Electron. Commun. Probab. **18** (2013), no. 53, 1–7.
DOI: 10.1214/ECP.v18-2633
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A simple observation on random matrices with continuous diagonal entries

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Abstract

Let T be an $n \times n$ random matrix, such that each diagonal entry $T_{i,i}$ is a continuous random variable, independent from all the other entries of T. Then for every $n \times n$ matrix A and every $t \geq 0$

$$\mathbb{P}\Big[|\det(A+T)|^{1/n} \leq t\Big] \leq 2bnt,$$

where $b > 0$ is a uniform upper bound on the densities of $T_{i,i}$.

Keywords: Random matrices; Singular values; Small ball probability; Determinant.
AMS MSC 2010: 60B20; 15B52.
Submitted to ECP on February 25, 2013, final version accepted on June 14, 2013.

1 introduction

In this note we are interested in the following question: Given an $n\times n$ random matrix T, what is the probability that T is invertible, or at least "close" to being invertible? One natural way to measure this property is to estimate the following small ball probability

$$\mathbb{P}\Big[s_n(T) \leq t\Big],$$

where $s_n(T)$ is the smallest singular value of T,

$$s_n(T) \stackrel{\text{def}}{=} \inf_{\|x\|_2=1} \|Tx\|_2 = \frac{1}{\|T^{-1}\|}.$$

In the case when the entries of T are i.i.d random variables with appropriate moment assumption, the problem was studied in [3, 11, 12, 15, 17]. We also refer the reader to the survey [10]. In particular, in [12] it is shown that if the entries of T are i.i.d subgaussian random variables, then

$$\mathbb{P}\Big[s_n(T) \leq t\Big] \leq C\sqrt{n}t + e^{-cn}, \tag{1.1}$$

where c, C depend on the moments of the entries.

Several cases of dependent entries have also been studied. A bound similar to (1.1) for the case when the rows are independent log-concave random vectors was obtained

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in [1, 2]. Another case of dependent entries is when the matrix is symmetric, which was studied in [5, 6, 7, 8, 9, 19]. In particular, in [5] it is shown that if the above diagonal entries of T are continuous and satisfy certain regularity conditions, namely that the entries are i.i.d subgaussian and satisfy certain smoothness conditions, then

$$\mathbb{P}\Big[s_n(T) \le t\Big] \le C\sqrt{n}t.$$

The regularity assumptions were completely removed in [6] at the cost of a $n^{3/2}$ (The result in [6] still assumes bounded density and independence of the entries in the non-symmetric part). On the other hand, in the discrete case, the result of [19] shows that if T is, say, symmetric whose above diagonal entries are i.i.d Bernoulli random variables, then

$$\mathbb{P}\Big[s_n(T) = 0\Big] \le e^{-n^c},$$

where c is an absolute constant.

A more general case is the so called *Smooth Analysis* of random matrices, where now we replace the matrix T by $A+T$, where A being an arbitrary deterministic matrix. The first result in this direction can be found in [13], where it is shown that if T is a random matrix with i.i.d standard normal entries, then

$$\mathbb{P}\Big[s_n(A+T) \le t\Big] \le C\sqrt{n}t. \tag{1.2}$$

Further development in this direction can be found in [18], where estimates similar to (1.2) are given in the case when T is a Bernoulli random matrix, and in [6, 8, 9], where T is symmetric.

An alternative way to measure the invertibility of a random matrix T is to estimate $\det(T)$, which was studied in [4, 14, 16] (when the entries are discrete distributions). Here we show that if the diagonal entries are independent continuous random variables, we can easily get a small ball estimate for $\det(A+T)$, where A being an arbitrary deterministic matrix.

Theorem 1.1. *Let T be an $n \times n$ random matrix, such that each diagonal entry $T_{i,i}$ is a continuous random variable, independent from all the other entries of T. Then for every $n \times n$ matrix A and every $t \ge 0$*

$$\mathbb{P}\Big[|\det(A+T)|^{1/n} \le t\Big] \le 2bnt,$$

where $b > 0$ is a uniform upper bound on the densities of $T_{i,i}$.

We remark that the proof works if we replace the determinant by the permanent of the matrix (see [4] for the difference between the notions).

Now, we use Theorem 1.1 to get a small ball estimate on the norm and smallest singular value of a random matrix.

Corollary 1.2. *Let T be a random matrix as in Theorem 1.1. Then*

$$\mathbb{P}\Big[\|T\| \le t\Big] \le (2bt)^n, \tag{1.3}$$

and

$$\mathbb{P}\Big[s_n(T) \le t\Big] \le (2b)^{\frac{n}{2n-1}} \left(\mathbb{E}\|T\|\right)^{\frac{n-1}{2n-1}} t^{\frac{1}{2n-1}}. \tag{1.4}$$

Corollary 1.2 can be applied to the case when the random matrix T is symmetric, under very weak assumptions on the distributions and the moments of the entries and under *no independence* assumptions on the above diagonal entries.

Finally, in Section 3 we show that in the case of 2×2 matrices, we use an ad-hoc argument to obtain a better bound than the one obtained in Theorem 1.1. We do not know what is the right order when the dimension is higher.

2 Proof of Theorem 1.1

Before we give the proof of Theorem 1.1, we fix some notation. First, let $M = A + T$, and let M_k be the matrix M after erasing the last $n-k$ rows and last $n-k$ columns. Also, let Ω_k be the σ-algebra generated by the entries of M_k *except* $M_{k,k}$.

Proof of Theorem 1.1. We have

$$|\det(M_k)| = \Big| M_{k,k} \det(M_{k-1}) + f_k \Big|,$$

where f_k is measurable with respect to Ω_k. We also have

$$\begin{aligned} &\mathbb{P}\Big[|\det(M_k)| \leq \varepsilon_k\Big] \\ &\quad \leq \mathbb{P}\Big[|\det(M_k)| \leq \varepsilon_k \wedge |\det(M_{k-1})| \geq \varepsilon_{k-1}\Big] + \mathbb{P}\Big[|\det(M_{k-1})| \leq \varepsilon_{k-1}\Big]. \end{aligned}$$

Now,

$$\begin{aligned} &\mathbb{P}\Big[|\det(M_k)| \leq \varepsilon_k \wedge |\det(T_{k-1}| \geq \varepsilon_{k-1}\Big] \\ &= \mathbb{E}\left[\mathbb{P}\left[|M_{k,k}\det(M_{k-1}) + f_k| \leq \varepsilon_k \Big| \Omega_k\right] \cdot \mathbb{1}_{\{|\det(M_{k-1})| \geq \varepsilon_{k-1}\}}\right] \\ &\leq \sup_{\gamma \in \mathbb{R}} \mathbb{P}\left[|M_{k,k} + \gamma| \leq \frac{\varepsilon_k}{\varepsilon_{k-1}}\right] \leq 2b\frac{\varepsilon_k}{\varepsilon_{k-1}}, \end{aligned}$$

where the last inequality follows from the fact for a continuous random variable X we always have

$$\sup_{\gamma \in \mathbb{R}} \mathbb{P}\Big[|X + \gamma| \leq t\Big] \leq 2bt, \tag{2.1}$$

where $b > 0$ is an upper bound on the density of X.

Thus, we get

$$\mathbb{P}\Big[|\det(M_k)| \leq \varepsilon_k\Big] \leq 2b\frac{\varepsilon_k}{\varepsilon_{k-1}} + \mathbb{P}\Big[|\det(M_{k-1})| \leq \varepsilon_{k-1}\Big],$$

Also, note that

$$\mathbb{P}\Big[|\det(M_1)| \leq \varepsilon_1\Big] = \mathbb{P}\Big[|T_{1,1} + A_{1,1}| \leq \varepsilon_1\Big] \overset{(2.1)}{\leq} 2b\varepsilon_1.$$

Therefore,

$$\mathbb{P}\Big[|\det(M_n)| \leq \varepsilon_n\Big] \leq 2b\left[\varepsilon_1 + \sum_{k=2}^{n} \frac{\varepsilon_k}{\varepsilon_{k-1}}\right].$$

Choosing $\varepsilon_j = t^j$, the result follows. □

Corollary 1.2 now follows immediately.

Proof of Corollary 1.2. Let $s_1(T) \geq \cdots \geq s_n(T)$ be the singular values of T. We have

$$s_1(T) = \|T\| = \sup_{\|x\|_2=1} \|Tx\|_2 = \sup_{\|x\|_2=\|y\|_2=1} \langle Tx, y\rangle \geq \max_{1\leq i\leq n} |T_{i,i}|.$$

Thus, by (2.1),

$$\mathbb{P}\Big[s_1(T) \leq t\Big] \leq \mathbb{P}\Big[\max_{1\leq i\leq n} |T_{i,i}| \leq t\Big] \leq (2bt)^n,$$

which proves (1.3).

To prove (1.4), note that

$$|\det(T)| = \prod_{i=1}^{n} s_i(T) \leq s_1(T)^{n-1} s_n(T) \leq \|T\|^{n-1} s_n(T). \tag{2.2}$$

Thus,

$$\mathbb{P}\Big[s_n(T) \leq t\Big] \leq \mathbb{P}\Big[s_n(T) \leq t \wedge \|T\| \leq \beta\Big] + \mathbb{P}\Big[\|T\| > \beta\Big] \tag{2.3}$$

For the first term, we have by (2.2) and Theorem 1.1,

$$\mathbb{P}\Big[s_n(T) \leq t \wedge \|T\| \leq \beta\Big] \leq \mathbb{P}\Big[\det(T) \leq \beta^{n-1} t\Big] \leq 2b\beta^{\frac{n-1}{n}} t^{1/n}.$$

Also,

$$\mathbb{P}\Big[\|T\| > \beta\Big] \leq \frac{\mathbb{E}\|T\|}{\beta}. \tag{2.4}$$

Thus, by (2.3) and (2.4),

$$\mathbb{P}\Big[s_n(T) \leq t\Big] \leq 2b\beta^{\frac{n-1}{n}} t^{1/n} + \frac{\mathbb{E}\|T\|}{\beta}.$$

Optimizing over β gives (1.4). □

3 The case of 2×2 matrices

As discussed in the introduction, we show that for 2×2 matrices the small ball estimate on the determinant obtained in Theorem 1.1 is not sharp. To do that, we use the well known fact that if X and Y are continuous random variables with joint density function $f_{X,Y}(\cdot,\cdot)$ then $X \cdot Y$ has a density function which is given by

$$f_{X\cdot Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|},$$

where f_X, f_Y are the density functions of X, Y, respectively.

We thus have the following.

Proposition 3.1. *Assume that X and Y are independent continuous random variables, with $f_X \leq b$, $f_Y \leq b$. Then $f_{X\cdot Y}$, the density function of $X \cdot Y$ satisfies*

$$f_{X\cdot Y}(z) \leq \begin{cases} 2b + 2b^2 |\log(|z|)| & |z| \leq 1, \\ 2b & |z| \geq 1. \end{cases}$$

Proof. Assume first that $|z| \le 1$. Write

$$f_{X\cdot Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|}$$
$$= \int_{|w|\le|z|} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|} + \int_{|z|\le|w|\le 1} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|} + \int_{|w|\ge 1} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|}. \quad (3.1)$$

Since X and Y are independent, $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$. We estimate each term of (3.1) separately.

$$\int_{|w|\le|z|} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le b \int_{|w|\le|z|} f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} = b \int_{|y|\ge 1} f_Y(y) \frac{dy}{|y|} \le b \quad (3.2)$$

$$\int_{|z|\le|w|\le 1} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le b^2 \int_{|z|\le|w|\le 1} \frac{dw}{|w|} = 2b^2 |\log(|z|)| \quad (3.3)$$

$$\int_{|w|\ge 1} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le b \int_{|w|\ge 1} f_X(w) \frac{dw}{|w|} \le b. \quad (3.4)$$

Plugging (3.2), (3.3) and (3.4) into (3.1), the result follows for $|z| \le 1$.

Now, if $|z| \ge 1$, then write

$$f_{X\cdot Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(w, \frac{z}{w}\right) \frac{dw}{|w|}$$
$$= \int_{|w|\le|z|} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} + \int_{|w|\ge|z|} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|}. \quad (3.5)$$

For the first term, we have

$$\int_{|w|\le|z|} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le b \int_{|y|\ge 1} f_Y(y) \frac{dy}{|y|} \le b. \quad (3.6)$$

And, for the second, by (3.4)

$$\int_{|w|\ge|z|} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le \int_{|w|\ge 1} f_X(w) \cdot f_Y\left(\frac{z}{w}\right) \frac{dw}{|w|} \le b. \quad (3.7)$$

Plugging (3.6) and (3.7) into (3.5), the result follows. ☐

Using Proposition 3.1, we immediately obtain the following:

Corollary 3.2. *Let X and Y be independent continuous random variables. Then for every $t \in (0,1)$ and every $\gamma \in \mathbb{R}$,*

$$\mathbb{P}\Big[|X \cdot Y + \gamma| < t\Big] \le 4bt + 4b^2 t(1 + |\log t|),$$

where $b > 0$ is a uniform upper bound on their densities.

Proof. Note that the function

$$g(z) = \left(2b + 2b^2|\log(|z|)|\right) \mathbb{1}_{\{|z|\le 1\}} + 2b\mathbb{1}_{\{|z|>1\}}$$

satisfies $g(|z_1|) \le g(|z_2|)$ whenever $|z_1| \ge |z_2|$. Thus, we have for every $\gamma \in \mathbb{R}$, $t \in (0,1)$,

$$\int_{\gamma-t}^{\gamma+t} g(z)dz \le \int_{-t}^{t} g(z)dz = \int_{-t}^{t} \left(2b + 2b^2|\log(|z|)|\right) dz = 4bt + 4b^2 t(1 + |\log t|).$$

Thus, by Proposition 3.1 we have

$$\mathbb{P}\Big[|X \cdot Y - \gamma| < t\Big] \le \int_{\gamma-t}^{\gamma+t} g(z)dz \le 4bt + 4b^2 t(1 + |\log t|).$$

☐

We also obtain the following corollary.

Corollary 3.3. *Let $T = \{T_{i,j}\}_{i,j\leq 2}$ be a random matrix such that $T_{1,1}$ and $T_{2,2}$ are continuous random variables, each independent of all the other entries of T. Then for every $t \in (0,1)$*

$$\mathbb{P}\Big[|\det(T)|^{1/2} \leq t\Big] \leq 4bt^2 + 4b^2t^2(1+2|\log t|),$$

where $b > 0$ is a uniform upper bound on the densities of $T_{1,1}$, $T_{2,2}$.

Proof. We have,

$$\begin{aligned}
\mathbb{P}\Big[|\det(T)| \leq t\Big] &= \mathbb{P}\Big[|T_{1,1}\cdot T_{2,2} - T_{1,2}\cdot T_{2,1}| \leq t\Big] \\
&= \mathbb{E}\left[\mathbb{P}\Big[|T_{1,1}\cdot T_{2,2} - T_{1,2}\cdot T_{2,1}| \leq t\Big| T_{1,2}, T_{2,1}\Big]\right] \\
&\leq \sup_{\gamma\in\mathbb{R}} \mathbb{P}\Big[|T_{1,1}\cdot T_{2,2} + \gamma| < t\Big] \\
&\leq 4bt + 4b^2t(1+|\log t|),
\end{aligned}$$

where in the last inequality we used Corollary 3.2. Replacing t by t^2, the result follows. □

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Acknowledgments. We thank Alexander Litvak and Nicole Tomczak-Jaegermann for helpful discussions and comments.