrow{\ r\ } & D \\
{\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle d}\downarrow \\
A & \xrightarrow{\ a\ } & X & \xrightarrow{\ s\ } & Y
\end{array}
\right) \circ w
\subset
-d \circ \{r, g, w\} + s \circ \left\{ \begin{array}{c} b \\ a \end{array},\ \begin{array}{c} g \\ f \end{array},\ w \right\}
$$

is valid and moreover if $d \circ r \circ \mathcal{A}(g \circ w) = \{1_{d \circ r \circ g \circ w}\}$ *then equality holds.*

COROLLARY 4.6. *In the homotopy commutative diagram*

$$
\begin{array}{ccccccc}
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B & & \\
& & {\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & \\
& & A & \xrightarrow{\ a\ } & X & \xrightarrow{\ s\ } & Y
\end{array}
$$

suppose that $g \circ w \simeq o$, $f \circ w \simeq o$, $s \circ a \simeq o$ *and* $s \circ b \simeq o$. *Then the equation*

$$s \circ \left\{ \begin{matrix} b \\ a \end{matrix} \, , \, \begin{matrix} g \\ f \end{matrix} \, , w \right\} = \left\{ s, \, \begin{matrix} b \\ a \end{matrix} \, , \, \begin{matrix} g \\ f \end{matrix} \, \right\} \circ w$$

is valid.

COROLLARY 4.7. (cf. [17, Proposition 1.4]) *For given 1-morphisms*

$$W \xrightarrow{\; w \;} C \xrightarrow{\; f \;} A \xrightarrow{\; a \;} X \xrightarrow{\; s \;} Y$$

assume that $f \circ w \simeq o$, $a \circ f \simeq o$ *and* $s \circ a \simeq o$. *Then the equality* $-s \circ \{a, f, w\} = \{s, a, f\} \circ w$ *is valid.*

Proof. We consider the homotopy commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{\; w \;} & C & \longrightarrow & * & \longrightarrow & * \\ \big\downarrow & & f\big\downarrow & & \big\downarrow & & \big\downarrow \\ * & \longrightarrow & A & \xrightarrow{\; a \;} & X & \xrightarrow{\; s \;} & Y \end{array}$$

and note the identifications $\boxdot \left(\begin{array}{ccccc} W & \xrightarrow{\; w \;} & C & \longrightarrow & * \\ \big\downarrow & & f\big\downarrow & & \big\downarrow \\ * & \longrightarrow & A & \xrightarrow{\; a \;} & X \end{array} \right) = -\{a, f, w\}$

and $\boxdot \left(\begin{array}{ccccc} C & \longrightarrow & * & \longrightarrow & * \\ f\big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \xrightarrow{\; a \;} & X & \xrightarrow{\; s \;} & Y \end{array} \right) = \{s, a, f\}$. The result follows. $\square$

COROLLARY 4.8.

(1) (cf. [13, Lemma 3]) *In the homotopy commutative diagram*

$$\begin{array}{ccccccc} W & \xrightarrow{\; w \;} & C & \xrightarrow{\; g \;} & B & \xrightarrow{\; r \;} & D \\ \big\downarrow & & & & b\big\downarrow & & \big\downarrow d \\ & & * & \longrightarrow & X & \xrightarrow{\; s \;} & Y \end{array}$$

suppose that $g \circ w \simeq o$ *and* $r \circ g \simeq o$. *Then the inclusion*

$$\left\{ \begin{matrix} d \\ s \end{matrix} \, , \, \begin{matrix} r \\ b \end{matrix} \, , g \right\} \circ w \subset -d \circ \{r, g, w\} + s \circ \{b, g, w\}$$

is valid, with equality if $d \circ r \circ \mathcal{A}(g \circ w) = \{1_{d \circ r \circ g \circ w}\}$.

(2) *In the homotopy commutative diagram*

$$
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \longrightarrow & * \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
U & \xrightarrow[\ u\ ]{} & A & \xrightarrow[\ a\ ]{} X & \xrightarrow[\ s\ ]{} Y
\end{array}
$$

suppose that $a \circ u \simeq o$ and $s \circ a \simeq o$. Then the inclusion

$$
s \circ \left\{ a, \ \begin{matrix} f \\ u \end{matrix} \ , \ \begin{matrix} w \\ h \end{matrix} \right\} \subset \{s, a, f\} \circ w - \{s, a, u\} \circ h
$$

is valid, with equality if $\mathcal{A}(s \circ a) \circ u \circ h = \{1_{s \circ a \circ u \circ h}\}$.

5. Vertical lemmas; Condition (M)

The lemma below states a basic property valid in any 2-category with zeros; its validity follows from the Interchange Axiom. We will make fundamental use of it in this section.

LEMMA 5.1. *In a 2-category $\mathcal{C}$ with zeros suppose given 2-morphisms*

$$
\begin{array}{ccc}
& \overset{o}{\underset{f}{\overbrace{\Uparrow F}}} & \overset{o}{\underset{g}{\overbrace{\Uparrow G}}} \\
\bullet & \bullet & \bullet
\end{array}
$$

with zero 1-morphisms as indicated. Then $gF = GF = Gf : g \circ f \Rightarrow o$.

PROPOSITION 5.2. *Suppose all vertical and horizontal pair composites are null homotopic in the following homotopy commutative diagram of 1-morphisms in $\mathcal{C}$.*

$$
\begin{array}{ccccc}
W' & \xrightarrow{\ w'\ } & C' & & \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle f'} & & \\
W & \xrightarrow{\ w\ } & C & \xrightarrow{\ g\ } & B \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\
U & \xrightarrow[\ u\ ]{} & A & \xrightarrow[\ a\ ]{} & X
\end{array}
$$

Then the relations

$$a \circ \Box\Box \left(\begin{array}{ccc} W' & \xrightarrow{h'} & W & \xrightarrow{h} & U \\ {\scriptstyle w'} \downarrow & & {\scriptstyle w} \downarrow & & {\scriptstyle u} \downarrow \\ C' & \xrightarrow{f'} & C & \xrightarrow{f} & A \end{array} \right) \subset$$

$$- \Box\Box \left(\begin{array}{ccc} W & \xrightarrow{w} & C & \xrightarrow{g} & B \\ {\scriptstyle h} \downarrow & & {\scriptstyle f} \downarrow & & {\scriptstyle b} \downarrow \\ U & \xrightarrow{u} & A & \xrightarrow{a} & X \end{array} \right) \circ h' + \{b, g \circ f', w'\}$$

and

$$\Box\Box \left(\begin{array}{ccc} W & \xrightarrow{w} & C & \xrightarrow{g} & B \\ {\scriptstyle h} \downarrow & & {\scriptstyle f} \downarrow & & {\scriptstyle b} \downarrow \\ U & \xrightarrow{u} & A & \xrightarrow{a} & X \end{array} \right) \circ h' \subset$$

$$\{b, g \circ f', w'\} - a \circ \Box\Box \left(\begin{array}{ccc} W' & \xrightarrow{h'} & W & \xrightarrow{h} & U \\ {\scriptstyle w'} \downarrow & & {\scriptstyle w} \downarrow & & {\scriptstyle u} \downarrow \\ C' & \xrightarrow{f'} & C & \xrightarrow{f} & A \end{array} \right)$$

are valid as subsets of $\mathcal{A}(o : W' \to X)$.

Proof. We prove only the first inclusion; the second inclusion admits a similar proof. An arbitrary element of

$$a \circ \Box\Box \left(\begin{array}{ccc} W' & \xrightarrow{h'} & W & \xrightarrow{h} & U \\ {\scriptstyle w'} \downarrow & & {\scriptstyle w} \downarrow & & {\scriptstyle u} \downarrow \\ C' & \xrightarrow{f'} & C & \xrightarrow{f} & A \end{array} \right)$$

is represented by a composite 2-morphism of the following form.

By selecting homotopies $K : o \Rightarrow g \circ w$ and $G : a \circ f \Rightarrow b \circ g$ we may rewrite this diagram in the form indicated below.

The fact that the composite 2-morphism in this diagram is equal to that in the previous diagram follows since $(a \circ u)M = H(h \circ h')$ by Lemma 5.1 and because $-Gw + bK - bK + Gw = 1_{a \circ f \circ w}$. But the latter diagram represents
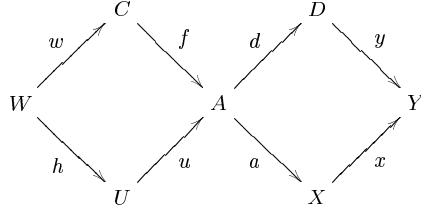
$$-(-bK + Gw + aF + Hh)h' + [-b(-gJ + Kh') + (Gf' + aN)w']$$

which is readily recognizable as an element of

$$- \ \square\!\square \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h \downarrow \quad\quad f \downarrow \quad\quad b \downarrow \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) \ \circ \ h' \ + \ \{b, g \circ f', w'\}.$$

Thus the claimed inclusion is verified. $\qquad\qquad\square$

DEFINITION 5.3. *Let the diagram*

be homotopy commutative. Such "abutting squares" are said to satisfy **Condition (M)** *if the following conditions hold. For any choice of homotopies* $F : u \circ h \Rightarrow f \circ w$, $\xi \in \mathcal{A}(d \circ u)$ *and* $G : x \circ a \Rightarrow y \circ d$, *it must be possible to rewrite the composite 2-morphism*
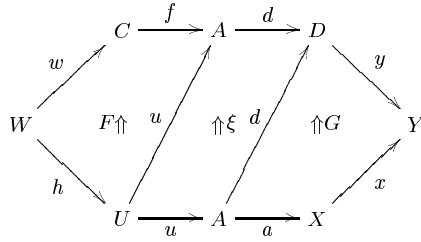
in the form
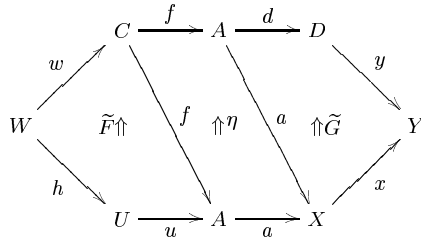
for some homotopies $\widetilde{F}$, η *and* $\widetilde{G}$ *as indicated; and vice-versa.*

EXAMPLE 5.4. (a) *For the squares of Definition 5.3, if it happens that* $y \circ \mathcal{A}(d \circ u) \circ h = \{1_{y \circ d \circ u \circ h}\}$ *and* $x \circ \mathcal{A}(a \circ f) \circ w = \{1_{x \circ a \circ f \circ w}\}$ *then Condition (M) is valid (by the Interchange Law for 2-mor-*

phisms in any 2-category). For example the squares
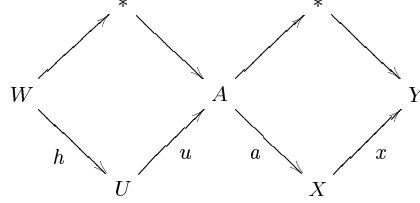
satisfy Condition (M) for this reason.

(b) *It is straightforward to check that the squares*
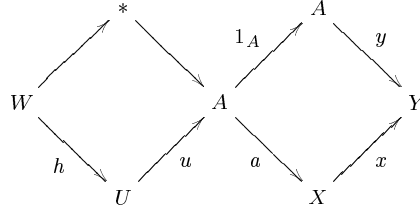
satisfy Condition (M).

THEOREM 5.5. ("The 3×3 Equality") *In the following homotopy commutative diagram of 1-morphisms in $\mathcal{C}$ suppose that all vertical and horizontal pair composites are null homotopic.*
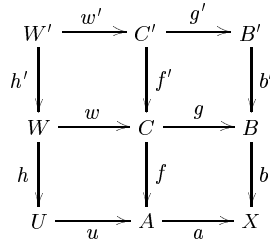
Also assume that Condition (M) is satisfied for the upper left and lower right squares. Then the equality

$$= -\square\square \begin{pmatrix} C' \xrightarrow{f'} C \xrightarrow{f} A \\ g' \downarrow \quad\quad \downarrow g \quad\quad \downarrow a \\ B' \xrightarrow{b'} B \xrightarrow{b} X \end{pmatrix} \circ w' - a \circ \square\square \begin{pmatrix} W' \xrightarrow{h'} W \xrightarrow{h} U \\ w' \downarrow \quad\quad \downarrow w \quad\quad \downarrow u \\ C' \xrightarrow{f'} C \xrightarrow{f} A \end{pmatrix}$$

is valid as subsets of $\mathcal{A}(o : W' \to X)$.

Proof. An element of the left side of the above stated equality may be represented by the following diagram of composite 2-morphisms.
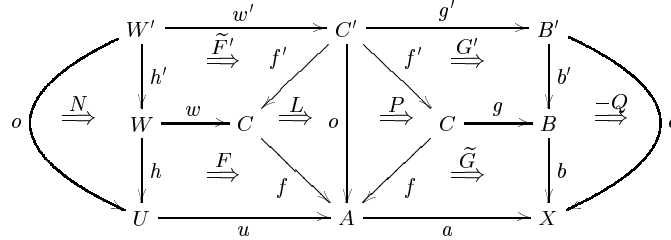

$$(*)$$

Note that $H' - K \in \mathcal{A}(g \circ w)$. Hence by application of Condition (M) there exist $\eta \in \mathcal{A}(f \circ f')$ and homotopies $\widetilde{G} : a \circ f \Rightarrow b \circ g$ and $\widetilde{F'} : w \circ h' \Rightarrow f' \circ w'$ so that the equation

$$(b \circ g)F' + b(H' - K)h' + G(w \circ h') = \widetilde{G}(f' \circ w') + a\eta w' + (a \circ f)\widetilde{F'}$$

holds. Now let $L : f \circ f' \Rightarrow o$ be a homotopy and set $P = \eta - L : o \Rightarrow f \circ f'$. Then $a\eta w' = aPw' + aLw'$. Also let $N : o \Rightarrow h \circ h'$ and $Q : o \Rightarrow b \circ b'$ be homotopies. Then $H(h \circ h') = (a \circ u)N$ and

$Q(g' \circ w') = (b \circ b')K'$ by Lemma 5.1. From these observations it follows that diagram $(*)$ may be rewritten as follows.


This latter diagram represents an element which can be recognized as belonging to the right side of the above stated equality. A similar argument shows that the reverse inclusion also holds and thus the claimed equality is established. $\square$

REMARK 5.6. *We observe that Corollaries 4.6 and 4.7 may also be obtained from Theorem 5.5. For example the homotopy commutative* 3×3 *diagram*
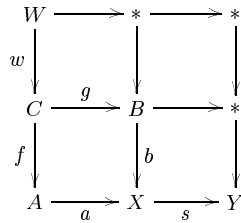
satisfies Condition (M) (by Example 5.4(a)) and yields Corollary 4.6. Theorem 5.5 also yields similar but slightly different versions of other corollaries to Theorem 4.4, as stated in the next result.

COROLLARY 5.7. *Let the following* 3×3 *diagrams be homotopy commutative with all vertical and horizontal composites null homotopic.*
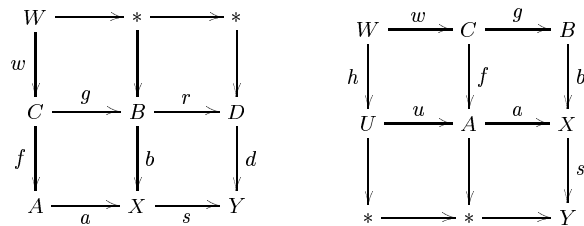
(1) *In the diagram on the left if* $d \circ \mathcal{A}(r \circ g) \circ w = \{1_{d \circ r \circ g \circ w}\}$ *then*

$$d \circ \{r, g, w\} + \square\square \left(\begin{array}{ccc} C \xrightarrow{g} B \xrightarrow{r} D \\ f\downarrow \quad b\downarrow \quad d\downarrow \\ A \xrightarrow{a} X \xrightarrow{s} Y \end{array} \right) \circ w = -s \circ \left\{ \begin{array}{cc} a & f \\ b & g \end{array}, w \right\}.$$

And if furthermore $A = *$ *then*

$$d \circ \{r, g, w\} + \left\{ \begin{array}{cc} d & r \\ s & b \end{array}, g \right\} \circ w = s \circ \{b, g, w\}.$$
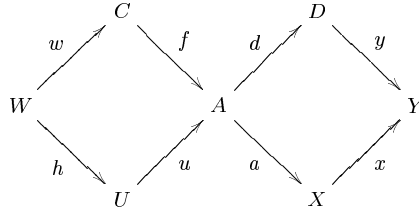
(2) *In the diagram on the right if* $s \circ \mathcal{A}(a \circ u) \circ h = \{1_{s \circ a \circ u \circ h}\}$ *then*

$$s \circ \square\square \left(\begin{array}{ccc} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad f\downarrow \quad b\downarrow \\ U \xrightarrow{u} A \xrightarrow{a} X \end{array} \right) + \{s, a, u\} \circ h = \left\{ s, \begin{array}{cc} b & g \\ a & f \end{array} \right\} \circ w.$$

And if furthermore $B = *$ *then*

$$s \circ \left\{ a, \begin{array}{cc} f & w \\ u & h \end{array} \right\} + \{s, a, u\} \circ h = \{s, a, f\} \circ w.$$

PROPOSITION 5.8. *In the homotopy commutative diagram*

suppose that each of the composites $a \circ f$, $d \circ u$, $a \circ u$ *and* $d \circ f$ *is null homotopic. If this diagram satisfies Condition* (M) *then the equation*

$$y \circ \left\{ d, \begin{array}{cc} f & w \\ u & h \end{array} \right\} + \left\{ \begin{array}{cc} y & d \\ x & a \end{array}, u \right\} \circ h =$$

$$\left\{ \begin{array}{cc} y & d \\ x & a \end{array}, f \right\} \circ w + x \circ \left\{ a, \begin{array}{cc} f & w \\ u & h \end{array} \right\}$$

holds.

Proof. For, under our hypotheses, we may apply Theorem 5.5 to the following 3×3 diagram.

$$
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \longrightarrow & * \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ d\ } & D \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle y} \\
* & \longrightarrow & X & \xrightarrow{\ x\ } & Y
\end{array}
$$

We note the following identifications

$$
\boxdot\left(
\begin{array}{ccccc}
W & \xrightarrow{\ w\ } & C & \longrightarrow & * \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow \\
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ d\ } & D
\end{array}
\right)
= \left\{ d, \ \begin{matrix} f \\ u \end{matrix} \ , \ \begin{matrix} w \\ h \end{matrix} \right\}
$$

$$
\boxdot\left(
\begin{array}{ccccc}
U & \xrightarrow{\ u\ } & A & \xrightarrow{\ d\ } & D \\
\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle y} \\
* & \longrightarrow & X & \xrightarrow{\ x\ } & Y
\end{array}
\right)
= \left\{ \begin{matrix} y \\ x \end{matrix} \ , \ \begin{matrix} d \\ a \end{matrix} \ , u \right\}
$$

$$
\boxdot\left(
\begin{array}{ccccc}
C & \xrightarrow{\ f\ } & A & \xrightarrow{\ a\ } & X \\
\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle x} \\
* & \longrightarrow & D & \xrightarrow{\ y\ } & Y
\end{array}
\right)
= \left\{ \begin{matrix} x \\ y \end{matrix} \ , \ \begin{matrix} a \\ d \end{matrix} \ , f \right\}
= - \left\{ \begin{matrix} y \\ x \end{matrix} \ , \ \begin{matrix} d \\ a \end{matrix} \ , f \right\}
$$

$$
\boxdot\left(
\begin{array}{ccccc}
W & \xrightarrow{\ h\ } & U & \longrightarrow & * \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u} & & \downarrow \\
C & \xrightarrow{\ f\ } & A & \xrightarrow{\ a\ } & X
\end{array}
\right)
= \left\{ a, \ \begin{matrix} u \\ f \end{matrix} \ , \ \begin{matrix} h \\ w \end{matrix} \right\}
= - \left\{ a, \ \begin{matrix} f \\ u \end{matrix} \ , \ \begin{matrix} w \\ h \end{matrix} \right\}
$$

and thus the equality given by Theorem 5.5 for the above 3×3 diagram just reduces to the equality claimed in the present proposition. $\qquad\square$

6. The topological case

We now consider the box bracket for the case when $\mathcal{C}$ is the category of based topological spaces $\mathcal{T}op_*$. We recall that $\mathcal{T}op_*$ is a

2-category with zeros. Its objects are based topological spaces, the
1-morphisms are based maps and the 2-morphisms are track classes
of based homotopies. (The book [7] is a good general reference for
the track viewpoint.) With regard to the 2-morphisms we follow
the custom of working with representative homotopies even though
this causes a slight conflict with the notation used in the previous
sections. Thus in $\mathcal{T}op_*$ we let $F : f \Rightarrow g : X \to Y$ denote a homo-
topy; its track class (the actual 2-morphism it represents) is denoted
$\{F\} : f \Rightarrow g : X \to Y$. We use $F \sim F'$ to indicate that two ho-
motopies $F, F' : f \Rightarrow g : X \to Y$ are track equivalent. Vertical
composition of track classes of homotopies $F : f \Rightarrow g : X \to Y$ and
$G : g \Rightarrow h : X \to Y$ is given by $\{G\} + \{F\} := \{G + F\}$ where

$$(G + F)(x, t) = \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $x \in X$. If also $H : u \Rightarrow v : Y \to Z$ is a homotopy then horizontal
composition is determined by $\{H\}\{F\} := \{HF\}$ where

$$(HF)(x, t) = \begin{cases} uF(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ H(g(x), 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $x \in X$. Of course the zero object of $\mathcal{T}op_*$ is the one point space,
denoted $*$, and the zero map $o : X \to Y$ is specified by requiring
that its image equal the base point of the space Y. There is a unique
factorization $o : X \to * \to Y$.

Assume given a homotopy commutative square of maps in $\mathcal{T}op_*$

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\alpha]{} & X \end{array}$$

with homotopy $F : \alpha \circ f \Rightarrow \beta \circ g$ as indicated. We define the map
$\mu_F : \mathcal{M}(f, g) \to X$ by $\mu_F[x, t] = F(x, t)$ for $x \in C$ and $0 \leq t \leq 1$,
with $\mu_F(a) = \alpha(a)$ for $a \in A$ and $\mu_F(b) = \beta(b)$ for $b \in B$. Here $\mathcal{M}$
denotes the reduced double mapping cylinder functor. As is well-
known the homotopy class of μ_F depends only on the track class of
F.

In particular, given a square of the form

$$
\begin{array}{ccc}
W & \longrightarrow & * \\
\downarrow & \overset{L}{\Longrightarrow} & \downarrow \\
* & \longrightarrow & X
\end{array}
$$

there is an induced map $\mu_L : \Sigma W \to X$. The following proposition holds.

PROPOSITION 6.1. *The function* $d : \mathcal{A}(o : W \to X) \to \pi(\Sigma W, X)$ *given by*

$$
W \overset{o}{\underset{o}{\overset{\curvearrowright}{\underset{\curvearrowleft}{\Downarrow \{L\}}}}} X \qquad \mapsto \qquad [\mu_L] : \Sigma W \to X
$$

is an isomorphism of groups. (The notation [ ] *denotes homotopy class.) Furthermore if maps* $h' : W' \to W$ *and* $f : X \to Y$ *are given then each of the squares*

$$
\begin{array}{ccc}
\mathcal{A}(o{:}W{\to}X) & \overset{d}{\longrightarrow} & \pi(\Sigma W,X) \\
(\ )\circ h' \downarrow & & \downarrow (\Sigma h')^{\#} \\
\mathcal{A}(o{:}W'{\to}X) & \overset{d}{\longrightarrow} & \pi(\Sigma W',X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{A}(o{:}W{\to}X) & \overset{d}{\longrightarrow} & \pi(\Sigma W,X) \\
f\circ(\ ) \downarrow & & \downarrow f_{\#} \\
\mathcal{A}(o{:}W{\to}Y) & \overset{d}{\longrightarrow} & \pi(\Sigma W,Y)
\end{array}
$$

is commutative. That is, the relations $\mu_L \circ \Sigma h' \simeq \mu_{Lh'}$ *and* $f \circ \mu_L \simeq \mu_{fL}$
are valid for all $\{L\} \in \mathcal{A}(o : W \to X)$.

REMARK 6.2. *Let* $f : W \to X$ *be null homotopic. For each homotopy* $F : o \Rightarrow f$ *the composite*

$$
\phi_{\{F\}} : \mathcal{A}(f) \xrightarrow{(\ )^{\{F\}}} \mathcal{A}(o : W \to X) \xrightarrow{\ d\ } \pi(\Sigma W, X)
$$

consists of isomorphisms of groups. Here the first isomorphism is given by

$$
\xi \mapsto \xi^{\{F\}} : = -\{F\} + \xi + \{F\}
$$

for all $\xi \in \mathcal{A}(f)$ *and moreover is independent of* $\{F\}$ *if the group* $\mathcal{A}(o : W \to X)$ *is abelian (see [6, Proposition 6.4]).*

We turn now to a description of a box bracket when viewed under the isomorphism of Proposition 6.1. So let

$$\mathcal{B} = \square\square \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\downarrow \qquad f\downarrow \qquad b\downarrow \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right)$$

be a box bracket in $\mathcal{T}op_*$ and let

be a diagram of representative homotopies for a fixed but arbitrary element

$$\theta = -b\{K\} + \{G\}w + a\{F\} + \{H\}h$$

of $\mathcal{B}$. In view of Proposition 2.2 we have

$$d(\mathcal{B}) = \ d(b \circ \mathcal{A}(o : W \to B)) + d([\mathcal{A}(a \circ f) \circ w]^{\{-Gw+bK\}}) + d(\theta)$$
$$+ \ d([a \circ \mathcal{A}(f \circ w)]^{\{aF+Hh\}}) + d(\mathcal{A}(o : U \to X) \circ h)$$

since d is an isomorphism. We proceed to identify each of the summands in this expression. By the last part of Proposition 6.1 we know that the equalities

$$d(b \circ \mathcal{A}(o : W \to B)) \ = \ b \circ \pi(\Sigma W, B)$$
$$d(\mathcal{A}(o : U \to X) \circ h) \ = \ \pi(\Sigma U, X) \circ \Sigma h$$

are valid. And using the notation introduced in Remark 6.2 we observe that the equalities

$$d([\mathcal{A}(a \circ f) \circ w]^{\{-Gw+bK\}}) \ = \ \phi_{\{-Gw+bK\}}(\mathcal{A}(a \circ f) \circ w)$$
$$d([a \circ \mathcal{A}(f \circ w)]^{\{aF+Hh\}}) \ = \ \phi_{\{aF+Hh\}}(a \circ \mathcal{A}(f \circ w))$$

hold. However without additional assumptions it seems difficult to characterize further these subgroups of $\pi(\Sigma W, X)$.

It remains to describe $d(\theta)$. We will give an extension-coextension representation, in analogy with the classical Toda bracket (cf [17, Proposition 1.7]). We construct the following diagram, in which Y denotes the double mapping cylinder $\mathcal{M}(i_1 \circ f, g)$. The various maps i_0 and i_1 are inclusions at parameter $t = 0$ and $t = 1$ respectively.
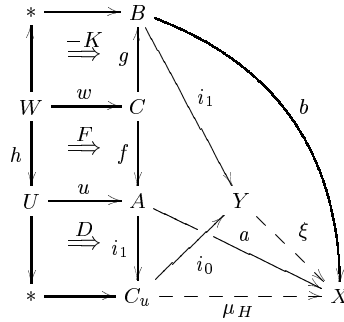
First the map μ_H satisfies $\mu_H \circ i_1 = a$ and is induced by the homotopy D defining the mapping cone C_u. Consequently the map ξ is induced by the homotopy $G : \mu_H \circ i_1 \circ f = a \circ f \Rightarrow b \circ g$. Also let $\rho : \Sigma W \to Y$ denote the map induced functorially from the double mapping cylinder of the first vertical column to that of the second vertical column. The map ξ is an "extension" and the map ρ is a "coextension" as indicated in the diagram below.
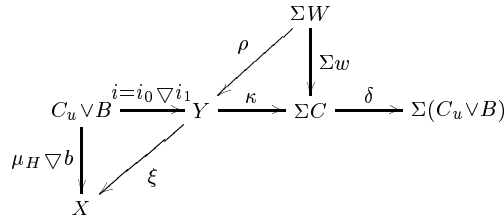
In this diagram the horizontal sequence is a cofibration sequence where explicitly $\delta = \Sigma(i_{C_u} \circ i_1 \circ f) - \Sigma(i_B \circ g)$. It is immediate from these constructions that we have:

PROPOSITION 6.3. $d(\theta) = [\mu_{(-bK+Gw+aF+Hh)}] = [\xi \circ \rho]$.

We summarize the above discussion in the next result.

THEOREM 6.4. *Let notation be as above. When a box bracket in* $\mathcal{T}op_*$ *is regarded as a subset of* $\pi(\Sigma W, X)$ *then the equality*

$$\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\ g\ } B \\ h\downarrow \quad\ f\downarrow \quad\ b\downarrow \\ U \xrightarrow{\ u\ } A \xrightarrow{\ a\ } X \end{array} \right) = \begin{array}{l} b \circ \pi(\Sigma W, B) \\[4pt] + \phi_{\{-Gw+bK\}}(\mathcal{A}(a \circ f) \circ w) \\[4pt] + [\xi \circ \rho] + \phi_{\{aF+Hh\}}(a \circ \mathcal{A}(f \circ w)) \\[4pt] + \pi(\Sigma U, X) \circ \Sigma h \end{array}$$

is valid. Moreover if the composites $b \circ g \circ w$ *and* $a \circ u \circ h$ *are admissible (in the sense of* [6, Definition 6.3]*) then* $b \circ \pi(\Sigma W, B) + \phi_{\{-Gw+bK\}}(\mathcal{A}(a \circ f) \circ w)$ *and* $\phi_{\{aF+Hh\}}(a \circ \mathcal{A}(f \circ w)) + \pi(\Sigma U, X) \circ \Sigma h$ *are subgroups of* $\pi(\Sigma W, X)$ *and consequently the box bracket is a double coset of these subgroups. Furthermore if* $\pi(\Sigma W, X)$ *is abelian then the box bracket is a coset of the subgroup*

$$b \circ \pi(\Sigma W, B) + \phi_{\{-Gw+bK\}}(\mathcal{A}(a \circ f) \circ w)+$$
$$\phi_{\{aF+Hh\}}(a \circ \mathcal{A}(f \circ w)) + \pi(\Sigma U, X) \circ \Sigma h.$$

EXAMPLE 6.5. *We examine the classical Toda bracket itself arising from a composite* $W \xrightarrow{\ w\ } C \xrightarrow{\ f\ } A \xrightarrow{\ a\ } X$ *of maps in* $\mathcal{T}op_*$ *satisfying* $f \circ w \simeq o$ *and* $a \circ f \simeq o$. *By Proposition 3.1 we know that*

$$\boxdot \left(\begin{array}{ccc} W \xrightarrow{\ w\ } C \xrightarrow{\quad} * \\ \downarrow \quad\ f\downarrow \quad\ \downarrow \\ * \xrightarrow{\quad} A \xrightarrow{\ a\ } X \end{array} \right) = -\{a, f, w\}.$$

Hence applying Theorem 6.4 it follows that

$$-\{a, f, w\} = \phi_{\{-Gw\}}(\mathcal{A}(a \circ f) \circ w) + [\xi \circ \rho] + \phi_{\{aF\}}(a \circ \mathcal{A}(f \circ w))$$

as a subset of $\pi(\Sigma W, X)$ *where* $G : a \circ f \Rightarrow o$ *and* $F : o \Rightarrow f \circ w$ *are homotopies. These homotopies are also used in constructing the*

maps ξ and ρ by means of the following extension-coextension diagram.

For this special case we observe that the horizontal cofibration sequence can begin one term to the left, as indicated, but of course this is not possible for the general case. Furthermore

$$
\begin{aligned}
\phi_{\{aF\}}\big(a \circ \mathcal{A}(f \circ w)\big) &= d\big((a \circ \mathcal{A}(f \circ w))^{\{aF\}}\big) \\
&= d\big(a \circ \mathcal{A}(f \circ w)^{\{F\}}\big) \\
&= d(a \circ \mathcal{A}(o : W \to A)) \\
&= a \circ \pi(\Sigma W, A)
\end{aligned}
$$

and similarly $\phi_{\{-Gw\}}(\mathcal{A}(a \circ f) \circ w) = \pi(\Sigma C, X) \circ \Sigma w$. *Thus we obtain*

$$
-\{a, f, w\} = \pi(\Sigma C, X) \circ \Sigma w + [\xi \circ \rho] + a \circ \pi(\Sigma W, A)
$$

or equivalently

$$
\{a, f, w\} = a \circ \pi(\Sigma W, A) + [-(\xi \circ \rho)] + \pi(\Sigma C, X) \circ \Sigma w.
$$

This last equality just corresponds to the classical computation of the indeterminacy of the topological Toda bracket as a double coset, together with an extension-coextension description of each of its elements. Of course the indeterminacy of $\{a, f, w\}$ in the setting of a general 2-category with zeros was available previously either by application of Proposition 1.3 to Example 1.5, or in [6, Proposition 8.2(a)], where a direct argument was given.

7. Some examples

In this final section we present some examples illustrative of the techniques we have developed. We follow [17] for notation in regard to the homotopy groups of spheres. The first result determines completely a Toda bracket which was only partially computed in [17].

PROPOSITION 7.1. $\{\eta_4, 2\iota_5, \sigma'''\} = \mu_4 + \{\eta_4 \circ \varepsilon_5\}$ *in* π_{13}^4.

Proof. Using (5.9), (5.4) and Proposition 1.2 of [17] we have

$$\eta_4 \circ \nu_5 \circ \varepsilon_8 = E\nu' \circ \eta_7 \circ \varepsilon_8 \in \{\eta_4, 2\iota_5, \eta_5\} \circ \eta_7 \circ \varepsilon_8 \subset \{\eta_4, 2\iota_5, \eta_5^2 \circ \varepsilon_7\}.$$

The indeterminacy of this last Toda bracket is

$$\eta_4 \circ \pi_{16}^5 + \pi_6^4 \circ \eta_6^2 \circ \varepsilon_8 = \{\eta_4 \circ \zeta_5, \eta_4 \circ \nu_5 \circ \varepsilon_8\}$$

for $\pi_{16}^5 = \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2 = \{\zeta_5\} \oplus \{\nu_5 \circ \overline{\nu}_8\} \oplus \{\nu_5 \circ \varepsilon_8\}$ and $\pi_6^4 = \mathbb{Z}/2 = \{\eta_4^2\}$, with $\eta_4 \circ \nu_5 \circ \overline{\nu}_8 = E\nu' \circ \eta_7 \circ \overline{\nu}_8 = E\nu' \circ \nu_7^3 = 0$ by the relations $\eta_5 \circ \overline{\nu}_6 = \nu_5^3$ [17, (7.3)] and $\nu' \circ \nu_6 = 0$ [17, Proposition 5.11], and with $\eta_4^4 \circ \varepsilon_8 = \eta_4 \circ 4\nu_5 \circ \varepsilon_8 = 0$ by the relation $\eta_5^3 = 4\nu_5$ [17, (5.5)]. We remark that $\eta_4 \circ \zeta_5 \equiv E\nu' \circ \mu_7 \mod E\nu' \circ \eta_7 \circ \varepsilon_8$ by Proposition 2.2(5) of [15]. It follows that

$$\{\eta_4, 2\iota_5, \eta_5^2 \circ \varepsilon_7\} = \{\eta_4 \circ \zeta_5, \eta_4 \circ \nu_5 \circ \varepsilon_8\} \subset E\pi_{15}^3.$$

Now by Lemma 6.5 of [17] there exists an integer x such that $x\nu_4^3 + \mu_4$ belongs to $\{\eta_4, 2\iota_5, \sigma'''\}$. Composing with the element ν_{13}, we have

$$x\nu_4^4 + \mu_4 \circ \nu_{13} \in \{\eta_4, 2\iota_5, \sigma'''\} \circ \nu_{13} \subset \{\eta_4, 2\iota_5, \sigma''' \circ \nu_{12}\} = \{\eta_4, 2\iota_5, \eta_5^2 \circ \varepsilon_7\}$$

by Proposition 1.2 of [17] and the relation $\sigma''' \circ \nu_{12} = \eta_5^2 \circ \varepsilon_7$ of [11, Proposition 9(1)]. We conclude that $x\nu_4^4 + \mu_4 \circ \nu_{13}$ is a suspension element. Applying the homomorphism $H : \pi_{16}^4 \to \pi_{16}^7$ thus yields the equation

$$0 = H(x\nu_4^4 + \mu_4 \circ \nu_{13}) = xH(\nu_4) \circ \nu_7^3 = x\nu_7^3$$

since $H(\nu_4) = \iota_7$ by Lemma 5.4 of [17]. Because the element ν_7^3 generates a $\mathbb{Z}/2$-summand of π_{16}^7 we deduce $x \equiv 0 \mod 2$ and thus $\mu_4 \in \{\eta_4, 2\iota_5, \sigma'''\}$. The indeterminacy of $\{\eta_4, 2\iota_5, \sigma'''\}$ is

$$\eta_4 \circ \pi_{13}^5 + \pi_6^4 \circ E\sigma''' = \{\eta_4 \circ \varepsilon_5, \ \eta_4^2 \circ E\sigma'''\} = \{\eta_4 \circ \varepsilon_5\}$$

by [17, Theorem 7.1], [17, Proposition 5.3], and [17, (7.4)]. This completes the proof.   □

PROPOSITION 7.2. $\left\{\begin{matrix} E\nu' \circ \eta_7 & \eta_8 \\ \eta_4 & \nu_5 \circ \eta_8 \end{matrix}, \nu_9\right\} = \{\nu_4^3, \eta_4 \circ \varepsilon_5\}$ *in* π_{13}^4.

Proof. By [5, Proposition 3.4(iii)] or Proposition 4.3(2) and Proposition 3.2(3), we have

$$\left\{\begin{matrix} E\nu' \circ \eta_7 & \eta_8 \\ \eta_4 & \nu_5 \circ \eta_8 \end{matrix}, \nu_9\right\} \supset \left\{\begin{matrix} E\nu' \circ \eta_7 & \iota_8 \\ \eta_4 & \nu_5 \end{matrix}, \eta_8 \circ \nu_9\right\}$$

with the latter containing 0 since $\eta_8 \circ \nu_9 = 0$. Thus

$$\left\{\begin{matrix} E\nu' \circ \eta_7 & \eta_8 \\ \eta_4 & \nu_5 \circ \eta_8 \end{matrix}, \nu_9\right\}$$

coincides with its indeterminacy which by [5, Corollary 5.8], is

$$E\nu' \circ \eta_7 \circ \pi_{13}S^8 + \pi_{10}S^4 \circ \nu_{10} + \eta_4 \circ \pi_{13}S^5$$
$$= E\nu' \circ \eta_7 \circ \pi_{13}S^8 + \pi_{10}^4 \circ \nu_{10} + \eta_4 \circ \pi_{13}S^5$$
$$= \{\nu_4^3, \eta_4 \circ \varepsilon_5\}$$

since $\pi_{13}S^8 = 0$, $\pi_{10}^4 = \mathbb{Z}/8 = \{\nu_4^2\}$ and $\pi_{13}S^5 = \mathbb{Z}/2 = \{\varepsilon_5\}$. Thus the proposition is established. $\square$

PROPOSITION 7.3. *The equality*

$$\boxdot\left(\begin{matrix} S^{12} \xrightarrow{\nu_9} S^9 \xrightarrow{\eta_8} S^8 \\ \sigma''' \downarrow \quad\quad \downarrow \nu_5 \circ \eta_8 \quad \downarrow E\nu' \circ \eta_7 \\ S^5 \xrightarrow{2\iota_5} S^5 \xrightarrow{\eta_4} S^4 \end{matrix}\right) = \mu_4 + \{\nu_4^3, \eta_4 \circ \varepsilon_5\}$$

holds in $\pi_{13}S^4 = (\mathbb{Z}/2)^3 = \{\mu_4\} \oplus \{\nu_4^3\} \oplus \{\eta_4 \circ \varepsilon_5\}$. *In particular this box bracket does not contain zero.*

Proof. Note that $\nu_5 \circ \eta_8 \circ \nu_9 = 0 = (2\iota_5) \circ \sigma'''$ so that by Proposition 3.3(2) the box bracket decomposes into the sum of a matrix Toda bracket and an ordinary Toda bracket. Hence and by Propositions 7.1 and 7.2 we have

$$\boxdot\left(\begin{matrix} S^{12} \xrightarrow{\nu_9} S^9 \xrightarrow{\eta_8} S^8 \\ \sigma''' \downarrow \quad\quad \downarrow \nu_5 \circ \eta_8 \quad \downarrow E\nu' \circ \eta_7 \\ S^5 \xrightarrow{2\iota_5} S^5 \xrightarrow{\eta_4} S^4 \end{matrix}\right)$$

$$= \left\{ \begin{matrix} E\nu' \circ \eta_7 & \eta_8 \\ \eta_4 & \nu_5 \circ \eta_8 \end{matrix} , \nu_9 \right\} + \{\eta_4, 2\iota_5, \sigma'''\}$$
$$= \{\nu_4^3, \eta_4 \circ \varepsilon_5\} + \mu_4 + \{\eta_4 \circ \varepsilon_5\}$$
$$= \mu_4 + \{\nu_4^3, \eta_4 \circ \varepsilon_5\}.$$

By the structure of $\pi_{13}S^4$, $\mu_4 \notin \{\nu_4^3, \eta_4 \circ \varepsilon_5\}$ and hence the proposition is obtained. $\square$

PROPOSITION 7.4. *In* $\pi_{25}^6 = \mathbb{Z}/32 \oplus \mathbb{Z}/8 = \{\overline{\sigma}_6\} \oplus \{\overline{\zeta}_6\}$ *the box bracket*

$$\square \left(\begin{array}{ccc} S^{24} \xrightarrow{\sigma_{17}} S^{17} \xrightarrow{\overline{\nu}_9} S^9 \\ \eta_{23} \downarrow \qquad \downarrow \eta_{16} \qquad \downarrow \nu_6 \\ S^{23} \xrightarrow{\sigma_{16}} S^{16} \xrightarrow{\nu_6 \circ \sigma_9} S^6 \end{array} \right)$$

is nontrivial. Note that this box bracket has the property that both central pair composites $\eta_{16} \circ \sigma_{17}$ *and* $\nu_6 \circ \sigma_9 \circ \eta_{16}$ *are essential; hence Proposition 3.3 may not be used to decompose it (into Toda brackets and matrix Toda brackets).*

Proof. The diagram

$$\begin{array}{cccc} S^{27} \xrightarrow{\nu_{24}} S^{24} \xrightarrow{\sigma_{17}} S^{17} \xrightarrow{\overline{\nu}_9} S^9 \\ \downarrow \qquad \downarrow \eta_{23} \qquad \downarrow \eta_{16} \qquad \downarrow \nu_6 \\ * \xrightarrow{\quad} S^{23} \xrightarrow{\sigma_{16}} S^{16} \xrightarrow{\nu_6 \circ \sigma_9} S^6 \end{array}$$

is homotopy commutative since $\eta_5 \circ \nu_6 = 0$ [17, (5.9)], $\eta_{10} \circ \sigma_{11} = \sigma_{10} \circ \eta_{17}$ [17, Lemma 6.4], $\nu_6 \circ \overline{\nu}_9 = \nu_6 \circ \varepsilon_9$ [17, (7.17)], and $\nu_5 \circ \varepsilon_8 = \nu_5 \circ \sigma_8 \circ \eta_{15}$ [17, p. 152]. Also $\sigma_{12} \circ \nu_{19} = 0$ [17, p. 72], $\overline{\nu}_6 \circ \sigma_{14} = 0$ [17, Lemma 10.7], and $\nu_6 \circ \sigma_9^2 = 0$ [11, Proposition 7(3)]. Hence all horizontal composites are nullhomotopic.

Since $\pi_{28}^{17} = \mathbb{Z}/8 = \{\zeta_{17}\}$ by [17, Theorem 7.4], $\pi_9^6 = \mathbb{Z}/8 = \{\nu_6\}$ by [17, Proposition 5.6] and $\overline{\nu}_6 \circ \zeta_{14} = \pm 8\overline{\sigma}_6$ by [15, Proposition 2.20(6)], it follows that $\nu_6 \circ \overline{\nu}_9 \circ \pi_{28}^{17} = 0$. Therefore Corollary 4.5(2) may be applied to obtain the following equality.

$$\square \left(\begin{array}{ccc} S^{24} \xrightarrow{\sigma_{17}} S^{17} \xrightarrow{\overline{\nu}_9} S^9 \\ \eta_{23} \downarrow \qquad \downarrow \eta_{16} \qquad \downarrow \nu_6 \\ S^{23} \xrightarrow{\sigma_{16}} S^{16} \xrightarrow{\nu_6 \circ \sigma_9} S^6 \end{array} \right) \circ \nu_{25}$$

$$= -\nu_6 \circ \{\overline{\nu}_9, \sigma_{17}, \nu_{24}\} + \nu_6 \circ \sigma_9 \circ \left\{ \begin{array}{cc} \eta_{16} & \sigma_{17} \\ \sigma_{16} & \eta_{23} \end{array}, \nu_{24} \right\}$$

Now $\{\overline{\nu}_9, \sigma_{17}, \nu_{24}\} = \overline{\sigma}_9$ (cf [14, Proposition 3.4(3), p. 59]) with $\nu_6 \circ \overline{\sigma}_9 \neq 0$ by [12, Theorem B]. Also $\left\{ \begin{array}{cc} \eta_{16} & \sigma_{17} \\ \sigma_{16} & \eta_{23} \end{array}, \nu_{24} \right\} \subset \pi_{28}^{16} = 0$ by [17, Theorem 7.6]. It follows that

$$\boxed{\boxplus} \left(\begin{array}{ccc} S^{24} \xrightarrow{\sigma_{17}} S^{17} \xrightarrow{\overline{\nu}_9} S^9 \\ \eta_{23}\Big\downarrow \quad \eta_{16}\Big\downarrow \quad \nu_6\Big\downarrow \\ S^{23} \xrightarrow{\sigma_{16}} S^{16} \xrightarrow{\nu_6 \circ \sigma_9} S^6 \end{array} \right) \circ \nu_{25} = -\nu_6 \circ \overline{\sigma}_9 \neq 0$$

and consequently $0 \notin \boxplus \left(\begin{array}{ccc} S^{24} \xrightarrow{\sigma_{17}} S^{17} \xrightarrow{\overline{\nu}_9} S^9 \\ \eta_{23}\Big\downarrow \quad \eta_{16}\Big\downarrow \quad \nu_6\Big\downarrow \\ S^{23} \xrightarrow{\sigma_{16}} S^{16} \xrightarrow{\nu_6 \circ \sigma_9} S^6 \end{array} \right).$ $\qquad\square$

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