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# Stochastic Frank-Wolfe: Unified Analysis and Zoo of Special Cases

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## Abstract

The Conditional Gradient (or Frank-Wolfe) method is one of the most well-known methods for solving constrained optimization problems appearing in various machine learning tasks. The simplicity of iteration and applicability to many practical problems helped the method to gain popularity in the community. In recent years, the Frank-Wolfe algorithm received many different extensions, including stochastic modifications with variance reduction and coordinate sampling for training of huge models or distributed variants for big data problems. In this paper, we present a unified convergence analysis of the Stochastic Frank-Wolfe method that covers a large number of particular practical cases that may have completely different nature of stochasticity, intuitions and application areas. Our analysis is based on a key parametric assumption on the variance of the stochastic gradients. But unlike most works on unified analysis of other methods, such as SGD, we do not assume an unbiasedness of the real gradient estimation. We conduct analysis for convex and non-convex problems due to the popularity of both cases in machine learning. With this general theoretical framework, we not only cover rates of many known methods, but also develop numerous new methods. This shows the flexibility of our approach in developing new algorithms based on the Conditional Gradient approach. We also demonstrate the properties of the new methods through numerical experiments.

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## 1 INTRODUCTION

In this paper, we are interested in the constrained optimization problem

$$\min_{\mathcal{X} \subset \mathbb{R}^d} f(x), \quad (1)$$

where  $\mathcal{X}$  is a convex set. This problem is a cornerstone of applied mathematics, including machine learning. The problem (1) is at the heart of model training, from classical regressions (Shalev-Shwartz and Ben-David, 2014) to neural networks (Goodfellow et al., 2016). There are many approaches for solving (1). When projection onto a set is expensive (e.g. projection onto the nuclear norm-ball require a full singular value decomposition) or not possible at all (e.g., dual structural SVMs (Lacoste-Julien et al., 2013)), the Frank-Wolfe method (Frank and Wolfe, 1956), also known as Conditional Gradient (see a big survey (Braun et al., 2022) for more details), is a good option for dealing with (1). This approach is based on considering a linear minimization problem on  $\mathcal{X}$ . The Frank-Wolfe algorithm is one of the classical optimization methods, but it is still relevant even now. Particularly it finds applications in submodular optimization (Bach, 2011), multi-class classification (Hazan and Luo, 2016), vision (Miech et al., 2017; Bojanowski et al., 2014), group fused lasso (Bleakley and Vert, 2011), reduced rank nonparametric regression (Foygel et al., 2012), trace-norm based tensor completion ((Liu et al., 2013)), variational inference (Krishnan et al., 2015) and routing (LeBlanc et al., 1975), among others.

Current world reality encourages avoiding the deterministic setting of (1) and favoring the various stochastic ones. For instance, we often meet problems (1) with an expectation form of target function:  $f(x) = \mathbb{E}_{\xi \sim \mathcal{D}}[f(x, \xi)]$ . Here  $\mathcal{D}$  is usually associated with unknown distribution, like in terms of machine learning, it corresponds to some nature of the data. In such a setting, it is clearly impossible to compute the full deterministic gradient. But, despite the fact that data distribution is unknown, we can sample from  $\mathcal{D}$  and

replace the expectation form with approximation via Monte Carlo integration:  $f(x) = \frac{1}{n} \sum_{i=1}^n f_i(x)$  with  $n$  samples. However, modern application problems become increasingly larger and more computationally complex. Therefore, even for this sum-type problem, computing the full gradient is expensive and should be avoided. We can consider absolutely different randomization techniques in methods to be computationally efficient, e.g., in SGD-type methods stochasticity can be achieved by choosing data batches (Roux et al., 2012; Defazio et al., 2014; Johnson and Zhang, 2013; Nguyen et al., 2017a), computing gradient coordinates (Nesterov, 2012; Nesterov and Stich, 2017; Richtárik and Takáč, 2013; Qu and Richtárik, 2016), or even by using compression operators (Seide et al., 2014a; Alistarh et al., 2017; Wangni et al., 2018; Mishchenko et al., 2019), client samplings (Cho et al., 2020; Nguyen et al., 2020; Ribero and Vikalo, 2020; Chen et al., 2020) in distributed and federated settings (Konečný et al., 2016; Kairouz et al., 2019). Following the trend, we consider a stochastic version of the Conditional Gradient method:

$$\begin{aligned} s^k &= \arg \min_{s \in \mathcal{X}} \langle g^k, s - x^k \rangle, \\ x^{k+1} &= (1 - \eta_k) x^k + \eta_k s^k, \end{aligned} \quad (2)$$

where  $g^k$  is some stochastic estimator of the real gradient  $\nabla f(x^k)$ .

Over its long history, the Frank-Wolfe method received a huge number of different modifications in the form (2), most of them in the last decade. Here one can note modifications related to variance reduction (Reddi et al., 2016; Hazan and Luo, 2016; Qu et al., 2018; Yurtsever et al., 2019; Gao and Huang, 2020; Négier et al., 2020; Lu and Freund, 2021; Weber and Sra, 2022; Beznosikov et al., 2023), coordinate randomization (Lacoste-Julien et al., 2013; Sahu et al., 2019), and distributed computation (Bellet et al., 2015; Wang et al., 2016; Hou et al., 2022). However, all of these separate practical variants of the Conditional Gradient method have different intuitions of convergence, various formal techniques of proving and do not always cover the same cases of assumptions on the target function  $f$ . Moreover, there remains a rather large gap in what can still be done in the creation of Frank-Wolfe’s modifications. Namely, there are advanced SGD-type methods that have not yet been adapted for use within the Frank-Wolfe iteration. These include some new coordinate approaches and approaches to finite-sum problems. Meanwhile, these two techniques help to address the main bottleneck of distributed algorithms – expensive communications. All of mentioned issues lead us to the two key questions of this paper:

1. *Can we conduct a novel general analysis of the Stochastic Frank-Wolfe unifying special cases and providing the ability to design new*

*extensions?*

2. *What new stochastic modifications of the classical Conditional Gradient can we possibly invent based on this unified analysis?*

## 1.1 Our contribution

### • Unified analysis of Stochastic Frank-Wolfe.

We propose a general assumption on the stochastic estimator  $g^k$  from Stochastic Frank-Wolfe (2) – see Assumption 2.1. Below we note in more details that our assumption is broad and encompasses many special cases, in particular, those that could not be analyzed in a unified way before. Under Assumption 2.1 we present general convergence results for the problem (1).

### • Convex and non-convex cases.

Motivated by various applications primarily from machine learning, we provide the unified analysis in the convex (Theorem 2.2 and non-convex (Theorem 2.3) cases of the target function  $f$ . This is also interesting for special cases, since the authors of some papers do not give an analysis in both setups.

### • Without assumptions of unbiasedness.

In our key Assumption 2.1, we bound the variance  $\mathbb{E}[\|g^k - \nabla f(x^k)\|^2]$  using universal letter constants and an additional auxiliary sequence. Similar assumptions are made in papers on analysis of the SGD family methods (Gorbunov et al., 2020; Li and Richtárik, 2020; Khaled et al., 2020). But these works also additionally assume that the stochastic gradient  $g^k$  is unbiased, i.e.,  $\mathbb{E}[g^k | x^k] = \nabla f(x^k)$ . We avoid this assumption, it extends the class of methods that can be considered under our assumptions compared to the works around the SGD-type methods. For example, it allows to prove the convergence of distributed methods with biased/greedy compression (Stich and Karimireddy, 2019; Richtarik et al., 2021) or SARAH-based variance reduced methods (Nguyen et al., 2017b).

### • Vast number of new special cases.

The previous point already gives an indication of the breadth and flexibility of the approach. Our general theoretical framework allows us to analyze different variants of the classical Frank-Wolfe method. Guided by algorithmic advances for solving unconstrained minimization problems we present a new method with coordinate randomization (SEGA FW), a new variance-reduced method (L-SVRG FW, SARAH FW, SAGA FW), new distributed methods with unbiased compression (DIANA FW, MARINA FW) and biased compression (EF21 FW), and others. Although the SGD-type analogs of these methods are known for solving primarily unconstrained minimization problems (Hanzely et al., 2018; Kovalev et al., 2020; Gorbunov et al., 2021; Richtarik et al., 2021), they were never integrated into

the Frank-Wolfe iteration for solving projection-free constrained problems. We also demonstrate that our general theorems allow to obtain convergence for methods that are combinations of the two basic approaches, i.e., SAGA SARAH FW and others. Moreover, we presents absolutely new methods that are not found in the literature on SGD. This algorithm uses special coordinate randomization. JAGUAR is a new coordinate method. Q-L-SVRG FW is a new distributed method with unbiased compression. This method is based on non-distributed L-SVRG FW: instead of the randomness of choosing a batch/term number, randomness from compression is used. PP-L-SVRG FW is a new distributed method with client sampling also based on non-distributed L-SVRG FW.

- **Sharp rates for known special cases.** For the known methods fitting our framework our general theorems either recover the best rates known for these methods. These methods include SARAH FW, SAGA SARAH FW (Beznosikov et al., 2023).

- **Numerical experiments.** In numerical experiments, we illustrate the most important properties of the new methods. The results corroborate our theoretical findings.

Throughout the paper, we provide necessary comparisons with closely related work.

## 1.2 Technical preliminaries

**Notations.** We use  $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$  to denote standard inner product of  $x, y \in \mathbb{R}^d$ , where  $x_i$  corresponds to the  $i$ -th component of  $x$  in the standard basis in  $\mathbb{R}^d$ . It induces  $\ell_2$ -norm in  $\mathbb{R}^n$  in the following way:  $\|x\| := \sqrt{\langle x, x \rangle}$ . Operator  $\mathbb{E}[\cdot]$  denotes full mathematical expectation and operator  $\mathbb{E}[\cdot | x^k]$  express conditional mathematical expectation w.r.t. all randomness coming from the  $k$ th iteration of (2). We introduce  $f^*$  as a solution of the problem (1), i.e. a global minimum of  $f$  on the convex set  $\mathcal{X}$ . For the non-convex function  $f$  the solution  $f^*$  may not be unique. We also define  $\Delta_0 := f(x^0) - f^*$ , where  $x^0$  is a starting point of (2).

Throughout the paper, we assume that the target  $f$  from (1) satisfies the following assumptions.

**Assumption 1.1** *The function  $f : \mathcal{X} \rightarrow \mathbb{R}$ , is  $L$ -smooth on  $\mathcal{X}$ , i.e., there exists a constant  $L > 0$  such that  $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$  for all  $x, y \in \mathcal{X}$ .*

**Assumption 1.2** *The function  $f : \mathcal{X} \rightarrow \mathbb{R}$ , is convex, i.e.,  $f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$  for all  $x, y \in \mathcal{X}$ .*

While we always need the assumption on smoothness of  $f$ , we abandon the convexity in one of the main theorems. The next assumption is also key for the

design and analysis of Frank-Wolfe-type methods.

**Assumption 1.3** *The set  $\mathcal{X}$  is convex and compact with a diameter  $D$ , i.e., for any  $x, y \in \mathcal{X}$ ,*

$$\|x - y\| \leq D.$$

For some particular cases of the method (2), we need to introduce additional objects and assumptions on them. This will be done in the corresponding sections.

## 2 MAIN THEOREMS

In this section, we first present the central part of our approach that allows us to conduct a general analysis of the algorithms (Assumption 2.1), then we provide convergence analysis for both convex and non-convex cases.

### 2.1 Unified assumption

First, we introduce the central part of our approach, all subsequent analysis is based on the following assumption on the stochastic gradients  $g^k$  :

**Assumption 2.1** *Let  $\{x^k\}_{k=0}^K$  be the iterates produced by Stochastic Frank-Wolfe (see (2)). Let there exist constants  $A, B, C, E \geq 0$ ,  $\rho_1, \rho_2 \in (0; 1]$  and a (possibly) random sequence  $\{\sigma_k\}_{k \geq 0}$  such that the following inequalities hold*

$$\mathbb{E}[\|g^k - \nabla f(x^k)\|^2 | x^k] \leq (1 - \rho_1)\|g^{k-1} - \nabla f(x^{k-1})\|^2 + A\sigma_{k-1}^2 + \eta_{k-1}^2 B D^2 + C, \quad (3)$$

$$\mathbb{E}[\sigma_k^2 | x^k] \leq (1 - \rho_2)\sigma_{k-1}^2 + \eta_{k-1}^2 E D^2. \quad (4)$$

The inequality 3 bound the second moment of stochastic estimation  $g^k$ . The sequence  $\{\sigma_k^2\}_{k \geq 0}$  is needed to capture the variance, which can be reduced during the algorithm's work process. Constants  $\rho_1, \rho_2$  show how quick this reduction is regarding the previous iteration.  $B, E$  provide the information of convergence depending on the previous step size and set's geometry (Assumption 1.3). Finally, constant  $C$  stands for the remaining noise that cannot be reduced as  $\sigma_k$ .

Proposed assumption takes into account specificity of the Frank-Wolfe analysis in terms of upper bound containing component  $D^2$  as irremovable part of such type inequalities.

### 2.2 Convergence results

**Convex case.** The following theorem describes the convergence rate of stochastic Frank-Wolfe (2) based methods under the convexity of  $f$ :

**Theorem 2.2** *Let Assumptions 1.1, 1.2, 1.3 and 2.1 be satisfied. For any  $K$  choose step sizes  $\{\eta_k\}_{k \geq 1}$  as follows:*

$$\begin{aligned} \text{if } K \leq d, & \quad \eta_k = \frac{1}{d}, \\ \text{if } K > d \text{ and } k < k_0, & \quad \eta_k = \frac{1}{d}, \end{aligned}$$

if  $K > d$  and  $k \geq k_0$ ,  $\eta_k = \frac{2}{2d+k-k_0}$ ,

where  $d = \frac{2}{\min(\rho_1, \rho_2)}$ ,  $k_0 = \lceil \frac{K}{2} \rceil$ . Then the output of Stochastic Frank-Wolfe after  $K$  iterations satisfies

$$\mathbb{E}[r_{K+1}] = \mathcal{O}\left(r_0 \exp\left(-\frac{K}{2d}\right) + \frac{LD^2}{K+d} + \frac{D^2}{K+d} \sqrt{\frac{B\rho_2+AE}{\rho_1\rho_2}} + \sqrt{\frac{KD^2}{K+d} \frac{C\rho_2}{\rho_1\rho_2}}\right),$$

where the Lyapunov function  $r_k$  is defined by  $r_k := f(x^k) - f^* + M_1 \|\nabla f(x^k) - g^k\|^2 + M_2 \sigma_k^2$  with  $M_1, M_2 > 0$ .

The proof is provided in Section B.2. Note, that with zero noises ( $C = 0$  in Assumption 2.1) this theorem reflects sublinear convergence  $\mathcal{O}\left(\frac{1}{K}\right)$ .

**Non-convex case.** To obtain our convergence results for the non-convex objective function we introduce the Frank-Wolfe gap function (Jaggi, 2013) as a convergence criterion:

$$\mathbf{gap}(y) = \max_{x \in \mathcal{X}} \langle \nabla f(y), y - x \rangle$$

Such type of criterion is standard for analyzing the convergence of constrained optimization algorithms in the non-convex case (Reddi et al., 2016). It is shown in (Lacoste-Julien, 2016) that FW gap is an affine invariant generalization of standard convergence criterion  $\|\nabla f(y)\|$  and therefore a meaningful measure of non-stationarity. In terms of FW gap we derive the following general convergence result:

**Theorem 2.3** *Let the Assumptions 1.1, 1.3 and 2.1 be satisfied. Then, there exist constants  $M_1, M_2$  such that for any  $K$  there exist constant  $\{\eta_k\}_{k \geq 1} \equiv \frac{1}{\sqrt{K}}$  for (2), thus*

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{r_0}{\sqrt{K}} + \frac{D^2}{\sqrt{K}} \left[L + \sqrt{\frac{B\rho_2+AE}{\rho_1\rho_2}} + \sqrt{D^2 \frac{C\rho_2}{\rho_1\rho_2}}\right],\right)$$

where  $r_0 := f(x^0) - f^* + M_1 \|g^0 - \nabla f(x^0)\|^2 + M_2 \sigma_0^2$  with  $M_1, M_2 > 0$ .

See the proof in Section B.2. Note, that in the case of zero noises ( $C = 0$  in Assumption 2.1) this result establishes as  $\mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$ .

### 3 WIDE VARIETY OF SPECIAL METHODS

In this section, we fulfill the promises made in the introduction and show how many existing and new techniques fit our framework. Due to space restrictions, the comparison and the full listing of algorithms are described particularly in Section A.

#### 3.1 Stochastic methods

As already mentioned in Section 1, in modern applications we often deal with finite-sum optimization prob-

lems (so-called empirical risk minimization):

$$\min_{x \in \mathcal{X}} f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x). \quad (5)$$

An important detail of this setting is that calling the full gradient of  $f$  is expensive, only small batches  $\frac{1}{b} \sum_{i=1}^b \nabla f_i(x)$  can be typically used. Therefore, for the theoretical analysis, we need not only the smoothness of the function  $f$ , but also of all summands  $f_i$ .

**Assumption 3.1** *Each function  $f_i : \mathcal{X} \rightarrow \mathbb{R}$  is  $L_i$ -smooth on  $\mathcal{X}$ , i.e., there exist constants  $\{L_i\} > 0$  such that  $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|$  for all  $x, y \in \mathcal{X}$ . We also define  $\tilde{L}$  as  $\tilde{L}^2 = \frac{1}{n} \sum_{i=1}^n L_i^2$ .*

**L-SVRG FW.** One of the most popular stochastic algorithms for (5) with  $\mathcal{X} \equiv \mathbb{R}^d$  is SVRG (Johnson and Zhang, 2013). We consider its loopless variant (Kovalev et al., 2020) called L-SVRG that uses SVRG idea but is a bit more friendly for theoretical analysis. In more details, we need to compute  $g^k$  as follows:

$$w^{k+1} = \begin{cases} x^k, & \text{with probability } p, \\ w^k, & \text{with probability } 1-p, \end{cases} \quad (6)$$

$g^{k+1} = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - \nabla f_i(w^{k+1})] + \nabla f(w^{k+1})$ , where batches of indexes  $S_k$  size of  $b$  are generated uniformly and independently. The essence of this approach is that the probability  $p$  is taken close to zero, then the full gradient at the point  $w^{k+1}$  are computed quite rarely and in most cases we use the approximation  $g^{k+1}$  via stochastic gradients on mini-batches of random indexes  $S_k$  size of  $b$ .

**Lemma 3.2** *Under assumptions 1.1, 1.2, 3.1 the algorithm (2) + (6) satisfies assumption 2.1 with:  $\rho_1 = 1$ ,  $A = \frac{\tilde{L}^2}{b} (1 - \frac{p}{2})$ ,  $B = \frac{8\tilde{L}^2}{pb}$ ,  $C = 0$ ,  $\sigma_k^2 = \|x^k - w^k\|^2$ ,  $\rho_2 = \frac{p}{2}$ ,  $E = \frac{8}{p}$ .*

Using this lemma, one can get the convergence of (2) + (6) in both convex and non-convex cases.

**Corollary 3.3** *For the algorithm (2)+(6) in the convex and non-convex cases accordingly the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{Kp}{8}\right) + \frac{LD^2}{K} \left[1 + \frac{\tilde{L}}{L} \frac{1}{p\sqrt{b}}\right]\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \frac{\tilde{L}}{L} \frac{1}{p\sqrt{b}}\right]\right).$$

See more details in Section B.3.1.

**SARAH FW.** Another common algorithm for solving the unconstrained version of (5) is SARAH (Nguyen et al., 2017b). In particular, it has better theoretical results in both convex (Nguyen et al., 2017b), non-convex (Li et al., 2021) target functions and also bits SVRG on practice. As in the previous method we look at the loopless version of SARAH (Li et al., 2021):

$$g^{k+1} = \begin{cases} \nabla f(x^{k+1}), & \text{with probability } p, \\ g^k + \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - \nabla f_i(x^k)], & \text{oth.,} \end{cases} \quad (7)$$

where batches of indexes  $S_k$  size of  $b$  are generated uniformly and independently. The methods main idea here is very close to SVRG since it also computes the full gradient only with small probability  $p$ . However, the approximation of the gradient  $g^{k+1}$  in SARAH is done not by the old point  $w^{k+1}$  as in SVRG, but more smoothly using the current and previous points:  $x^{k+1}$  and  $x^k$ .

**Lemma 3.4** *Under Assumptions (1.1), (1.2), (3.1) the algorithm (2)+(7) satisfies Assumption 2.1 with:  $\rho_1 = p$ ,  $A = 0$ ,  $B = \frac{1-p}{b}\tilde{L}^2$ ,  $C = 0$ ,  $\sigma_k = 0$ ,  $\rho_2 = 1$ ,  $E = 0$ .*

**Corollary 3.5** *For the algorithm (2)+(7) in the convex and non-convex cases respectively the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{Kp}{4}\right) + \frac{LD^2}{K} \left[1 + \frac{\tilde{L}}{L} \frac{1}{\sqrt{pb}}\right]\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \frac{\tilde{L}}{L} \frac{1}{\sqrt{pb}}\right]\right).$$

The details are provided in Section B.3.2. It is easy to see that with the same  $p$  the results for SARAH FW is better than for L-SVRG FW (Corollary 3.3).

**SAGA FW.** The final algorithm of provided lineup for (5) is SAGA (Defazio et al., 2014):

$$y_i^{k+1} = \begin{cases} \nabla f_i(x^k), & \text{for } i \in S_k, \\ y_i^k, & \text{for } i \notin S_k, \end{cases} \quad (8)$$

$$g^{k+1} = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - y_i^{k+1}] + \frac{1}{n} \sum_{j=1}^n y_j^{k+1},$$

where batches of indexes  $S_k$  size of  $b$  are generated uniformly and independently. The essence of the SAGA technique is different from SVRG and SARAH. In this case, we do not compute full gradients even rarely, but we need to store an additional set of vectors  $\{y_i^k\}_{i=1}^n$ . The vector  $y_i$  stores information about the last gradient of the function  $f_i$  that was computed during the operation of the algorithm. Thus one can state that we collect "delayed" full gradient in  $\frac{1}{n} \sum_{j=1}^n y_j^k$ .

**Lemma 3.6** *Under Assumptions 1.1, 1.2, 3.1 the algorithm (2)+(8) satisfies Assumption 2.1 with:  $\rho_1 = 1$ ,  $A = \frac{1}{b} \left(1 + \frac{b}{2n}\right)$ ,  $B = \frac{2\tilde{L}^2}{b} \left(1 + \frac{2n}{b}\right)$ ,  $C = 0$ ,  $\sigma_k^2 = \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^{k+1}\|^2$ ,  $\rho_2 = \frac{b}{2n}$ ,  $E = \frac{2n}{b}\tilde{L}^2$ .*

**Corollary 3.7** *For the algorithm (2)+(8) in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{Kb}{8n}\right) + \frac{LD^2}{K} \left[1 + \frac{\tilde{L}}{L} \frac{n}{b\sqrt{b}}\right]\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \frac{\tilde{L}}{L} \frac{n}{b\sqrt{b}}\right]\right).$$

The full statement together with its proof can be found in Section B.3.3. One more method (SAGA SARAH FW)

for the stochastic setting will be analyzed in Section 3.4. A comparison of the presented and already existing methods is provided in Section A.1. In particular, there we analyze Corollaries 3.3–3.7 with the substituted optimal parameter value of  $p$ .

### 3.2 Coordinate methods

Previous approaches reduce the cost of gradient computing by selecting small batches, but there are other strategies. In particular, wide range of algorithms use random sampling of coordinates for gradient evaluation (Nesterov, 2012; Richtárik and Takáč, 2013; Qu and Richtárik, 2016). This technique can also significantly decrease the computational cost. Then, in this section, we focus on methods, where gradient estimator stochastically depend on function's partial derivatives.

**SEGA FW.** The original algorithm developed in (Hanzely et al., 2018) covers a general setting, instead of which we use a slightly more simplified version. Particularly, we update  $g^k$  as follows:

$$h^{k+1} = h^k + e_{i_k}(\nabla_{i_k} f(x^k) - h_{i_k}^k),$$

$$g^{k+1} = d(\nabla_{i_k} f(x^{k+1}) - h_{i_k}^{k+1})e_{i_k} + h^{k+1}, \quad (9)$$

where coordinate  $i_k$  is chosen uniformly and independently. The idea of this approach is in some sense close to SAGA. We also have some memory buffer, but unlike SAGA, where we save the last calculated gradient on  $i$ th batch, here in  $h_i$  we save the last partial derivative  $\nabla_i f$  calculated for the  $i$ th coordinate.

**Lemma 3.8** *Under Assumptions 1.1, 1.2 the algorithm (2) +(9) satisfies Assumption 2.1 with:  $\rho_1 = 1$ ,  $A = d$ ,  $B = d^2 L^2$ ,  $C = 0$ ,  $\sigma_k^2 = \|h^{k+1} - \nabla f(x^k)\|^2$ ,  $\rho_2 = \frac{1}{2d}$ ,  $E = 3L^2 d$ .*

**Corollary 3.9** *For the algorithm (2)+(9) in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{K}{8d}\right) + \frac{LD^2}{K} \cdot d\sqrt{d}\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \cdot d\sqrt{d}\right).$$

See details in Section B.4.1. Despite the fact that SEGA has recommended itself as an effective coordinate method, theoretical estimation of demanded steps to converge has undesired component  $d\sqrt{d}$  (while the original algorithm has only  $d$ ). Therefore, it makes sense to introduce a new algorithm, which has theoretically better convergence.

**JAGUAR.** An important feature of SEGA is the fact that it uses unbiased gradient estimation:  $\mathbb{E}[g^k | x^k] = \nabla f(x^k)$ . On the one hand it is good and helps to simplify the theoretical analysis. But experience shows that stochastic methods with biased gradient approxi-

mation can outperform unbiased ones. The example of SARAH (biased) and SVRG (unbiased) supports this. Therefore, we propose to consider the following form of  $g^k$ :

$$g^{k+1} = e_{i_k}(\nabla_{i_k} f(x^{k+1}) - g_{i_k}^k) + g^k, \quad (10)$$

where coordinate  $i_k$  is chosen uniformly and independently.

**Lemma 3.10** *Under Assumptions 1.1, 1.2 the algorithm (2)+(10) satisfies Assumption 2.1 with:  $\rho_1 = 1, A = 0, B = 3dL^2, C = 0, \sigma_k^2 = 0, \rho_2 = 1, E = 0$ .*

**Corollary 3.11** *For the algorithm (2)+(10) in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned} \mathbb{E}[f(x^K) - f^*] &= \mathcal{O}\left(\Delta_0 \exp\left(-\frac{K}{8d}\right) + \frac{LD^2}{K} \cdot d\right), \\ \mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] &= \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \cdot d\right). \end{aligned}$$

One can find the full statement together with its proof in Section B.4.2. A comparison of the presented and already existing methods is presented in Section A.2.

### 3.3 Distributed methods with compression

In this section, we focus on distributed versions of Frank-Wolfe algorithm for solving finite-sum problems (5), where  $\{f_i\}_{i=1}^n$  are distributed across  $n$  devices connected with parameter-server in a centralized way and each device has an access only to  $f_i$ . Here we allow different machines to have different data distributions, i.e., we consider the heterogeneous data setting. For such type of problem, the bottleneck commonly is a communication cost (Konečný et al., 2016), which motivates to use compressed communication (Seide et al., 2014b). To formally describe compression we introduce the following definition.

**Definition 3.12** *Map  $\mathcal{Q} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is an unbiased compression operator, if there exist a constant  $\omega \geq 0$  such that for all  $x \in \mathbb{R}^d$*

$$\mathbb{E}[\mathcal{Q}(x)] = x, \quad \mathbb{E}[\|\mathcal{Q}(x) - x\|^2] \leq \omega \|x\|^2.$$

Examples of such operators are random coordinate selection or randomized roundings (Beznosikov et al., 2022). The usage of unbiased compression has been extensively studied. The first and simplest idea that comes to a mind is to apply compression directly on the gradient estimator when forwarding to the server (Alistarh et al., 2017). But this kind of approach has a problem, namely for a fixed step it guarantees convergence only to the neighborhood of the solution. Therefore, we propose to consider more advanced techniques that compress some difference that tends to zero during the course of the algorithm.

**DIANA FW.** We start with the DIANA technique (Mishchenko et al., 2019). Its essence lies in the fact that it maintains the "memory" variables  $h_i^k$  at each worker  $i$  and compresses gradient differences  $\nabla f_i(x^k) - h_i^k$ . In particular,

$$\begin{aligned} \Delta_i^k &= \mathcal{Q}(\nabla f_i^k - h_i^k), \quad h_i^{k+1} = h_i^k + \alpha \cdot \hat{\Delta}_i^k, \\ h^{k+1} &= h^k + \alpha \cdot \frac{1}{n} \sum_{i=1}^n \Delta_i^k, \\ g^{k+1} &= h^{k+1} + \frac{1}{n} \sum_{i=1}^n \Delta_i^{k+1}, \end{aligned} \quad (11)$$

where first two equations belong to the local computations and last two – to parameter-server computation. It is important to highlight that we need only compressed differences  $\Delta_i^k$  for the server updates.

**Lemma 3.13** *Under Assumptions 1.1, 1.2, 3.1 the algorithm (2) +(11) satisfies Assumption 2.1 with:  $\rho_1 = 1, A = \frac{\omega}{n^2}, B = \frac{2\omega(\omega+1)\tilde{L}^2}{n}, C = 0, \sigma_k^2 = \sum_{i=1}^n \|\nabla f_i(x^k) - h_i^k\|^2, \rho_2 = \frac{1}{2(1+\omega)}, E = 2(\omega+1)n\tilde{L}^2$ .*

**Corollary 3.14** *For the algorithm (2)+(11) in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned} \mathbb{E}[f(x^K) - f^*] &= \mathcal{O}\left(\Delta_0 \exp\left(-\frac{K}{8\omega}\right) + \frac{LD^2}{K} \left(1 + \frac{\tilde{L}}{L} \frac{\omega^{\frac{3}{2}}}{\sqrt{n}}\right)\right), \\ \mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] &= \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left(1 + \frac{\tilde{L}}{L} \frac{\omega^{\frac{3}{2}}}{\sqrt{n}}\right)\right). \end{aligned}$$

The full proof and convergence results are presented in Section B.5.1.

**MARINA FW.** Next, as a natural generalization of the idea of compressing gradient differences, we arrive at the fact that gradient estimator could be biased. We have already seen this idea take place with the examples of L-SVRG FW vs SARAH FW (Section 3.1), as well as SEGA FW vs JAGUAR (Section 3.2). Therefore, we consider the work by (Gorbunov et al., 2021), where the authors base their method on SARAH technique but with compression stochasticity instead of (7). Our algorithm utilizes this idea and performs the update rule (2) with

$$\begin{aligned} c_i^{k+1} &= \begin{cases} \nabla f_i(x^{k+1}) - g_i^k, & \text{with probability } p, \\ \mathcal{Q}(\nabla f_i(x^k) - \nabla f_i(x^{k-1})), & \text{otherwise,} \end{cases} \\ g_i^{k+1} &= g_i^k + c_i^{k+1} \\ g^{k+1} &= g^k + \frac{1}{n} \sum_{i=1}^n c_i^{k+1}, \end{aligned} \quad (12)$$

where  $g_i^{k+1}$  are computed on the local devices and  $g^{k+1}$  – on server side. In fact, with  $p$  close to zero, one can note that to compute  $g^{k+1}$  we typically need only compressed differences:  $g^{k+1} = g^k + \frac{1}{n} \sum_{i=1}^n \mathcal{Q}(\nabla f_i(x^{k+1}) - \nabla f_i(x^k))$ . But rarely (with probability  $p$ ) we need to send the full uncompressed gradients.

**Lemma 3.15** *Under Assumption 3.1 the algorithm (2) +(12) satisfies Assumption 2.1 with:  $\rho_1 = p,$*

$\rho_2 = 1$ ,  $A = 0$ ,  $B = \frac{(1-p)\omega L^2}{n}$ ,  $C = 0$ ,  $\sigma_k = 0$ ,  $E = 0$ .

**Corollary 3.16** *For the algorithm (2)+(12) in the convex and non-convex cases respectively the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{pK}{4}\right) + \frac{LD^2}{K}\left(1 + \sqrt{\frac{\omega}{pn}}\right)\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}}\left(1 + \sqrt{\frac{\omega}{pn}}\right)\right).$$

One can find the full statement with its proof in B.5.2 of Appendix.

**EF21 FW.** To complement our results of distributed methods with compression we now introduce biased compressors:

**Definition 3.17** *Map  $\mathcal{C} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a biased compression operator, if there exist a constant  $\delta \geq 1$  such that for all  $x \in \mathbb{R}^d$*

$$\mathbb{E}[\|\mathcal{C}(x) - x\|^2] \leq \left(1 - \frac{1}{\delta}\right) \|x\|^2.$$

This is a broader class than unbiased operators. Here, for example, one can find the greedy choice of coordinates (Alistarh et al., 2018), sparse decompositions (Vogels et al., 2019) and other operators interesting in practice (Beznosikov et al., 2022). Intuition suggests that using biased/greedy compressors may improve convergence over unbiased operators. But biased compression are less "suitable" in theory than unbiased ones. Indeed, one can construct a simple convex quadratic problem for which distributed SGD with Top1 compression diverges exponentially fast (Beznosikov et al., 2022). This issue can be resolved using error compensation technique (Seide et al., 2014a; Stich and Karimireddy, 2019; Qian et al., 2020). Then we consider one of the state-of-art algorithms with error compensation technique for biased compression (Richtarik et al., 2021):

$$g_i^k = g^{k-1} + \mathcal{C}(\nabla f_i(x^k) - g_i^{k-1}), \quad (13)$$

$$g^k = \frac{1}{n} \sum_{i=1}^n g_i^k = g^{k-1} + \frac{1}{n} \sum_{i=1}^n \mathcal{C}(\nabla f_i(x^k) - g_i^{k-1}).$$

Here as in (12) the computation of  $g_i^k$  takes place on the local devices and the computation of  $g^k$  – on the server and for that only compressed differences  $\mathcal{C}(\nabla f_i(x^{k+1}) - g_i^k)$  are needed. Note that (13) does not send uncompressed packages at all.

**Lemma 3.18** *Under Assumptions 1.1, 1.2, 3.1 the algorithm (2)+(13) satisfies Assumption 2.1 with:  $\rho_1 = 1$ ,  $A = 1$ ,  $B = 0$ ,  $C = 0$ ,  $\sigma_k^2 = \frac{1}{n} \sum_{i=1}^n \|g_i^k - \nabla f_i(x^k)\|^2$ ,  $\rho_2 = \frac{1}{2\delta}$ ,  $E = 2\delta\tilde{L}^2$ .*

**Corollary 3.19** *For the algorithm (2)+(13) in the convex and non-convex cases respectively the following*

*convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{K}{8\delta}\right) + \frac{LD^2}{K}\left[1 + \frac{\tilde{L}}{L}\delta\right]\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}}\left[1 + \frac{\tilde{L}}{L}\delta\right]\right).$$

The full statement together with its proof is provided in B.5.3. One more method with compression will be presented in the next section. A comparison of the presented and already existing methods is provided in Section A.3.

### 3.4 Combinations of different approaches

In this section, we show that combinations of already presented methods can also be analyzed using Assumption 2.1.

**SAGA SARAH FW.** Here we combine ideas of two approaches: SARAH FW and SAGA FW. Such method is preferable because it obtains benefits of both methods: better rates from SARAH and missing full gradient calculations from SAGA. We get the following gradient estimation:

$$g^k = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^k) - \nabla f_i(x^{k-1})] + (1 - \lambda)g^{k-1} + \lambda \left( \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k-1}) - y_i^k] + \frac{1}{n} \sum_{j=1}^n y_j^k \right),$$

$$y_i^{k+1} = \begin{cases} \nabla f_i(x^k), & i \in S_k, \\ y_i^k, & i \notin S_k, \end{cases} \quad (14)$$

where  $\lambda = \frac{b}{2n}$  and batches of indexes  $S_k$  size of  $b$  are generated uniformly and independently.

**Lemma 3.20** *Under Assumptions 1.1, 1.2, 3.1 the algorithm (2)+(14) satisfies Assumption 2.1 with:*

$$\rho_1 = \frac{b}{2n}, \quad A = \frac{b}{2n^2}, \quad B = \frac{2\tilde{L}^2}{b}, \quad C = 0, \quad \sigma_k^2 = \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^{k+1}\|^2, \quad \rho_2 = \frac{b}{2n}, \quad E = \frac{2n\tilde{L}^2}{b}.$$

**Corollary 3.21** *For the algorithm (2)+(14) in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O}\left(\Delta_0 \exp\left(-\frac{bK}{8n}\right) + \frac{LD^2}{K}\left[1 + \frac{\tilde{L}}{L}\frac{\sqrt{n}}{b}\right]\right),$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{\Delta_0}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}}\left[1 + \frac{\tilde{L}}{L}\frac{\sqrt{n}}{b}\right]\right).$$

The full statement together with its proof can be found in Section B.6.1. The estimates from Corollary 3.21 are the same as for SARAH FW (Corollary 3.5) and this estimates are the best in the literature on Stochastic Conditional Gradient – see Section A.1 for more details.

Also different combinations of techniques can be considered in the distributed case. In all approaches from Section 3.3, we considered that we have a full gradient available for each  $f_i$ . But for example we can assume that  $f_i$  also has the finite-sum form  $f_i = \sum_{j=1}^m f_{ij}$ .

Thus one can add the techniques from Section 3.1. We leave this in Appendix B. In the main part, we consider another illustration.

**Q-L-SVRG FW.** As we noted above, MARINA (12) is based on SARAH. Then we can do exactly the same manipulation with L-SVRG and convert it from a method for stochastic sum-type problem into a method with compression:

$$w^{k+1} = \begin{cases} x^k, & \text{with probability } p, \\ w^k, & \text{with probability } 1-p, \end{cases} \quad (15)$$

$$g^{k+1} = \frac{1}{n} \sum_{i=1}^n \mathcal{Q}(\nabla f_i(x^{k+1}) - \nabla f_i(w^{k+1})) + \nabla f(w^{k+1}),$$

The same way as in (12), to compute  $g^{k+1}$  on the server we typically need only compress differences  $\mathcal{Q}(\nabla f_i(x^{k+1}) - \nabla f_i(w^{k+1}))$ . But with small probability  $p$  we need to transfer on the server full packages and compute new  $\nabla f(w^{k+1})$ .

**Lemma 3.22** *Under Assumptions 1.1, 1.2, 3.1 the algorithm (2)+(15) satisfies Assumption 2.1 with:  $\rho_1 = 1$ ,  $A = \omega \tilde{L}^2 (1 - \frac{p}{2})$ ,  $B = \omega \tilde{L}^2 \left(1 + \frac{8(1-p)}{p}\right)$ ,  $C = 0$ ,  $\sigma_k^2 = \|x^k - w^k\|^2$ ,  $\rho_2 = \frac{p}{2}$ ,  $E = 1 + \frac{8(1-p)}{p}$ .*

**Corollary 3.23** *For the algorithm (2)+(15) in the convex and non-convex cases accordingly the following convergences take place:*

$$\mathbb{E}[f(x^K) - f^*] = \mathcal{O} \left( \Delta_0 \exp \left( -\frac{Kp}{8} \right) + \frac{\tilde{L}D^2}{K + \frac{1}{p}} \left[ 1 + \frac{\sqrt{\omega}}{p} \right] \right),$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \text{gap}(x^k) \right] = \mathcal{O} \left( \frac{\Delta_0}{\sqrt{K}} + \frac{\tilde{L}D^2}{\sqrt{K}} \left[ 1 + \frac{\sqrt{\omega}}{p} \right] \right).$$

See the proof in Section B.6.2.

## 4 EXPERIMENTS

In this section, we present experimental results to support our theoretical findings.

**Setup.** We consider the logistic regression problem:  $\min_{x \in \mathcal{X}} f(x) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-b_i \cdot x^T a_i))$ , (16) where  $x$  are weights and  $\{a_i, b_i\}_{i=1}^n$  are training data samples with  $a_i \in \mathbb{R}^d$  and  $b_i \in \{-1, 1\}$ . We choose  $\mathcal{X}$  as the  $l_1$ -ball with radius  $r = 2 \cdot 10^3$ . The linear minimization oracle, i.e.  $\arg \min_{s \in \mathcal{X}} \langle g, s \rangle$ , for this constraint set can be computed in the closed-form:  $s^* = -\text{sign}(g_i) e_i$  with  $i = \arg \max_j |g_j|$ . We take LibSVM datasets (Chang and Lin, 2011).

For further details about the experiments and additional experiments see Section C.

**Stochastic methods.** In this experiment, we test the performance of proposed stochastic Frank-Wolfe-based methods (SAGA SARAH FW, L-SVRG FW) and compare them to deterministic version of it. The performance is measured in number of full gradients computed. Hyperparameters of the methods are chosen according to

theory and all methods starts from zero. The results are provided in Figure 1. One can note that SAGA SARAH FW outperforms competitors. It confirms our theoretical conclusions perfectly.

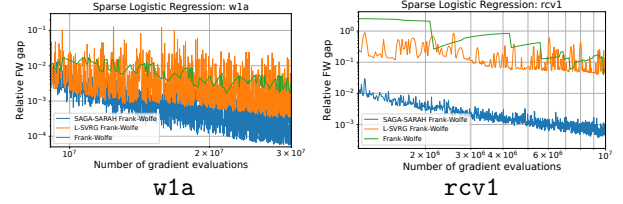


Figure 1: Comparison of methods for solving (16) in the stochastic case. SARAH FW, SAGA SARAH FW are considered. The comparison is made on LibSVM datasets w1a, rcv1.

In addition, we compared the methods with stochastic baselines chosen from (Lu and Freund, 2021), (Mokhtari et al., 2020), (Négiar et al., 2020):

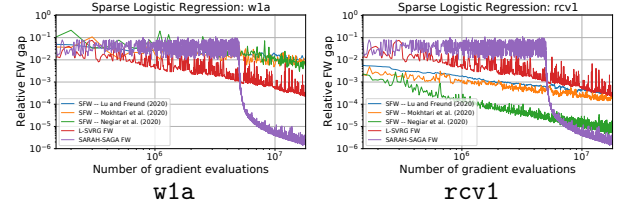


Figure 2: Comparison of stochastic methods with baselines. SARAH FW, SAGA SARAH FW are considered. The comparison is made on LibSVM datasets w1a, rcv1.

Based on the obtained results, one can see, SAGA SARAH FW overcomes all competitors on both datasets.

Finally for this section of methods we introduce a comparison with methods for constrained optimization with Euclidean projection instead of linear minimization. It is presented in Section C of Appendix.

**Coordinate methods.** In the second experiment, we check the convergence of alleged coordinate methods: SEGA FW, JAGUAR and compare them to the original Frank-Wolfe method. The performance is measured in number of full gradients computed. Hyperparameters of the methods are chosen according to theory and all methods starts from zero. The results are presented in Figure 3.

**Distributed methods.** In the last experiment, we consider the distributed setup of (16), in which we assume that the information about  $f_i$  is available for worker  $i$  only. Here we compare distributed methods proposed in this paper: MARINA Frank-Wolfe, VR-MARINA Frank-Wolfe, DIANA Frank-Wolfe and EF21 Frank-Wolfe. The performance is measured in number of bits communicated from workers to the



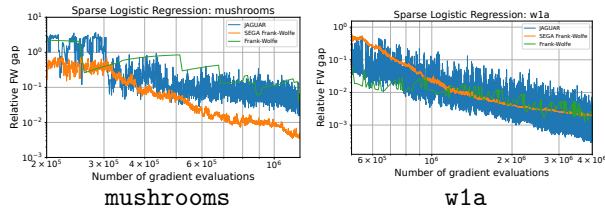


Figure 3: Comparison of methods for solving (16) in the coordinate case. SEGA FW, JAGUAR are considered. The comparison is made on LibSVM datasets mushrooms, w1a.

server. Hyperparameters of the methods are chosen according to theory and all methods starts from zero. In MARINA and DIANA algorithms RandK (random sparsification) compression is used, while in EF21 TopK ("greedy" sparsification) compression is implemented. Convergence performance is shown in Figure 4. The advantage of using compressed communication is clearly observable in every case.

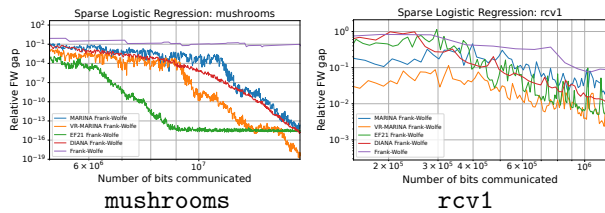


Figure 4: Comparison of methods for solving (16) in the distributed case. MARINA FW, VR-MARINA FW, EF21 FW, DIANA FW are considered. The comparison is made on LibSVM datasets mushrooms, rcv1.

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## Checklist

1. For all models and algorithms presented, check if you include:
  - (a) A clear description of the mathematical setting, assumptions, algorithm, and/or model. [Yes]
  - (b) An analysis of the properties and complexity (time, space, sample size) of any algorithm. [Yes]
  - (c) (Optional) Anonymized source code, with specification of all dependencies, including external libraries. [No]
2. For any theoretical claim, check if you include:
  - (a) Statements of the full set of assumptions of all theoretical results. [Yes]
  - (b) Complete proofs of all theoretical results. [Yes]
  - (c) Clear explanations of any assumptions. [Yes]
3. For all figures and tables that present empirical results, check if you include:
  - (a) The code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL). [Not Applicable, small experiments, easy and fast to rerun]
  - (b) All the training details (e.g., data splits, hyperparameters, how they were chosen). [Yes]
  - (c) A clear definition of the specific measure or statistics and error bars (e.g., with respect to the random seed after running experiments multiple times). [Not Applicable, small experiments]
  - (d) A description of the computing infrastructure used. (e.g., type of GPUs, internal cluster, or cloud provider). [Not Applicable, small experiments with simulation of real distributed setting – no need in large computational powers]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets, check if you include:
  - (a) Citations of the creator If your work uses existing assets. [Yes]
  - (b) The license information of the assets, if applicable. [Not Applicable]
  - (c) New assets either in the supplemental material or as a URL, if applicable. [Not Applicable]
  - (d) Information about consent from data providers/curators. [Not Applicable]
  - (e) Discussion of sensible content if applicable, e.g., personally identifiable information or offensive content. [Not Applicable]
5. If you used crowdsourcing or conducted research with human subjects, check if you include:
  - (a) The full text of instructions given to participants and screenshots. [Not Applicable]
  - (b) Descriptions of potential participant risks, with links to Institutional Review Board (IRB) approvals if applicable. [Not Applicable]
  - (c) The estimated hourly wage paid to participants and the total amount spent on participant compensation. [Not Applicable]

# Supplementary Material

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## A MISSING COMPARISON AND DETAILS

### A.1 Stochastic methods

Table 1: Summary of complexity results for finding an  $\varepsilon$ -solution **stochastic finite-sum non-distributed constrained minimization problems** (5) with  $n$  terms by projection free methods. Convergence is measured by the functional distance to the solution in the convex case and by the gap function in the non-convex case. Complexities are given in terms of the number of stochastic gradient calls.

blue = results of our paper (in particular, Reddi et al. (2016) do not give a proof in the convex setting).

Method	Link	Reference	CVX	nCVX
Frank-Wolfe	Alg. 1	(Frank and Wolfe, 1956), (Lacoste-Julien, 2016)	$\mathcal{O}(\frac{n}{\varepsilon})$	$\mathcal{O}(\frac{n}{\varepsilon^2})$
SVRG FW	Alg. 3	Reddi et al. (2016)	$\mathbf{X}$	$\mathcal{O}(n + \frac{n^{2/3}}{\varepsilon^2})$
SVRG FW	Alg. 1	Hazan and Luo (2016)	$\mathcal{O}(n + \frac{1}{\varepsilon^2})$	$\mathbf{X}$
SPIDER FW	Alg. 2	(Yurtsever et al., 2019)	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(n + \frac{\sqrt{n}}{\varepsilon^2})$
SFW	Alg. 1	Négiar et al. (2020)	$\mathcal{O}(\frac{n}{\varepsilon})$	$\mathbf{X}$
GSFW	Alg. 1	(Lu and Freund, 2021)	$\mathcal{O}(\frac{n}{\varepsilon})$	$\mathbf{X}$
SVRG FW	Alg. 3	Weber and Sra (2022)	$\mathbf{X}$	$\mathcal{O}(n + \frac{n^{2/3}}{\varepsilon^2})$
SAGA FW	Alg. 3	(Reddi et al., 2016)	$\tilde{\mathcal{O}}(n + \frac{n^{2/3}}{\varepsilon})$ <sup>(1)</sup>	$\mathcal{O}(n + \frac{n^{2/3}}{\varepsilon^2})$ <sup>(1)</sup>
SARAH FW	Alg. 2	(Beznosikov et al., 2023)	$\tilde{\mathcal{O}}(n + \frac{\sqrt{n}}{\varepsilon})$ <sup>(1)</sup>	$\mathcal{O}(n + \frac{\sqrt{n}}{\varepsilon^2})$ <sup>(1)</sup>
SAGA SARAH FW	Alg. 10	(Beznosikov et al., 2023)	$\tilde{\mathcal{O}}(n + \frac{\sqrt{n}}{\varepsilon})$ <sup>(1)</sup>	$\mathcal{O}(n + \frac{\sqrt{n}}{\varepsilon^2})$ <sup>(1)</sup>
L-SVRG FW	Alg. 1	<b>NEW</b> , (Kovalev et al., 2020)	$\tilde{\mathcal{O}}(n + \frac{n^{2/3}}{\varepsilon})$ <sup>(1)</sup>	$\mathcal{O}(n + \frac{n^{2/3}}{\varepsilon^2})$ <sup>(1)</sup>

<sup>(1)</sup> In the main part of the paper, we give these results with depending on the parameters  $p$  and  $b$ . The optimal choice of them are given in the original paper (Beznosikov et al., 2023) (SARAH FW and SAGA SARAH FW) or in the corresponding subsections of Section B (SAGA FW and L-SVRG FW).

It is important to note that in Table 1 we only covers works on the finite-sum stochastic optimization. However, there are many papers where the authors also consider a stochastic version of the FW algorithm, but under the (additional) assumption of bounded variance of the stochastic gradients (Hazan and Kale, 2012; Lan and Zhou, 2016; Reddi et al., 2016; Qu et al., 2018; Yurtsever et al., 2019; Mokhtari et al., 2020; Zhang et al., 2020).



## A.2 Coordinate methods

Table 2: Summary of complexity results for finding an  $\varepsilon$ -solution **constrained minimization problems** with dimension  $d$  by **coordinate** projection free methods. Convergence is measured by the functional distance to the solution in the convex case and by the gap function in the non-convex case. Complexities are given in terms of the number of coordinates derivatives computed.

blue = results of our paper.

Method	Link	Reference	CVX	nCVX
BCFW	Alg. 3	(Lacoste-Julien et al., 2013)	$\mathcal{O}(d + \frac{d}{\varepsilon})$	$\times$
SGFFW	Alg. 1	(Sahu et al., 2019) <sup>(1)</sup>	$\mathcal{O}(d + \frac{d}{\varepsilon})$	$\times$
SGFFW	Alg. 2	(Sahu et al., 2019) <sup>(1)</sup>	$\mathcal{O}(d + \frac{d}{\varepsilon^3})$	$\mathcal{O}(d + \frac{d^{4/3}}{\varepsilon^4})$
SEGA FW	Alg. 4	<b>NEW</b> , (Hanzely et al., 2018)	$\tilde{\mathcal{O}}(d + \frac{d\sqrt{d}}{\varepsilon})$	$\mathcal{O}(d + \frac{d^3}{\varepsilon^2})$
JAGUAR	Alg. 5	<b>NEW</b>	$\tilde{\mathcal{O}}(d + \frac{d}{\varepsilon})$	$\mathcal{O}(d + \frac{d^2}{\varepsilon^2})$

<sup>(1)</sup> Zero-order methods.

## A.3 Distributed methods

Table 3: Summary of complexity results for finding an  $\varepsilon$ -solution **distributed constrained minimization problems** (5) with  $n$  devices by projection free methods **with compression** with parameters  $\omega$  and  $\delta$  (see Definitions 3.12 and 3.17). Convergence is measured by the functional distance to the solution in the convex case and by the gap function in the non-convex case. Complexities are given in terms of the number of transmitted coordinates if we choose RandK and TopK as particular cases of compressors.

blue = results of our paper.

Method	Link	Reference	Any compression?	CVX	nCVX
dFW	Alg. 3	(Bellet et al., 2015)	$\times$	$\mathcal{O}(\frac{1}{\varepsilon})$	$\times$
DIANA FW	Alg. 7	<b>NEW</b> , (Mishchenko et al., 2019)	✓ (unbiased)	$\tilde{\mathcal{O}}(1 + \frac{1}{\varepsilon}(\frac{1}{\omega} + \frac{\sqrt{\omega}}{\sqrt{n}}))$	$\mathcal{O}(\frac{1}{\varepsilon^2}(\frac{1}{\omega} + \frac{\omega^2}{n}))$
MARINA FW	Alg. 8	<b>NEW</b> , (Gorbunov et al., 2021)	✓ (unbiased)	$\tilde{\mathcal{O}}(1 + \frac{1}{\varepsilon}(\frac{1}{\omega} + \frac{1}{\sqrt{n}}))$	$\mathcal{O}(\frac{1}{\varepsilon^2}(\frac{1}{\omega} + \frac{\varepsilon}{n}))$
VR MARINA FW	Alg. 12	<b>NEW</b> , (Gorbunov et al., 2021)	✓ (unbiased)	$\tilde{\mathcal{O}}(1 + \frac{1}{\varepsilon}(\frac{1}{\omega} + \frac{1}{\sqrt{n}}))$	$\mathcal{O}(\frac{1}{\varepsilon^2}(\frac{1}{\omega} + \frac{\varepsilon}{n}))$
EF21 FW	Alg. 9	<b>NEW</b> , (Richtarik et al., 2021)	✓	$\tilde{\mathcal{O}}(1 + \frac{1}{\varepsilon})$	$\mathcal{O}(\frac{\delta}{\varepsilon^2})$
Q-L-SVRG FW	Alg. 11	<b>NEW</b>	✓ (unbiased)	$\tilde{\mathcal{O}}(\sqrt[4]{\frac{n}{\omega^3}} + \frac{1}{\varepsilon}(\frac{1}{\omega} + \frac{1}{\sqrt[4]{n\omega}}))$	$\mathcal{O}(\frac{1}{\varepsilon^2}(\frac{1}{\omega} + \sqrt{\frac{\omega}{n}}))$

Here one can also highlight two papers on distributed Frank-Wolfe algorithms, but without compression (Wang et al., 2016; Hou et al., 2022).

## B MISSING METHODS, DETAILS AND PROOFS

In this section, we first provide complete proofs of our two main Theorems 2.2 and 2.3. Then we discuss the zoo of special cases, in particular we provide the full listing of algorithms, detailed convergence rates and proofs for methods from Section 3. Moreover, we present some statements and algorithms for them that are not encountered in the main part.

### B.1 Technical facts

In our proofs, we often apply following inequalities that hold for any  $a, b \in \mathbb{R}^d$  and  $\alpha > 0$ :

$$\|a + b\|^2 \leq (1 + \alpha)\|a\|^2 + \left(1 + \frac{1}{\alpha}\right)\|b\|^2, \quad (17)$$

$$2\langle a, b \rangle \leq \frac{1}{\alpha}\|a\|^2 + \alpha\|b\|^2. \quad (18)$$

**Lemma B.1** (Lemma 1.2.3 from (Nesterov, 2014)). *Let the Assumption 1.1 be satisfied. Then for all  $x, y \in \mathbb{R}^d$ :*

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2}\|x - y\|^2. \quad (19)$$

**Lemma B.2** (Lemma A.1 from (Lei et al., 2017)). *Let  $x_1, \dots, x_N \in \mathbb{R}^d$  be arbitrary vectors with*

$$\sum_{i=1}^N x_i = 0.$$

*Further let  $S$  be a uniform subset of  $[N]$  with size  $b$ . Then*

$$\mathbb{E} \left\| \frac{1}{b} \sum_{i \in S} x_i \right\|^2 \leq \frac{1}{bN} \sum_{i=1}^N \|x_i\|^2. \quad (20)$$

Our further analysis relies heavily on work conducted in (Stich, 2019), but it seems to us that there is a typo in the main part. Furthermore, as we will deal with nonreducible "noises", we will need to adjust proofs by a little. For convenience of the proof, we split the result into three following lemmas:

**Lemma B.3** *Consider two non-negative sequences  $\{r_k\}, \{\eta_k\}$ , that satisfy*

$$r_{k+1} \leq (1 - \eta_k)r_k + a\eta_k^2 + b, \quad (21)$$

*for all  $k \geq 0$ , constants  $a \geq 0, b \geq 0$  and for positive stepsizes  $\{\eta_k\}$  with  $\eta_k \leq \frac{1}{d}, d > 1$ . Then there exists  $\eta$ , such that  $\forall k \geq 0, \eta_k \equiv \eta$*

$$r_K < r_0 \exp\left[-\frac{K}{d}\right] + \frac{a}{d} + bK. \quad (22)$$

**Proof:**

Set  $\eta_k \equiv \eta, \forall k \geq 0$ . Unroll inequalities:

$$\begin{aligned} r_K &\leq (1 - \eta)r_{K-1} + a\eta^2 + b \leq (1 - \eta)^K r_0 + a\eta^2 \sum_{i=0}^{K-1} (1 - \eta)^i + bK \\ &\leq (1 - \eta)^K r_0 + a\eta^2 \sum_{k=0}^{\infty} (1 - \eta)^k + bK \leq (1 - \eta)^K r_0 + a\eta + bK. \end{aligned}$$

Put  $\eta = \frac{1}{d}$  and use, that for  $\forall x > 0, 0 < a < 1 \rightarrow (1 - a)^x < \exp(-ax)$ . Hence,

$$r_K < r_0 \exp\left[-\frac{K}{d}\right] + \frac{a}{d} + bK. \quad (23)$$

□

**Lemma B.4** *Let the non-negative sequence  $\{r_k\}$  satisfy the conditions of Lemma B.3. Then there are decreasing stepsizes  $\eta_k = \frac{2}{2d+k}$ , such that*

$$r_{K+1} \leq \frac{4d^2 r_0}{(2d+K)^2} + 4a \frac{K+1}{(2d+K)^2} + \frac{b(K+1)(24d^2 + 12dK + 2K^2 + K)}{6(2d+K)^2}. \quad (24)$$

**Proof:**

Rearrange (21):

$$0 \leq (1 - \eta_k)r_k - r_{k+1} + a\eta_k^2 + b.$$

Divide both parts by  $\eta_k$ :

$$0 \leq \frac{1 - \eta_k}{\eta_k} r_k - \frac{r_{k+1}}{\eta_k} + a\eta_k + \frac{b}{\eta_k} = (2d+k-2) \frac{r_k}{2} - (2d+k) \frac{r_{k+1}}{2} + a \frac{2}{2d+k} + b \frac{2d+k}{2}.$$

Multiply both parts by  $2 \cdot (2d+k)$  and use, that  $x \cdot (x-2) \leq (x-1)^2$ :

$$\begin{aligned} 0 &\leq (2d+k-2)(2d+k)r_k - (2d+k)^2 r_{k+1} + 4a + b(2d+k)^2 \\ &\leq (2d+k-1)^2 r_k - (2d+k)^2 r_{k+1} + 4a + b(2d+k)^2. \end{aligned}$$

We obtain a telescoping sum, hence:

$$\begin{aligned} 0 &\leq (2d-1)^2 r_0 - (2d+k)^2 r_{K+1} + 4a(K+1) + b \sum_{i=0}^K (2d+i)^2 \\ &\leq 4d^2 r_0 - (2d+k)^2 r_{K+1} + 4a(K+1) + \frac{b(K+1)(24d^2 + 12dK + 2K^2 + K)}{6}. \end{aligned}$$

Finally, we can gain the original inequality by rearranging the terms.  $\square$

**Lemma B.5** *Let  $\{r_k\}$  satisfy (21). Then there exist stepsizes  $\eta_k$ ,*

$$\begin{aligned} \text{if } K &\leq d, & \eta_k &= \frac{1}{d}, \\ \text{if } K &> d \text{ and } k < k_0, & \eta_k &= \frac{1}{d}, \\ \text{if } K &> d \text{ and } k \geq k_0, & \eta_k &= \frac{2}{2d+k-k_0}, \end{aligned}$$

where  $k_0 = \lfloor \frac{K}{2} \rfloor$ , such that

$$r_{K+1} = \mathcal{O} \left( r_0 \exp \left( -\frac{K}{2d} \right) + \frac{a}{d+K} + bK \right). \quad (25)$$

**Proof:**

If  $K \leq d$  we'll take  $\eta_k \equiv \frac{1}{d}$ , hence using Lemma B.3 we obtain

$$r_K \leq r_0 \exp \left[ -\frac{K}{d} \right] + \frac{a}{d} + bK \leq r_0 \exp \left[ -\frac{K}{d} \right] + \frac{a}{K} + bK.$$

As  $\frac{a}{K} / \frac{a}{d+K} = \frac{d+K}{K} \leq \frac{2d}{K} \leq 2d$  and  $\exp(-\frac{K}{d}) / \exp(-\frac{K}{2d}) = \exp(-\frac{K}{2d}) < 1$ , thus (25) is correct for  $K \leq d$ .

If  $K > d$  for  $k \leq k_0 := \lfloor \frac{K}{2} \rfloor$  we'll take  $\eta_k \equiv \frac{1}{d}$  and for  $k > k_0$ :  $\eta_k = \frac{2}{2d+k-k_0}$ . Therefore, according to (22) we obtain

$$r_{k_0} \leq r_0 \exp \left( -\frac{k_0}{d} \right) + \frac{a}{d} + bk_0.$$

Using Lemma B.4 with  $k_1 = K - k_0$  we get

$$r_{K+1} \leq \frac{4d^2 r_{k_0}}{(2d+k_1)^2} + 4a \frac{K+1}{(2d+k_1)^2} + \frac{b(k_1+1)(24d^2 + 12dk_1 + 2k_1^2 + k_1)}{6(2d+k_1)^2}.$$

Combining inequalities we derive

$$\frac{4d^2 r_{k_0}}{(2d+k_1)^2} \leq \frac{4d^2}{(2d+k_1)^2} \left( r_0 \exp \left( -\frac{k_0}{d} \right) + \frac{a}{d} + bk_0 \right) \leq r_0 \exp \left( -\frac{k_0}{d} \right) + \frac{4ad}{(2d+k_1)^2} + \frac{4bk_0 d^2}{(2d+k_1)^2}.$$

Since  $\frac{K}{2} \leq \lceil \frac{K}{2} \rceil < K$  and  $\frac{K}{4} < \lfloor \frac{K}{2} \rfloor < K$  we can replace all  $k_0, k_1$  with  $K$  in fractions in O-notation. We gain the replacement in the exponent, as  $-\lceil \frac{K}{2} \rceil \leq -\frac{K}{2}$ . Thus,

$$\begin{aligned} r_{K+1} &= \mathcal{O} \left( r_0 \exp \left( -\frac{K}{2d} \right) + \frac{ad}{(d+K)^2} + \frac{d^2bK}{(d+K)^2} + \frac{aK}{(d+K)^2} + bK \frac{d^2 + K^2}{(d+K)^2} \right) \\ &= \mathcal{O} \left( r_0 \exp \left( -\frac{K}{2d} \right) + \frac{a}{K+d} + bK \frac{d^2 + K^2}{(d+K)^2} \right), \end{aligned}$$

where second equality is implied by  $d, K > 0$ ,  $d^2 + K^2 \leq (d+K)^2$ . □

## B.2 Unified main theorems

In this section, we provide complete proofs of our main results.

**Lemma B.6** *If  $x^k$  is upgraded due to Algorithm (2), then for all  $\alpha > 0$ :*

$$\mathbb{E}[f(x^{k+1}) - f(x^*)] \leq (1 - \eta_k)\mathbb{E}[f(x^k) - f(x^*)] + \frac{\alpha}{L}\mathbb{E}[\|\nabla f(x^k) - g^k\|^2] + \frac{L\eta_k^2}{\alpha}\mathbb{E}[\|s^k - x^*\|^2] + \frac{L\eta_k^2}{2}\mathbb{E}[\|s^k - x^k\|^2].$$

**Proof:**

First, we assume that  $f(x)$  satisfies Assumptions 1.1 and 1.2:

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2.$$

With update of  $x^{k+1}$ , according to Algorithm (2):

$$\begin{aligned} f(x^{k+1}) - f(x^*) &\leq f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), s^k - x^k \rangle + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2 \\ &= f(x^k) - f(x^*) + \eta_k \langle g^k, s^k - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2. \end{aligned}$$

The optimal choice of  $s^k$  in Algorithm (2) gives  $\langle g^k, s^k - x^k \rangle \leq \langle g^k, x^* - x^k \rangle$ . Then

$$\begin{aligned} f(x^{k+1}) - f(x^*) &\leq f(x^k) - f(x^*) + \eta_k \langle g^k, x^* - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2 \\ &= f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x^* - x^k \rangle + \eta_k \langle g^k - \nabla f(x^k), x^* - x^k \rangle \\ &\quad + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2 \\ &= f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x^* - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2. \end{aligned}$$

Using Young's inequality here we state for any positive  $\alpha$ :

$$\begin{aligned} f(x^{k+1}) - f(x^*) &\leq f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x^* - x^k \rangle + \frac{\alpha}{L}\|\nabla f(x^k) - g^k\|^2 \\ &\quad + \frac{L\eta_k^2}{\alpha}\|s^k - x^*\|^2 + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2 \end{aligned}$$

Since  $f$  satisfies Assumption 1.2 we have  $\langle \nabla f(x^k), x^* - x^k \rangle \leq -(f(x^k) - f(x^*))$ . Thus,

$$\begin{aligned} f(x^{k+1}) - f(x^*) &\leq f(x^k) - f(x^*) - \eta_k(f(x^k) - f(x^*)) + \frac{\alpha}{L}\|\nabla f(x^k) - g^k\|^2 \\ &\quad + \frac{L\eta_k^2}{\alpha}\|s^k - x^*\|^2 + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2 \\ &= (1 - \eta_k)(f(x^k) - f(x^*)) + \frac{\alpha}{L}\|\nabla f(x^k) - g^k\|^2 \\ &\quad + \frac{L\eta_k^2}{\alpha}\|s^k - x^*\|^2 + \frac{L\eta_k^2}{2}\|s^k - x^k\|^2. \end{aligned}$$

Taking the full mathematical expectation finishes the proof.  $\square$

Now we are ready to proof our main result. We start with the convex case, where  $f$  satisfies 1.2. For readers convenience, we restate the theorems below.

**Theorem B.7** (Theorem 2.2). *Let the Assumptions 1.1, 1.2 and 2.1 be satisfied. Then there exist  $\eta_k \leq \min(\rho_1, \rho_2)$  for Algorithm (2) and constants  $M_1, M_2, \alpha$  such that:*

$$r_{K+1} = \mathcal{O}\left(r_0 \exp\left(-\frac{K \min(\rho_1, \rho_2)}{4}\right) + \frac{LD^2}{K + \frac{1}{\min(\rho_1, \rho_2)}} + D^2 \sqrt{\frac{B\rho_2 + AE}{\rho_1\rho_2 \left(K + \frac{1}{\min(\rho_1, \rho_2)}\right)^2} + \frac{K}{K + \frac{1}{\min(\rho_1, \rho_2)}} \frac{C\rho_2}{\rho_1\rho_2 D^2}}\right),$$

where  $r_k = \mathbb{E}_k[f(x^k) - f(x_*) + M_1\|g^k - \nabla f(x^k)\|^2 + M_2\sigma_k^2]$ .

**Proof:**

Since Assumption 2.1 holds, we can derive the following inequality:

$$r_{k+1} = \mathbb{E}[f(x^{k+1}) - f(x^*) + M_1\|\nabla f(x^{k+1}) - g^{k+1}\|^2 + M_2\sigma_{k+1}^2] \stackrel{(B.6)}{\leq} (1 - \eta_k)\mathbb{E}[f(x^k) - f(x^*)]$$

$$\begin{aligned}
 & + \frac{\alpha}{L} \mathbb{E}[\|\nabla f(x^k) - g^k\|^2] + \left(\frac{L}{\alpha} + \frac{L}{2}\right) D^2 \eta_k^2 + M_1(1 - \rho_1) \|g^k - \nabla f(x^k)\|^2 \\
 & + M_1 A \sigma_k^2 + M_1 \eta_k^2 B D^2 + M_1 C + M_2(1 - \rho_2) \sigma_k^2 + M_2 \eta_k^2 E D^2 \\
 & = (1 - \eta_k) \mathbb{E}[f(x^k) - f(x^*)] + \left(\frac{\alpha}{L} + M_1(1 - \rho_1)\right) \mathbb{E}[\|\nabla f(x^k) - g^k\|^2] + (M_1 A + M_2(1 - \rho_2)) \sigma_k^2 \\
 & + \left(\frac{L}{\alpha} + \frac{L}{2} + M_1 B + M_2 E\right) \eta_k^2 D^2 + M_1 C.
 \end{aligned}$$

With constants  $M_1 = \frac{2\alpha}{\rho_1 L}$ ,  $M_2 = \frac{2M_1 A}{\rho_2}$  we get

$$\begin{aligned}
 & \mathbb{E}[f(x^{k+1}) - f(x^*)] + M_1 \mathbb{E}[\|\nabla f(x^{k+1}) - g^{k+1}\|^2] + M_2 \sigma_{k+1}^2 \leq (1 - \eta_k) \mathbb{E}[f(x^k) - f(x^*)] \\
 & + \left(1 - \frac{\rho_1}{2}\right) M_1 \mathbb{E}[\|\nabla f(x^k) - g^k\|^2] + \left(1 - \frac{\rho_2}{2}\right) M_2 \sigma_k^2 + \left(\frac{L}{2} + \frac{L}{\alpha} + \frac{2B}{\rho_1 L} \alpha + \frac{4AE}{\rho_1 \rho_2 L} \alpha\right) \eta_k^2 D^2 + \frac{2C}{\rho_1 L} \alpha.
 \end{aligned}$$

Using Lemma B.5 with  $d = \frac{2}{\min(\rho_1, \rho_2)}$  and  $r_k = \mathbb{E}_k[f(x^k) - f(x_*) + M_1 \|\nabla f(x^k) - g^k\|^2 + M_2 \sigma_k^2]$  we obtain

$$r_{K+1} = \mathcal{O} \left( r_0 \exp\left(-\frac{K}{2d}\right) + \frac{LD^2}{2(K+d)} + \frac{\frac{L}{\alpha} + \alpha \frac{2B\rho_2 + 4AE}{\rho_1 \rho_2 L}}{K+d} D^2 + K \alpha \frac{2C\rho_2}{\rho_1 \rho_2 L} \right).$$

This estimation holds for any  $\alpha > 0$ , and thus to obtain optimal estimation on the  $K$ -th iteration we shall minimize this to  $\alpha$ . It is easy to see, that minimum of  $x\alpha + \frac{y}{\alpha}$  is located at  $\alpha = \sqrt{\frac{y}{x}}$  and equals to  $2\sqrt{xy}$ . Therefore taking optimal  $\alpha$  as

$$\alpha = \sqrt{\frac{\frac{LD^2}{K+d}}{\frac{2B\rho_2 + 4AE}{\rho_1 \rho_2 L(K+d)} D^2 + K \frac{2C\rho_2}{\rho_1 \rho_2 L}}},$$

we get

$$r_{K+1} = \mathcal{O} \left( r_0 \exp\left(-\frac{K}{2d}\right) + \frac{LD^2}{K+d} + D^2 \sqrt{\frac{2B\rho_2 + 4AE}{\rho_1 \rho_2 (K+d)^2} + \frac{K}{K+d} \frac{2C\rho_2}{\rho_1 \rho_2 D^2}} \right).$$

It completes the proof.  $\square$

**Theorem B.8** (Theorem 2.3). *Let the Assumptions 1.1 and 2.1 be satisfied. Then, there exist  $\eta_k$  for Algorithm (2), that*

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{r_0}{\sqrt{K}} + \frac{D^2}{\sqrt{K}} \left[ L + \sqrt{\frac{B\rho_2 + AE}{\rho_1 \rho_2} + K \frac{C\rho_2}{D^2 \rho_1 \rho_2}} \right] \right). \quad (26)$$

**Proof:**

With the Assumption 1.1 we can use (19):

$$f(x^{k+1}) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2.$$

With update of  $x^{k+1}$ , according to 2:

$$\begin{aligned}
 f(x^{k+1}) - f(x^*) & \leq f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), s^k - x^k \rangle + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2 \\
 & = f(x^k) - f(x^*) + \eta_k \langle g^k, s^k - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2.
 \end{aligned}$$

The optimal choice of  $s^k$  in 2 gives  $\langle g^k, s^k - x^k \rangle \leq \langle g^k, x - x^k \rangle$  for all  $x \in \mathcal{X}$ . Then,

$$\begin{aligned}
 f(x^{k+1}) - f(x^*) & \leq f(x^k) - f(x^*) + \eta_k \langle g^k, x - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2 \\
 & = f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x - x^k \rangle + \eta_k \langle g^k - \nabla f(x^k), x - x^k \rangle \\
 & + \eta_k \langle \nabla f(x^k) - g^k, s^k - x^k \rangle + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2 \\
 & = f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x - x^k \rangle + \eta_k \langle \nabla f(x^k) - g^k, s^k - x \rangle + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2.
 \end{aligned}$$

Using Young's inequality (18) here we state for any positive  $\alpha$ :

$$\begin{aligned} f(x^{k+1}) - f(x^*) &\leq f(x^k) - f(x^*) + \eta_k \langle \nabla f(x^k), x - x^k \rangle + \frac{\alpha}{L} \|\nabla f(x^k) - g^k\|^2 \\ &\quad + \frac{L\eta_k^2}{\alpha} \|s^k - x\|^2 + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2. \end{aligned}$$

After small rearrangements one can get:

$$\begin{aligned} \eta_k \langle \nabla f(x^k), x^k - x \rangle &\leq f(x^k) - f(x^*) - (f(x^{k+1}) - f(x^*)) + \frac{\alpha}{L} \|\nabla f(x^k) - g^k\|^2 \\ &\quad + \frac{L\eta_k^2}{\alpha} \|s^k - x\|^2 + \frac{L\eta_k^2}{2} \|s^k - x^k\|^2. \end{aligned}$$

Maximizing over  $\mathcal{X}$ , taking the full mathematical expectation and bounding the distances by diameter, we get:

$$\begin{aligned} \eta_k \mathbb{E} \left[ \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq \mathbb{E} \left[ f(x^k) - f(x^*) \right] - \mathbb{E} \left[ f(x^{k+1}) - f(x^*) \right] + \frac{\alpha}{L} \mathbb{E} \left[ \|\nabla f(x^k) - g^k\|^2 \right] \\ &\quad + \eta_k^2 \frac{LD^2}{\alpha} + \eta_k^2 \frac{LD^2}{2}. \end{aligned}$$

After multiplying 2.1 by the positive constants  $M_1, M_2$  (which we will define below) and summarizing with previous inequality we have:

$$\begin{aligned} \eta_k \mathbb{E} \left[ \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq \mathbb{E} \left[ f(x^k) - f(x^*) + \left( 1 - \rho_1 + \frac{\alpha}{M_1 L} \right) M_1 \|\nabla f(x^k) - g^k\|^2 \right. \\ &\quad \left. + \left( 1 - \rho_2 + \frac{M_1 A}{M_2} \right) M_2 \sigma_k^2 \right] \\ &\quad - \mathbb{E} \left[ f(x^{k+1}) - f(x^*) + M_1 \|\nabla f(x^{k+1}) - g^k\|^2 + M_2 \sigma_{k+1}^2 \right] \\ &\quad + D^2 \eta_k^2 \left( \frac{L}{2} + \frac{L}{\alpha} + M_1 B + M_2 E \right) + M_1 C. \end{aligned}$$

With  $M_1 = \frac{\alpha}{L\rho_1}, M_2 = \frac{M_1 A}{\rho_2}$  and summarizing over all  $k$  from 0 to  $K-1$  we have:

$$\begin{aligned} \sum_{k=0}^{K-1} \eta_k \mathbb{E} \left[ \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq f(x^0) - f(x^*) + \|\nabla f(x^0) - g^0\|^2 + \|\sigma_0\|^2 \\ &\quad + D^2 \left( \frac{L}{2} + \frac{L}{\alpha} + \alpha \left( \frac{B}{\rho_1 L} + \frac{AE}{\rho_1 \rho_2 L} \right) \right) \sum_{k=0}^{K-1} \eta_k^2 + KM_1 C. \end{aligned}$$

Assuming  $\min_{k \leq K} \eta_k = \eta_{min}$ , we gain:

$$\begin{aligned} \sum_{k=0}^{K-1} \eta_k \mathbb{E} \left[ \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq f(x^0) - f(x^*) + \|\nabla f(x^0) - g^0\|^2 + \|\sigma_0\|^2 \\ &\quad + D^2 \left( \frac{L}{2} + \frac{L}{\alpha} + \alpha \left( \frac{B}{\rho_1 L} + \frac{AE}{\rho_1 \rho_2 L} + \frac{C}{\rho_1 L \eta_{min}^2 D^2} \right) \right) \sum_{k=0}^{K-1} \eta_k^2. \end{aligned}$$

With  $\alpha = \sqrt{\frac{LD^2}{\frac{1}{\rho_1 L} (BD^2 + \frac{C}{\eta_{min}^2}) + \frac{A}{\rho_1 \rho_2 L} ED^2}}$  one can obtain:

$$\begin{aligned} \sum_{k=0}^{K-1} \eta_k \mathbb{E} \left[ \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq f(x^0) - f(x^*) + \|\nabla f(x^0) - g^0\|^2 \\ &\quad + D^2 \left( \frac{L}{2} + 2 \sqrt{\frac{B\rho_2 + AE}{\rho_1 \rho_2} + \frac{1}{\eta_{min}^2} \frac{C\rho_2}{D^2 \rho_1 \rho_2}} \right) \sum_{k=0}^{K-1} \eta_k^2. \end{aligned}$$

If we take  $\eta_k = \frac{1}{\sqrt{K}}$ , then  $\eta_{min} = \frac{1}{\sqrt{K}}$ . Divide both sides by  $\sqrt{K}$ , then:

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{K} \max_{x \in \mathcal{X}} \langle \nabla f(x^k), x^k - x \rangle \right] &\leq \frac{f(x^0) - f(x^*) + \|\nabla f(x^0) - g^0\|^2 + \|\sigma_0\|^2}{\sqrt{K}} + \\ &\quad + \frac{D^2}{\sqrt{K}} \left( \frac{L}{2} + 2 \sqrt{\frac{B\rho_2 + AE}{\rho_1 \rho_2} + K \frac{C\rho_2}{D^2 \rho_1 \rho_2}} \right). \end{aligned}$$

Finally, we obtain the needed estimation.  $\square$

### B.3 Stochastic methods

In this section, we provide the detailed convergence rates and proofs for some specific methods (see Section 3.1) in the finite-sum case (5) of the constrained optimization problem (1).

In the following, we prove that some specific methods, i.e., L-SVRG, SARAH and SAGA satisfy our unified Assumption 2.1 and thus can be captured by our unified analysis. Then, we plug their corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rates for these methods.

#### B.3.1 L-SVRG Frank-Wolfe

We first restate our Lemma 3.2 for L-SVRG FW method (Algorithm 1) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 1 L-SVRG Frank-Wolfe

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**Input:** initial  $x^0, w^0 = x^0, g^0 = \nabla f(x^0)$  step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$ , probability  $p \in (0, 1]$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Update  $w^{k+1} = \begin{cases} x^k, & \text{with probability } p \\ w^k, & \text{with probability } 1 - p \end{cases}$

    Generate batch  $S_k$  with size  $b$

$g^{k+1} = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - \nabla f_i(w^{k+1})] + \nabla f(w^{k+1})$

**end for**

---

**Lemma B.9** (Lemma 3.2). *Under Assumption 3.1 Algorithm 1 satisfies Assumption 2.1 with*

$$\rho_1 = 1, \quad A = \frac{\tilde{L}^2}{b} \left(1 - \frac{p}{2}\right), \quad B = \frac{8\tilde{L}^2}{pb}, \quad C = 0,$$

$$\sigma_k^2 = \|x^k - w^k\|^2, \quad \rho_2 = \frac{p}{2}, \quad E = \frac{8}{p}.$$

**Proof:**

According to Lemma 3 from (Li and Richtárik, 2020) we get an estimation:

$$\mathbb{E}_k[\|g^k\|^2] \leq \frac{\tilde{L}^2}{b} \|x^k - w^k\|^2 + \|\nabla f(x^k)\|^2.$$

Since  $g^k$  is unbiased gradient estimator previous inequality turns to:

$$\mathbb{E}_k[\|\nabla f(x^k) - g^k\|^2] \leq \frac{\tilde{L}^2}{b} \|x^k - w^k\|^2.$$

Considering Algorithm 1 we have:

$$\begin{aligned} \mathbb{E}_k[\|x^k - w^k\|^2] &= p\mathbb{E}_k[\|x^k - x^{k-1}\|^2] + (1-p)\mathbb{E}_k[\|x^k - w^{k-1}\|^2] \\ &= p\eta_{k-1}^2\mathbb{E}_k[\|s^{k-1} - x^{k-1}\|^2] + (1-p)\mathbb{E}_k[\|x^{k-1} + \eta_k(s^{k-1} - x^{k-1}) - w^{k-1}\|^2] \\ &= \eta_{k-1}^2\mathbb{E}_k[\|s^{k-1} - x^{k-1}\|^2] + (1-p)\|x^{k-1} - w^{k-1}\|^2 \\ &\quad + 2\eta_k(1-p)\mathbb{E}_k[\langle x^{k-1} - w^{k-1}, s^{k-1} - x^{k-1} \rangle] \end{aligned} \tag{27}$$

$$\begin{aligned} &= \eta_{k-1}^2\mathbb{E}_k[\|s^{k-1} - x^{k-1}\|^2] + (1-p)\|x^{k-1} - w^{k-1}\|^2 \\ &\quad + 2(1-p)\mathbb{E}_k[\langle x^{k-1} - w^{k-1}, \eta_k(s^{k-1} - x^{k-1}) \rangle]. \end{aligned} \tag{28}$$

According to Young's inequality for any positive  $\beta$  there are:

$$\langle x^{k-1} - w^{k-1}, (s^{k-1} - x^{k-1})\eta_k \rangle \leq \beta\|x^{k-1} - w^{k-1}\|^2 + \frac{1}{\beta}\eta_{k-1}^2\|s^{k-1} - x^{k-1}\|^2.$$



Hence,

$$\begin{aligned} \mathbb{E}_k[\|x^k - w^k\|^2] &\leq \eta_{k-1}^2 D^2 + (1-p)\|x^{k-1} - w^{k-1}\|^2 + 2(1-p)\beta\|x^{k-1} - w^{k-1}\|^2 \\ &\quad + 2\frac{1-p}{\beta}\|\eta_{k-1}(s^{k-1} - x^{k-1})\|^2 \end{aligned} \quad (29)$$

$$\leq \left(1 + \frac{2(1-p)}{\beta}\right)\eta_{k-1}^2 D^2 + (1-p)(1+2\beta)\|x^{k-1} - w^{k-1}\|^2. \quad (30)$$

Finally, choose  $\beta = \frac{p}{4}$ . It leads to:

$$(1-p)(1+2\beta) \leq \left(1 - \frac{p}{2}\right).$$

Then,

$$\mathbb{E}_k[\|x^k - w^k\|^2] \leq \frac{8}{p}\eta_{k-1}^2 D^2 + \left(1 - \frac{p}{2}\right)\|x^{k-1} - w^{k-1}\|^2,$$

and

$$\mathbb{E}_k[\|\nabla f(x^k) - g^k\|^2] \leq \frac{8\tilde{L}^2}{pb}\eta_{k-1}^2 D^2 + \frac{\tilde{L}^2}{b}\left(1 - \frac{p}{2}\right)\|x^{k-1} - w^{k-1}\|^2.$$

This finishes the proof.  $\square$

**Corollary B.10** (Corollary 3.3). *Suppose that Assumption 1.1 holds. For Algorithm 1 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[r_{K+1}] = \mathcal{O}\left(\left(f(x_0) - f(x_*)\right)\exp\left(-\frac{Kp}{8}\right) + \frac{LD^2}{K + \frac{1}{p}}\left[1 + \frac{\tilde{L}}{L}\frac{1}{p\sqrt{b}}\right]\right).$$

$$\mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] = \mathcal{O}\left(\frac{f(x_0) - f(x_*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}}\left[1 + \frac{\tilde{L}}{L}\frac{1}{p\sqrt{b}}\right]\right).$$

**Proof:**

It suffices to plug parameters from Lemma B.9 into Theorems 2.2 and 2.3  $\square$

We proved results for L-SVRG FW depending on the parameters  $p$  and  $b$  to be tuned. Let us find the optimal choices of  $p$  and  $b$  for L-SVRG FW. To find the optimal choice of  $p$ , one can note that on average we call the stochastic gradients  $(pn + 2b)$  times. In more details, at each iteration we compute a batch size of  $b$  in two points  $x^k$  and  $w^k$  and with probability  $p$  we call the full gradient in the new point  $w^k$ . From Corollary 3.3, we know the estimate on the number of iterations of L-SVRG FW, then we can get an estimate on the number of the stochastic gradient calls by multiplying this result by  $(pn + 2b)$ . Then the new estimate can be optimized first by  $p$  (in the convex case, we need to minimize  $(1 + \frac{1}{p\sqrt{b}})(pn + 2b)$ ) and obtain that the optimal  $p \sim b^{1/4}/n^{1/2}$ . Then with already optimized  $p$ , the estimate on the number of the stochastic gradient calls can additionally be optimized by  $b$  (actually we need to minimize  $b^{1/4}n^{1/2} + b + n/b^{1/2}$ ) and find the optimal  $b \sim n^{2/3}$ . The final result for L-SVRG FW is presented in Table 1.

### B.3.2 SARAH Frank-Wolfe

We first restate our Lemma 3.4 for SARAH FW method (Algorithm 2) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

**Algorithm 2** SARAH Frank-Wolfe
 

---

**Input:** initial  $x^0, g^0 = \nabla f(x^0)$  step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$ , probability  $p \in (0, 1]$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Generate batch  $S_k$  with size  $b$

    Update  $g^{k+1} = \begin{cases} \nabla f(x^{k+1}), & \text{with probability } p \\ g^{k+1} + \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - \nabla f_i(x^k)], & \text{with probability } 1 - p \end{cases}$

**end for**

---

**Lemma B.11** (Lemma 3.4). *Under Assumption 3.1 Algorithm 2 satisfies Assumption 2.1 with:*

$$\rho_1 = p, \quad A = 0, \quad B = \frac{1-p}{b} \tilde{L}^2, \quad C = 0,$$

$$\sigma_k = 0, \quad \rho_2 = 1, \quad E = 0.$$

**Proof:**

Using Lemma 3 from (Li et al., 2021) we can obtain:

$$\begin{aligned} \mathbb{E}_k [\|\nabla f(x^k) - g^k\|^2] &\leq (1-p) \|\nabla f(x^{k-1}) - g^{k-1}\|^2 + \frac{1-p}{b} \tilde{L}^2 \|x^k - x^{k-1}\|^2 \\ &\leq (1-p) \|\nabla f(x^{k-1}) - g^{k-1}\|^2 + \frac{1-p}{b} \tilde{L}^2 \eta_{k-1}^2 D^2. \end{aligned}$$

□

**Corollary B.12** (Corollary 3.5). *For Algorithm 2 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[r_{K+1}] = \mathcal{O} \left( (f(x^0) - f(x^*)) \exp\left(-\frac{Kp}{4}\right) + \frac{LD^2}{K + \frac{1}{p}} \left[1 + \frac{\tilde{L}}{L} \sqrt{\frac{1-p}{pb}}\right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \frac{\tilde{L}}{L} \sqrt{\frac{1-p}{pb}}\right] \right).$$

**Proof:**

It suffices to plug parameters from Lemma B.11 into Theorems 2.2 and 2.3. □

The choices of  $p$  and  $b$  for SARAH FW are presented in the original paper (Beznosikov et al., 2023). The final result for SARAH FW is presented in Table 1.

### B.3.3 SAGA Frank-Wolfe

We first restate our Lemma 3.6 for SAGA FW method (Algorithm 3) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

**Algorithm 3** SAGA Frank-Wolfe

---

**Input:** initial  $x^0, \forall i \in [n] y_i^0 = \nabla f_i(x^0), g^0 = \nabla f(x^0)$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Generate batch  $S_k$  with size  $b$

    Update  $y_i^{k+1} = \begin{cases} \nabla f_i(x^k), & \text{for } i \in S_k \\ y_i^k, & \text{for } i \notin S_k \end{cases}$

    Update  $g^{k+1} = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - y_i^{k+1}] + \frac{1}{n} \sum_{j=1}^n y_j^{k+1}$

**end for**

---

**Lemma B.13** (Lemma 3.6). *Under Assumption 3.1 Algorithm 3 satisfies Assumption 2.1 with:*

$$\rho_1 = 1, \quad A = \frac{1}{b} \left(1 + \frac{b}{2n}\right), \quad B = \frac{\tilde{L}^2}{b} \left(1 + \frac{2n}{b}\right), \quad C = 0,$$

$$\sigma_k^2 = \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^{k+1}\|^2, \quad \rho_2 = \frac{b}{2n}, \quad E = \frac{2n}{b} \tilde{L}^2.$$

**Proof:**

We bound the difference between estimator and exact gradient:

$$\begin{aligned} \mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k \left[ \left\| \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^k) - y_i^k] + \frac{1}{n} \sum_{j=1}^n y_j^k - \nabla f(x^k) \right\|^2 \right] \\ &= \mathbb{E}_k \left[ \left\| \frac{1}{b} \left( \sum_{i \in S_k} [\nabla f_i(x^k) - y_i^k] - \left( \frac{1}{n} \sum_{j=1}^n [\nabla f_j(x^k) - y_j^k] \right) \right) \right\|^2 \right] \\ &\stackrel{(B.2)}{\leq} \frac{1}{bn} \sum_{j=1}^n \left\| \nabla f_j(x^k) - y_j^k - \left( \frac{1}{n} \sum_{i=1}^n [\nabla f_i(x^k) - y_i^k] \right) \right\|^2 \\ &\leq \frac{1}{bn} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^k\|^2 \\ &\leq \frac{1}{bn} (1 + \alpha) \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(x^{k-1})\|^2 + \frac{1}{bn} \left(1 + \frac{1}{\alpha}\right) \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 \\ &\leq \frac{\tilde{L}^2}{b} (1 + \alpha) \eta_{k-1}^2 D^2 + \frac{1}{b} \left(1 + \frac{1}{\alpha}\right) \sigma_{k-1}^2 \end{aligned}$$

for  $\forall \alpha > 0$  (in particular, we can put  $\alpha = \frac{2n}{b}$  to obtain the needed estimates). The second inequality holds, since  $\frac{1}{n} \sum_{i=1}^n$  can be described, as an expected value. And  $\mathbb{E}\|x - \mathbb{E}x\|^2 \leq \mathbb{E}\|x\|^2$ . Then we need to bound the second term:

$$\begin{aligned} \mathbb{E}_k [\sigma_k^2] &= \mathbb{E}_k \left[ \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^{k+1}\|^2 \right] = \left(1 - \frac{b}{n}\right) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^k\|^2 \\ &= \left(1 - \frac{b}{n}\right) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(x^{k-1}) + \nabla f_j(x^{k-1}) - y_j^{k-1}\|^2 \\ &\leq \left(1 - \frac{b}{n}\right) (1 + \beta) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^{k-1}\|^2 + \left(1 - \frac{b}{n}\right) \left(1 + \frac{1}{\beta}\right) \tilde{L}^2 \|x^k - x^{k-1}\|^2. \end{aligned}$$

With  $\beta = \frac{b}{2n}$  we have:

$$\mathbb{E}_k[\sigma_k^2] \leq \left(1 - \frac{b}{2n}\right) \sigma_{k-1}^2 + \frac{2n}{b} \tilde{L}^2 \eta_{k-1}^2 D^2.$$

This finishes the proof  $\square$

**Corollary B.14** (Corollary 3.7). *For Algorithm 3 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[r_{K+1}] = \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{Kb}{8n} \right) + \frac{LD^2}{K + \frac{2n}{b}} \left[ 1 + \frac{\tilde{L}}{L} \frac{n}{b\sqrt{b}} \right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + \frac{\tilde{L}}{L} \frac{n}{b\sqrt{b}} \right] \right).$$

**Proof:**

It suffices to plug parameters from Lemma B.11 into Theorems 2.2 and 2.3.  $\square$

To find the optimal choice of  $b$ , one can note that we call the stochastic gradients  $2b$  times. In more details, at each iteration we compute a batch size of  $b$  in two points  $x^k$  and  $w^k$ . From Corollary 3.7, we know the estimate on the number of iterations of **SAGA FW**, then we can get an estimate on the number of the stochastic gradient calls by multiplying this result by  $2b$ . We need to minimize  $b(1 + nb^{-3/2})$  and obtain that the optimal  $b \sim n^{2/3}$ . One can notice that Algorithm 3 is required to store  $n$  extra vectors  $\{y_i\}$  requiring  $\mathcal{O}(nd)$  extra memory. The final result for **SAGA FW** is presented in Table 1.

## B.4 Coordinate methods

In this section we provide the detailed convergence rates and proofs for specific methods (see Section 3.2) of the constrained optimization problem (1). These methods use partial derivatives with respect to coordinates instead of taking gradients of terms of finite sums.

In the following, we prove that some specific methods, i.e., **SEGA** and **JAGUAR** (new proposed method) satisfy our unified Assumption 2.1 and thus can be captured by our unified analysis. Then, we plug their corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rates.

### B.4.1 SEGA Frank-Wolfe

We first restate our Lemma 3.8 for **SEGA FW** method (Algorithm 4) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 4 SEGA Frank-Wolfe

---

**Input:** initial  $x^0, h^0 = \nabla f(x^0), g^0 = \nabla f(x^0)$  step sizes  $\{\eta_k\}_{k \geq 0}$

**for**  $k = 0, 1, 2, \dots, K-1$  **do**

Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

Sample  $i_k \in [d]$  uniformly at random

Set  $h^{k+1} = h^k + e_{i_k} (\nabla_{i_k} f(x^k) - h_{i_k}^k)$

Update  $g^{k+1} = de_{i_k} (\nabla_{i_k} f(x^{k+1}) - h_{i_k}^{k+1}) + h^{k+1}$

**end for**

---

**Lemma B.15** (Lemma 3.8). *Under Assumptions 1.1 Algorithm 4 satisfies Assumption 2.1 with:*

$$\rho_1 = 1, A = d, B = d^2 L^2, C = 0,$$

$$\sigma_k^2 = \|h^{k+1} - \nabla f(x^k)\|^2, \rho_2 = \frac{1}{2d}, E = 3L^2 d.$$

**Proof:**

We first bound the difference between estimator and exact gradient:

$$\begin{aligned}
\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k [\|de_{i_k} e_{i_k}^T (\nabla f(x^k) - h^k) + h^k - \nabla f(x^k)\|^2] \\
&= \mathbb{E}_k [\|(I - de_{i_k} e_{i_k}^T)(h^k - \nabla f(x^k))\|^2] \\
&= \mathbb{E}_k [(h^k - \nabla f(x^k))^T (I - de_{i_k} e_{i_k}^T)^T (I - de_{i_k} e_{i_k}^T) (h^k - \nabla f(x^k))] \\
&= (h^k - \nabla f(x^k))^T \mathbb{E}_k [I - 2de_{i_k} e_{i_k}^T + d^2 e_{i_k} e_{i_k}^T] (h^k - \nabla f(x^k)) \\
&= (h^k - \nabla f(x^k))^T [I - 2 \cdot I + d \cdot I] (h^k - \nabla f(x^k)) \\
&= (d-1) \|h^k - \nabla f(x^k)\|^2 \\
&\leq (d-1)(1+\alpha) \|h^k - \nabla f(x^{k-1})\|^2 + (d-1) \left(1 + \frac{1}{\alpha}\right) \eta_{k-1}^2 L^2 D^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}_k [\|h^{k+1} - \nabla f(x^k)\|^2] &= \mathbb{E}_k [\|h^k + e_{i_k} e_{i_k}^T (\nabla f(x^k) - h^k) - \nabla f(x^k)\|^2] \\
&= \mathbb{E}_k [\|(I - e_{i_k} e_{i_k}^T)(h^k - \nabla f(x^k))\|^2] \\
&= \left(1 - \frac{1}{d}\right) \|h^k - \nabla f(x^k)\|^2 \\
&\leq \left(1 - \frac{1}{d}\right) (1+\beta) \|h^k - \nabla f(x^{k-1})\|^2 + \left(1 - \frac{1}{d}\right) \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2.
\end{aligned}$$

If  $\beta = \frac{1}{2d}$  then  $(1 - \frac{1}{d})(1 + \frac{1}{2d}) \leq 1 - \frac{1}{2d}$  and  $(1 - \frac{1}{d})(1 + 2d) \leq 2d$ , then as  $d \geq 1$ :

$$\mathbb{E}_k [\|h^{k+1} - \nabla f(x^k)\|^2] \leq \left(1 - \frac{1}{2d}\right) \|h^k - \nabla f(x^{k-1})\|^2 + 3dL^2 \eta_{k-1}^2 D^2.$$

Taking  $\alpha = \frac{1}{d}$ , we obtain the needed constants.  $\square$

**Corollary B.16** (Corollary 3.9). *For Algorithm 4 in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned}
\mathbb{E}[r_{K+1}] &= \mathcal{O} \left( (f(x^0) - f(x^*)) \exp\left(-\frac{K}{8d}\right) + \frac{LD^2}{K+d} [1 + d\sqrt{d}] \right). \\
\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] &= \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} [1 + d\sqrt{d}] \right).
\end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.15 into Theorems 2.2 and 2.3.  $\square$

### B.4.2 JAGUAR

We first restate our Lemma 3.10 for JAGUAR method (Algorithm 5) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

**Algorithm 5** JAGUAR

---

**Input:** initial  $x^0, g^0 = \nabla f(x^0)$ , step sizes  $\{\eta_k\}_{k \geq 0}$

**for**  $k = 0, 1, 2, \dots, K-1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Sample  $i_{k+1} \in [d]$  uniformly at random

    Update  $g^{k+1} = e_{i_{k+1}} (\nabla_{i_{k+1}} f(x^k) - g_{i_{k+1}}^k) + g^k$

**end for**

---

**Lemma B.17** (Lemma 3.10). *Under Assumptions 1.1 Algorithm 5 satisfies Assumption 2.1 with:*

$$\begin{aligned}
\rho_1 &= \frac{1}{2d}, A = 0, B = 3dL^2, C = 0, \\
\sigma_k^2 &= 0, \rho_2 = 1, E = 0.
\end{aligned}$$

**Proof:**

We first bound the difference between estimator and exact gradient:

$$\begin{aligned}
 \mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k [\|e_{i_k} e_{i_k}^T (\nabla f(x^{k-1}) - g^{k-1}) + g^{k-1} - \nabla f(x^k)\|^2] \\
 &= \mathbb{E}_k [\|e_{i_k} e_{i_k}^T (\nabla f(x^{k-1}) - g^{k-1}) + g^{k-1} - \nabla f(x^k) + \nabla f(x^{k-1}) - \nabla f(x^{k-1})\|^2] \\
 &= \mathbb{E}_k [\|(I - e_{i_k} e_{i_k}^T) (\nabla f(x^{k-1}) - g^{k-1}) + \nabla f(x^{k-1}) - \nabla f(x^k)\|^2] \\
 &\leq (1 + \beta) \mathbb{E}_k [\|(I - e_{i_k} e_{i_k}^T) (g^{k-1} - \nabla f(x^{k-1}))\|^2] + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2 \\
 &= (1 + \beta) \left(1 - \frac{1}{d}\right) \|g^{k-1} - \nabla f(x^{k-1})\|^2 + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2.
 \end{aligned}$$

If  $\beta = \frac{1}{2d}$  then  $(1 - \frac{1}{d})(1 + \frac{1}{2d}) \leq 1 - \frac{1}{2d}$  and then as  $d \geq 1$ :

$$\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] \leq \left(1 - \frac{1}{2d}\right) \|g^{k-1} - \nabla f(x^{k-1})\|^2 + 3d\eta_{k-1}^2 L^2 D^2.$$

This finishes the proof.  $\square$

**Corollary B.18** (Corollary 3.11). *For Algorithm 5 in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned}
 \mathbb{E}[r_{K+1}] &= \mathcal{O}\left(\left(f(x^0) - f(x^*)\right) \exp\left(-\frac{K}{8d}\right) + \frac{LD^2}{K+d} [1+d]\right). \\
 \mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] &= \mathcal{O}\left(\frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} [1+d]\right).
 \end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.17 into Theorems 2.2 and 2.3.  $\square$

**B.4.3 ZOJA**

We introduce ZOJA (Zero-Order JAGUAR) method and provide proof of its convergence. The essence of this method is that in some setting we cannot compute the direction derivatives, but only approximate through zero-order information:

$$\nabla_{i_{k+1}} f(x^k) \approx \tilde{\nabla}_{i_k} f(x^k) = \frac{f(x^k + \tau e_{i_{k+1}}) - f(x^k)}{\tau}.$$

This can be used instead of the real directional derivative in the JAGUAR FW method, but it is worth considering the error that arises due to the approximation. We derive corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

**Algorithm 6** ZOJA (Zero-Order JAGUAR)

**Input:** initial  $x^0, g^0 = \sum_{i=1}^d \frac{f(x^0 + \tau e_i) - f(x^0)}{\tau}$ , step sizes  $\{\eta_k\}_{k \geq 0}, \tau > 0$

**for**  $k = 0, 1, 2, \dots, K-1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Sample  $i_{k+1} \in [d]$  uniformly at random

    Compute  $\tilde{\nabla}_{i_k} f(x^k) = \frac{f(x^k + \tau e_{i_{k+1}}) - f(x^k)}{\tau}$

    Update  $g^{k+1} = e_{i_{k+1}} (\tilde{\nabla}_{i_{k+1}} f(x^k) - g_{i_{k+1}}^k) + g^k$

**end for**

---

**Lemma B.19** *Under Assumptions 1.1 Algorithm 6 satisfies Assumption 2.1 with:*

$$\begin{aligned}
 \rho_1 &= \frac{1}{4d}, A = 0, B =, 3dL^2 C = \frac{5dL^2\tau^2}{4}, \\
 \sigma_k^2 &= 0, \rho_2 = 1, E = 0.
 \end{aligned}$$

**Proof:**

We bound the difference between estimator and exact gradient:

$$\begin{aligned}
\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k \left[ \|e_{i_k} (\tilde{\nabla}_{i_{k-1}} f(x^{k-1}) - g_{i_{k-1}}^{k-1}) + g^{k-1} - \nabla f(x^k)\|^2 \right] \\
&= \mathbb{E}_k \left[ \|e_{i_k} (\tilde{\nabla}_{i_{k-1}} f(x^{k-1}) - g_{i_{k-1}}^{k-1}) + g^{k-1} - \nabla f(x^k) + \nabla f(x^{k-1}) - \nabla f(x^{k-1})\|^2 \right] \\
&\leq (1 + \beta) \mathbb{E}_k \left[ \|e_{i_k} (\tilde{\nabla}_{i_{k-1}} f(x^{k-1}) - g_{i_{k-1}}^{k-1}) + g^{k-1} - \nabla f(x^{k-1})\|^2 \right] + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2 \\
&= (1 + \beta) \mathbb{E}_k \left[ \|(I - e_{i_k} e_{i_k}^T)(g^{k-1} - \nabla f(x^{k-1})) + e_{i_k} (\tilde{\nabla}_{i_k} f(x^{k-1}) - \nabla_{i_k} f(x^{k-1}))\|^2 \right] \\
&\quad + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2 \\
&\leq (1 + \beta)(1 + \alpha) \left(1 - \frac{1}{d}\right) \|g^{k-1} - \nabla f(x^{k-1})\|^2 \\
&\quad + (1 + \beta) \left(1 + \frac{1}{\alpha}\right) \frac{1}{\tau^2} \|f(x^{k-1} + \tau e_{i_{k-1}}) - f(x^{k-1}) - \langle \nabla f(x^{k-1}), \tau e_{i_k} \rangle\|^2 \\
&\quad + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2 \\
&\stackrel{19}{\leq} (1 + \beta)(1 + \alpha) \left(1 - \frac{1}{d}\right) \|g^{k-1} - \nabla f(x^{k-1})\|^2 + (1 + \beta) \left(1 + \frac{1}{\alpha}\right) \frac{L^2 \tau^2}{4} \\
&\quad + \left(1 + \frac{1}{\beta}\right) \eta_{k-1}^2 L^2 D^2.
\end{aligned}$$

If  $\beta = \frac{1}{2d}$ , then  $(1 - \frac{1}{d})(1 + \frac{1}{2d}) \leq 1 - \frac{1}{2d}$ . And with  $\alpha = \frac{1}{4d}$ , we get

$$\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] \leq \left(1 - \frac{1}{4d}\right) \|g^{k-1} - \nabla f(x^{k-1})\|^2 + 3d\eta_{k-1}^2 L^2 D^2 + \frac{5dL\tau^2}{4}.$$

This finishes the proof.  $\square$

**Corollary B.20** *For Algorithm 6 in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned}
\mathbb{E}[r_{K+1}] &= \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{K}{8d} \right) + \frac{LD^2}{K+d} \left[ 1 + d \sqrt{1 + \frac{K}{K+d} \frac{\tau^2}{D^2}} \right] \right) \\
\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] &= \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + d \sqrt{1 + K \frac{\tau^2}{D^2}} \right] \right).
\end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.19 into Theorems 2.2 and 2.3.  $\square$

## B.5 Distributed methods

In this section, we provide the detailed convergence rates and proofs for specific methods (see Section 3.3) solving constrained optimization problem (1) with finite-sum form (5) in distributed/federated setting, i.e.,

$$\min_{x \in \mathcal{X}} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x) \right\}.$$

Each device  $i$  has an access only to  $f_i$ . Furthermore, we allow that different machine can have different data distribution, i.e., heterogeneous data setting.

As we have already emphasized (see Section 3.3), the bottleneck for such type of problem usually is communication cost. Therefore, we focus on methods with compressed communication.

In the following, we prove that some specific methods satisfy our unified Assumption 2.1 and thus can be captured by our unified analysis. Then, we plug their corresponding parameters into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rates for these methods.

### B.5.1 DIANA Frank-Wolfe

We first restate our main convergence lemma for DIANA FW method (Algorithm 7) and provide its proof for various stochastic gradients. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate, depending on the stochastic gradient we use. After that we provide readers with convergence rate for (2)+(11) method.

---

#### Algorithm 7 DIANA Frank-Wolfe

---

**Input:** initial point  $x^0, \forall i \in [n] h_i^0 = \nabla f_i(x^0), h^0 = \frac{1}{n} \sum_{i=1}^n h_i^0$ , step sizes  $\{\eta_k\}_{k \geq 0}, \alpha > 0$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Update  $h^{k+1} = h^k + \alpha \cdot \frac{1}{n} \sum_{i=1}^n \Delta_i^k$

**for**  $i = 1, \dots, n$  **do**

        Compress shifted local gradient  $\Delta_i^{k+1} = \mathcal{Q}(\nabla f_i(x^{k+1}) - h_i^{k+1})$  and send  $\Delta_i^{k+1}$  to the server

        Update local shift  $h_i^{k+1} = h_i^k + \alpha \cdot \mathcal{Q}(\nabla f_i(x^{k+1}) - h_i^{k+1})$

**end for**

    Aggregate received compressed gradient information  $g^{k+1} = h^{k+1} + \frac{1}{n} \sum_{i=1}^n \Delta_i^{k+1}$

**end for**

---

**Lemma B.21** Under Assumption 3.1 with  $\alpha = \frac{1}{1+\omega}$  Algorithm 7 satisfy Assumption 2.1 with:

$$\begin{aligned} \rho_1 &= 1, \quad A = \frac{\omega}{n^2}, \quad B = \frac{2\omega(\omega+1)\tilde{L}^2}{n}, \quad C = 0, \\ \sigma_k^2 &= \sum_{i=1}^n \|\nabla f_i(x^k) - h_i^k\|^2, \quad \rho_2 = \frac{1}{2(1+\omega)}, \quad E = 2(\omega+1)n\tilde{L}^2. \end{aligned}$$

**Proof:** Deriving inequalities from the proof of Theorem 7 from (Li and Richtárik, 2020), we get

$$\begin{aligned} \mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &\leq \frac{\omega}{n^2} \mathbb{E}_k \left[ \sum_{i=1}^n \|\nabla f_i(x^k) - h_i^k\|^2 \right] \\ \mathbb{E}_k \left[ \sum_{i=1}^n \|\nabla f_i(x^k) - h_i^k\|^2 \right] &\leq \left( 1 - 2\alpha + \frac{(1-\alpha)^2}{\beta} + \alpha^2(1+\omega) \right) \sum_{i=1}^n \mathbb{E}_k [\|\nabla f_i(x^{k-1}) - h_i^{k-1}\|^2] \\ &\quad + (1+\beta) \sum_{i=1}^n \mathbb{E}_k [\|\nabla f_i(x^k) - \nabla f_i(x^{k-1})\|^2] \end{aligned}$$

for  $\forall \beta > 0$ . Choose  $\beta = \frac{2\omega^2}{1+\omega}$ , then

$$\mathbb{E}_k \left[ \sum_{i=1}^n \|\nabla f_i(x^k) - h_i^k\|^2 \right] \leq \frac{\omega + \frac{1}{2}}{\omega + 1} \sum_{i=1}^n \mathbb{E}_k [\|\nabla f_i(x^{k-1}) - h_i^{k-1}\|^2] + \frac{2\omega^2 + \omega + 1}{\omega + 1} n\tilde{L}^2 \eta_{k-1}^2 D^2,$$



$$\leq \left(1 - \frac{1}{2(1+\omega)}\right) \sum_{i=1}^n \mathbb{E}_k \left[ \|\nabla f_i(x^{k-1}) - h_i^{k-1}\|^2 \right] + 2(\omega+1)n\tilde{L}^2\eta_{k-1}^2 D^2.$$

$$\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] \leq \frac{\omega}{n^2} \sum_{i=1}^n \mathbb{E}_k \left[ \|\nabla f_i(x^{k-1}) - h_i^{k-1}\|^2 \right] + 2\frac{\omega}{n}(\omega+1)\tilde{L}^2\eta_{k-1}^2 D^2.$$

This finishes the proof.  $\square$

**Corollary B.22** (Corollary 3.14). *For the algorithm (2)+(11) in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[r_{K+1}] = \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{K}{8(\omega+1)} \right) + \frac{LD^2}{K+\omega} \left[ 1 + \frac{\tilde{L}(\omega+1)\sqrt{\omega}}{L\sqrt{n}} \right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + \frac{\tilde{L}(\omega+1)\sqrt{\omega}}{L\sqrt{n}} \right] \right).$$

### B.5.2 MARINA Frank-Wolfe

We first restate our Lemma 3.15 for MARINA FW method (Algorithm 8) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 8 MARINA Frank-Wolfe

---

**Input:** initial  $x^0, \forall i \in [n] g_i^0 = \nabla f_i(x^0), g^0 = \nabla f(x^0)$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , probability  $p \in (0, 1]$

**for**  $k = 0, 1, 2, \dots, K-1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Broadcast  $g^k$  to all workers

**for**  $i = 1, \dots, n$  **do**

        Update  $c_i^{k+1} = \begin{cases} \nabla f_i(x^{k+1}) - g_i^k, & \text{with probability } p \\ \mathcal{Q}(\nabla f_i(x^{k+1}) - \nabla f_i(x^k)), & \text{with probability } 1 - p \end{cases}$

        Send  $c_i^{k+1}$  to the server

        Update  $g_i^{k+1} = g_i^k + c_i^{k+1}$

**end for**

    Set  $g^{k+1} = g^k + \frac{1}{n} \sum_{i=1}^n c_i^{k+1}$

**end for**

---

**Lemma B.23** (Lemma 3.15). *Under Assumption 3.1 Algorithm 8 satisfy Assumption 2.1 with:*

$$\rho_1 = p, \quad A = 0, \quad B = \frac{(1-p)\omega L^2}{n}, \quad C = 0,$$

$$\sigma_k = 0, \quad \rho_2 = 1, \quad E = 0.$$

**Proof:**

Using the Theorem 2.1 from (Gorbunov et al., 2021) we can obtain:

$$\begin{aligned} \mathbb{E} [\|g^{k+1} - \nabla f(x^{k+1})\|^2] &\leq \frac{(1-p)\omega L^2}{n} \mathbb{E} [\|x^{k+1} - x^k\|^2] + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2] \\ &= \frac{(1-p)\omega L^2}{n} \eta_k^2 \mathbb{E} [\|s^k - x^k\|^2] + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2] \\ &= \frac{(1-p)\omega L^2}{n} \eta_k^2 D^2 + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2]. \end{aligned}$$

$\square$

**Corollary B.24** (Corollary 3.16). *For algorithm 8 in the convex and non-convex cases the following conver-*

gences take place:

$$\begin{aligned}\mathbb{E}[r_{K+1}] &= \mathcal{O}\left(\left(f(x^0) - f(x^*)\right) \exp\left(-\frac{pK}{4}\right) + \frac{LD^2}{K + \frac{1}{p}} \left[1 + \sqrt{\frac{(1-p)\omega}{np}}\right]\right). \\ \mathbb{E}\left[\min_{0 \leq k \leq K-1} \mathbf{gap}(x^k)\right] &= \mathcal{O}\left(\frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \sqrt{\frac{(1-p)\omega}{pn}}\right]\right).\end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.11 into Theorems 2.2 and 2.3.  $\square$

We proved results for MARINA FW depending on the parameters  $p$  to be tuned. Let us find the optimal choice of  $p$  for MARINA FW. To find the optimal choice of  $p$ , one can note that on average we send  $pd + (1-p)k$  coordinates from a single worker per iteration. In more details, at each iteration with probability  $p$  we call the full gradient in the new point  $x^{k+1}$  or send the compressed difference of gradients. It is optimal to choose  $p$  from the condition  $pd = (1-p)k$ , hence  $p = \frac{k}{d+k}$ . From Corollary 3.16, we know the estimate on the number of iterations of MARINA FW, then we can get an estimate on the number of the stochastic gradient calls by multiplying this result by  $(pd + k)$ , also knowing that for RandK  $\omega \sim \frac{d}{k}$ . The final result for MARINA FW is presented in Table 3.

### B.5.3 EF21 Frank-Wolfe

We first restate our Lemma 3.18 for EF21 FW method (Algorithm 9) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 9 EF21 Frank-Wolfe

---

**Input:** initial  $x^0, \forall i \in [n] g_i^0 = \nabla f_i(x^0), g^0 = \frac{1}{n} \sum_{i=1}^n g_i^0$ , step sizes  $\{\eta_k\}_{k \geq 0}$   
**for**  $k = 0, 1, 2, \dots, K - 1$  **do**  
     Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$   
     Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$   
     Broadcast  $x^{k+1}$  to all workers  
     **for**  $i = 1, \dots, n$  **do**  
         Compress  $c_i^{k+1} = \mathcal{C}(\nabla f_i(x^{k+1}) - g_i^k)$  and send to the server  
         Update  $g_i^{k+1} = g_i^k + c_i^{k+1}$   
     **end for**  
     Update  $g^{k+1} = g^k + \frac{1}{n} \sum_{i=1}^n c_i^{k+1}$   
**end for**

---

**Lemma B.25** (Lemma 3.18). *Under Assumption 3.1 Algorithm 9 satisfies Assumption 2.1 with:*

$$\begin{aligned}\rho_1 &= 1, \quad A = 1, \quad B = 0, \quad C = 0, \\ \sigma_k^2 &= \frac{1}{n} \sum_{i=1}^n \|g_i^k - \nabla f_i(x^k)\|^2, \quad \rho_2 = \frac{\delta + 1}{2\delta^2}, \quad E = 2\delta\tilde{L}^2.\end{aligned}$$

**Proof:**

First, let us notice:

$$\mathbb{E}_k \left[ \|g^k - \nabla f(x^k)\|^2 \right] = \mathbb{E}_k \left[ \left\| \frac{1}{n} \sum_{i=1}^n \left( g_i^k - \nabla f_i(x^k) \right) \right\|^2 \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}_k \left[ \|g_i^k - \nabla f_i(x^k)\|^2 \right].$$

Similar to the Proof of Theorem 1 from (Richtarik et al., 2021), we can derive:

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n \mathbb{E}_k \left[ \|g_i^k - \nabla f_i(x^k)\|^2 \right] &= \frac{1}{n} \sum_{i=1}^n \mathbb{E}_k \left[ \|g_i^{k-1} + \mathcal{C}(\nabla f_i(x^k) - g_i^{k-1}) - \nabla f_i(x^k)\|^2 \right] \\ &\leq \left(1 - \frac{1}{\delta}\right) \frac{1}{n} \sum_{i=1}^n \|g_i^{k-1} - \nabla f_i(x^k)\|^2\end{aligned}$$

$$\leq \left(1 - \frac{1}{\delta}\right) (1 + \alpha) \frac{1}{n} \sum_{i=1}^n \|g_i^{k-1} - \nabla f_i(x^{k-1})\|^2 + \left(1 - \frac{1}{\delta}\right) \left(1 + \frac{1}{\alpha}\right) \tilde{L}^2 \eta_{k-1}^2 D^2.$$

for any  $\alpha > 0$ . Choose  $\alpha = \frac{1}{2\delta}$ , hence

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}_k \left[ \|g_i^k - \nabla f_i(x^k)\|^2 \right] \leq \left(1 - \frac{\delta + 1}{2\delta^2}\right) \frac{1}{n} \sum_{i=1}^n \|g_i^{k-1} - \nabla f_i(x^{k-1})\|^2 + 2\delta \tilde{L}^2 \eta_{k-1}^2 D^2.$$

□

**Corollary B.26** (Corollary 3.19). *For Algorithm 9 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E} \left[ r_{K+1} \right] = \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{K}{8\delta} \right) + \frac{LD^2}{K + \delta} \left[ 1 + \frac{\tilde{L}}{L} \delta \right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + \frac{\tilde{L}}{L} \delta \right] \right).$$

**Proof:**

It suffices to plug parameters from Lemma B.25 into Theorems 2.2 and 2.3.

□

## B.6 Combinations of different approaches

In this section, we provide the detailed convergence rates and proofs for specific methods (see Section 3.4) solving constrained optimization problem (1) using combinations of methods, presented earlier. This approach may outperform existing methods, as they combine advantages of both algorithms.

### B.6.1 SAGA SARAH Frank-Wolfe

We first restate our Lemma 3.20 for SAGA SARAH FW method (Algorithm 10) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 10 SAGA SARAH Frank-Wolfe

---

**Input:** initial  $x^0, y_i^0 = \nabla f_i(x^0), g^0 = \nabla f(x^0)$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , momentum  $\lambda$ , batch size  $b$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Generate batch  $S_k$  with size  $b$

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Update  $y_i^{k+1} = \begin{cases} \nabla f_i(x^k), & \text{for } i \in S_k, \\ y_i^k, & \text{for } i \notin S_k \end{cases}$

    Update  $g^{k+1} = \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^{k+1}) - \nabla f_i(x^k)] + (1 - \lambda)g^k + \lambda \left( \frac{1}{b} \sum_{i \in S_k} [\nabla f_i(x^k) - y_i^{k+1}] + \frac{1}{n} \sum_{j=1}^n y_j^{k+1} \right)$

**end for**

---

**Lemma B.27** (Lemma 3.20). *Under Assumption 3.1 Algorithm 10 satisfies Assumption 2.1 with:*

$$\begin{aligned} \rho_1 &= \frac{b}{2n}, \quad A = \frac{b}{2n^2}, \quad B = \frac{2\tilde{L}^2}{b}, \quad C = 0, \\ \sigma_k^2 &= \frac{1}{n} \sum_{j=1}^n \mathbb{E}[\|\nabla f_j(x^k) - y_j^{k+1}\|^2], \quad \rho_2 = \frac{b}{2n}, \quad E = \frac{2n\tilde{L}^2}{b}. \end{aligned}$$

**Proof:**

Using Lemma 2 from (Li et al., 2021) we can obtain:

$$\begin{aligned} \mathbb{E}_k [\|\nabla f(x^k) - g^k\|^2] &\leq (1 - \lambda)^2 \|\nabla f(x^{k-1}) - g^{k-1}\|^2 + \frac{2\lambda^2}{b} \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 \\ &\quad + \frac{2\tilde{L}}{b} \|x^k - x^{k-1}\|^2 \\ &\leq (1 - \lambda)^2 \|\nabla f(x^k) - g^k\|^2 + \frac{2\lambda^2}{b} \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 + \frac{2\tilde{L}}{b} \eta_{k-1}^2 D^2. \end{aligned}$$

Additionally Lemma 3 from (Li et al., 2021) with  $\beta_k = \frac{b}{2n}$  gives us:

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - y_j^{k+1}\|^2 &\leq \left(1 - \frac{b}{2n}\right) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 + \frac{2n\tilde{L}^2}{b} \|x^k - x^{k-1}\|^2 \\ &\leq \left(1 - \frac{b}{2n}\right) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 + \frac{2n\tilde{L}^2}{b} \eta_{k-1}^2 D^2. \end{aligned}$$

With  $\lambda = \frac{b}{2n}$  we have:

$$\mathbb{E}_k [\|\nabla f(x^k) - g^k\|^2] \leq \left(1 - \frac{b}{2n}\right) \|\nabla f(x^{k-1}) - g^{k-1}\|^2 + \frac{b}{2n^2} \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - y_j^k\|^2 + \frac{2\tilde{L}}{b} \eta_{k-1}^2 D^2.$$

That finishes the proof.  $\square$

**Corollary B.28** (Corollary 3.21). *For Algorithm 10 in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned}\mathbb{E}[r_{K+1}] &= \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{bK}{8n} \right) + \frac{LD^2}{K + \frac{n}{b}} \left[ 1 + \frac{\tilde{L}\sqrt{n}}{Lb} \right] \right). \\ \mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] &= \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + \frac{\tilde{L}\sqrt{n}}{Lb} \right] \right).\end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.27 into Theorems 2.2 and 2.3.  $\square$

The choice  $b$  for SAGA SARAH FW is presented in the original paper (Beznosikov et al., 2023).

### B.6.2 Q-L-SVRG Frank-Wolfe with compression

We first restate our Lemma 3.22 for Q-L-SVRG FW method (Algorithm 11) and provide its proof. Then we plug its corresponding parameters (i.e., specific values for  $A, B, C, E, \sigma_k^2, \rho_1, \rho_2$ ) into our unified Theorems 2.2 and 2.3 to obtain the detailed convergence rate.

---

#### Algorithm 11 Q-L-SVRG Frank-Wolfe

---

**Input:** initial  $x^0, w^0 = x^0$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$ , probability  $p \in (0, 1]$

```

for  $k = 0, 1, 2, \dots, K - 1$  do
  Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$ 
  Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k g^k$ 
  Update  $w^{k+1} = \begin{cases} x^k, & \text{with probability } p \\ w^k, & \text{with probability } 1 - p \end{cases}$ 
  Broadcast  $x^{k+1}$  to all workers
  for  $i = 1, \dots, n$  do
    Compress  $c_i^{k+1} = \mathcal{C}(\nabla f_i(x^{k+1}) - g_i^k)$  and send to the server
    Update  $g_i^{k+1} = g_i^k + c_i^{k+1}$ 
  end for
  Update  $g^{k+1} = \frac{1}{n} \sum_{i=1}^n \mathcal{Q}(\nabla f_i(x^{k+1}) - \nabla f_i(w^{k+1})) + \nabla f(w^{k+1})$ 
end for

```

---

**Lemma B.29** (Lemma 3.22). *Under Assumptions 3.1 Algorithm 11 satisfy Assumption 2.1 with:*

$$\begin{aligned}\rho_1 &= 1, \quad A = \frac{\omega \tilde{L}^2}{n} \left( 1 - \frac{p}{2} \right), \quad B = \frac{\omega \tilde{L}^2}{n} \left( 1 + \frac{8(1-p)}{p} \right), \quad C = 0, \\ \sigma_k^2 &= \|x^k - w^k\|^2, \quad \rho_2 = \frac{p}{2}, \quad E = 1 + \frac{8(1-p)}{p}.\end{aligned}$$

**Proof:**

$$\begin{aligned}\mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k \left[ \left\| \frac{1}{n} \sum_{i=1}^n \mathcal{Q}(\nabla f_i(x^k) - \nabla f_i(w^k)) + \nabla f(w^k) - \nabla f(x^k) \right\|^2 \right] \\ &= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}_k \left[ \left\| \mathcal{Q}(\nabla f_i(x^k) - \nabla f_i(w^k)) + \nabla f_i(w^k) - \nabla f_i(x^k) \right\|^2 \right] \\ &+ \frac{2}{n^2} \sum_{i < j} \mathbb{E}_k \left[ \left\langle \mathcal{Q}(\nabla f_i(x^k) - \nabla f_i(w^k)) + \nabla f_i(w^k) - \nabla f_i(x^k), \right. \right. \\ &\quad \left. \left. \mathcal{Q}(\nabla f_j(x^k) - \nabla f_j(w^k)) + \nabla f_j(w^k) - \nabla f_j(x^k) \right\rangle \right]\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{\omega}{n^2} \sum_{i=1}^n \mathbb{E}_k [\|\nabla f_i(x^k) - \nabla f_i(w^k)\|^2] \\
 &\leq \frac{\omega \tilde{L}^2}{n} \|x^k - w^k\|^2,
 \end{aligned} \tag{31}$$

since  $\mathcal{Q}(\nabla f_i(x^k) - \nabla f_i(w^k))$  and  $\mathcal{Q}(\nabla f_j(x^k) - \nabla f_j(w^k))$  are independent. According to (29) we derive:

$$\mathbb{E}_k[\|x^k - w^k\|^2] \leq \left(1 + \frac{2(1-p)}{\beta}\right) \eta_{k-1}^2 D^2 + (1-p)(1+2\beta) \|x^{k-1} - w^{k-1}\|^2.$$

Finally substituting it in (31) we get

$$\mathbb{E}_k[\|\nabla f(x^k) - g^k\|^2] \leq \frac{\omega \tilde{L}^2}{n} \left(1 + \frac{2(1-p)}{\beta}\right) \eta_{k-1}^2 D^2 + \frac{\omega \tilde{L}^2}{n} (1-p)(1+2\beta) \|x^{k-1} - w^{k-1}\|^2.$$

With  $\beta = \frac{p}{4}$  we have

$$\mathbb{E}_k[\|x^k - w^k\|^2] \leq \left(1 + \frac{8(1-p)}{p}\right) \eta_{k-1}^2 D^2 + \left(1 - \frac{p}{2}\right) \|x^{k-1} - w^{k-1}\|^2, \tag{32}$$

and

$$\mathbb{E}_k[\|\nabla f(x^k) - g^k\|^2] \leq \frac{\omega \tilde{L}^2}{n} \left(1 + \frac{8(1-p)}{p}\right) \eta_{k-1}^2 D^2 + \frac{\omega \tilde{L}^2}{n} \left(1 - \frac{p}{2}\right) \|x^{k-1} - w^{k-1}\|^2.$$

□

**Corollary B.30** (Corollary 3.23). *For Algorithm 11 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E}[r_{K+1}] = \mathcal{O} \left( (f(x^0) - f(x^*)) \exp\left(-\frac{Kp}{8}\right) + \frac{LD^2}{K + \frac{1}{p}} \left[1 + \frac{\tilde{L} \sqrt{\omega}}{L p \sqrt{n}}\right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[1 + \frac{\tilde{L} \sqrt{\omega}}{L p \sqrt{n}}\right] \right).$$

**Proof:**

It suffices to plug parameters from Lemma B.29 into Theorems 2.2 and 2.3. □

We proved results for Q-L-SVRG FW depending on the parameters  $p$  to be tuned. Let us find the optimal choice of  $p$  for Q-L-SVRG FW. To find the optimal choice of  $p$ , one can note that on average we send  $pd + \frac{d}{\omega}$  coordinates from a single worker per iteration. In more details, at each iteration with probability  $p$  we call the full gradient in the new point  $w^{k+1}$  and also send the compressed difference of gradient and its estimator. From Corollary 3.23, we know the estimate on the number of iterations of Q-L-SVRG FW, then we can get an estimate on the number of the stochastic gradient calls by multiplying this result by  $(pd + \frac{d}{\omega})$ . Then the new estimate can be optimized first by  $p$  (in fact, we need to minimize  $(1 + \frac{\sqrt{\omega}}{p\sqrt{n}})(p + \frac{1}{\omega})$ ) and obtain that the optimal  $p \sim (n\omega)^{-\frac{1}{4}}$ . The final result for Q-L-SVRG FW is presented in Table 3.

### B.6.3 VR-MARINA Frank-Wolfe

Developing the idea of combining different approaches, we present a modification of the MARINA FW, which is an adaptation from (Gorbunov et al., 2021). We first present the algorithm, and then provide detailed convergence result together with its proof.

**Algorithm 12** VR-MARINA Frank-Wolfe

---

**Input:** initial  $x^0$ ,  $g^0 = \nabla f(x^0)$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$ , probability  $p \in (0, 1]$

**for**  $k = 0, 1, 2, \dots, K - 1$  **do**

    Compute  $s^k = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k s^k$

    Broadcast  $g^k$  to all workers

**for**  $i = 1, \dots, n$  **do**

        Generate batch  $|S_k^i| = b$

        Update  $g_i^{k+1} = \begin{cases} \nabla f_i(x^{k+1}), & \text{with probability } p \\ g_i^k + \mathcal{Q} \left( \sum_{j \in S_k^i} (\nabla f_{ij}(x^{k+1}) - \nabla f_{ij}(x^k)) \right), & \text{with probability } 1 - p \end{cases}$

**end for**

    Set  $g^{k+1} = \frac{1}{n} \sum_{i=1}^n g_i^{k+1}$

**end for**

---

In this section, we assume that the local loss on each node has either a finite-sum form:

$$f_i(x) = \frac{1}{m} \sum_{j=1}^m f_{ij}(x).$$

**Assumption B.31** (Average  $\mathcal{L}$ -smoothness). *For all  $k \geq 0$  and  $i \in [n]$  the minibatch stochastic gradients difference  $\tilde{\Delta}_i^k = \frac{1}{b} \sum_{S_b^k} (\nabla f_{ij}(x^{k+1}) - \nabla f_{ij}(x^k))$  computed on the  $i$ -th machine satisfies:*

$$\begin{aligned} \mathbb{E} \left[ \tilde{\Delta}_i^k \mid x^k, x^{k+1} \right] &= \Delta_i^k, \\ \mathbb{E} \left[ \|\tilde{\Delta}_i^k - \Delta_i^k\|^2 \mid x^k, x^{k+1} \right] &\leq \frac{\mathcal{L}_i}{b} \|x^{k+1} - x^k\|^2, \end{aligned}$$

with some  $\mathcal{L}_i > 0$  and  $\Delta_i^k = \nabla f_i(x^{k+1}) - \nabla f_i(x^k)$ .

This assumption is satisfied in many cases. In particular, if  $S'_{i,k} = \{1 \dots, m\}$ , then  $\mathcal{L}_i = 0$ , and if  $S'_{i,k}$  consists of  $b'$  i.i.d. samples from the uniform distributions on  $\{1 \dots, m\}$  and  $f_{ij}$  are  $L_{ij}$ -smooth, then  $\mathcal{L}_i \leq \max_{j \in [m]} L_{ij}$ . Under this and the previously introduced assumptions, we derive the following result.

**Lemma B.32** *Under Assumptions 3.1, B.31 Algorithm 12 satisfies Assumption 2.1 with:*

$$\begin{aligned} \rho_1 &= p, \quad A = 0, \quad B = \frac{(1-p)}{n} (\omega L^2 + \frac{(1+\omega)\mathcal{L}^2}{b}), \quad C = 0, \\ \sigma_k &= 0, \quad \rho_2 = 1, \quad E = 0. \end{aligned}$$

**Proof:**

Using the Theorem 3.1 from (Gorbunov et al., 2021), we can obtain:

$$\begin{aligned} \mathbb{E}_k [\|g^{k+1} - \nabla f(x^{k+1})\|^2] &\leq \frac{(1-p)}{n} (\omega L^2 + \frac{(1+\omega)\mathcal{L}^2}{b}) \mathbb{E} [\|x^{k+1} - x^k\|^2] + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2] \\ &= \frac{(1-p)}{n} (\omega L^2 + \frac{(1+\omega)\mathcal{L}^2}{b}) \eta_k^2 \mathbb{E} [\|s^k - x^k\|^2] + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2] \\ &= \frac{(1-p)}{n} (\omega L^2 + \frac{(1+\omega)\mathcal{L}^2}{b}) \eta_k^2 D^2 + (1-p) \mathbb{E} [\|g^k - \nabla f(x^k)\|^2]. \end{aligned}$$

□

**Corollary B.33** *For Algorithm 12 in the convex and non-convex cases the following convergences take place:*

$$\mathbb{E} [r_{K+1}] = \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{pK}{4} \right) + \frac{D^2}{K + \frac{1}{p}} \left[ L + \sqrt{\frac{(1-p) (\omega L^2 + \frac{1+\omega}{b} \mathcal{L}^2)}{np}} \right] \right).$$

$$\mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] = \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{D^2}{\sqrt{K}} \left[ L + \sqrt{\frac{(1-p)(\omega L^2 + \frac{1+\omega}{b} \mathcal{L}^2)}{np}} \right] \right).$$

**Proof:**

It suffices to plug parameters from Lemma B.32 into Theorems 2.2 and 2.3.  $\square$

#### B.6.4 PP-L-SVRG Frank-Wolfe

Combining ideas of L-SVRG and distributed methods, one might introduce PP-L-SVRG, decentralized method, that is similar to SAGA when size of batch equals to 1, with only exception - there is no fixed size of batch.

---

#### Algorithm 13 PP-L-SVRG Frank-Wolfe

---

**Input:** initial  $x^0 = w^0$ , step sizes  $\{\eta_k\}_{k \geq 0}$ , batch size  $b$ , probability  $p \in (0, 1]$

**for**  $k = 0, 1, 2, \dots, K-1$  **do**

    Compute  $s = \arg \min_{s \in \mathcal{X}} \langle s, g^k \rangle$

    Update  $x^{k+1} = (1 - \eta_k)x^k + \eta_k g^k$

    Update  $w^{k+1} = \begin{cases} x^k, & \text{with probability } p \\ w^k, & \text{with probability } 1-p \end{cases}$

    Sample  $i_k \in [d]$  uniformly at random

    Update  $g^{k+1} = \nabla f_{i_k}(x^{k+1}) - \nabla f_{i_k}(w^{k+1}) + \nabla f(w^{k+1})$

**end for**

---

**Lemma B.34** Under Assumption 3.1 Algorithm 13 satisfies Assumption 2.1 with

$$\begin{aligned} \rho_1 &= 1, \quad A = 1 + \frac{p}{2}, \quad B = \tilde{L}^2 \left( 1 + \frac{2}{p} \right), \quad C = 0, \\ \sigma_k^2 &= \sum_{i=1}^n \|\nabla f_i(x^k) - w^{k+1}\|^2, \quad \rho_2 = \frac{p}{2}, \quad E = \tilde{L}^2 \frac{2}{p}. \end{aligned}$$

**Proof:**

We bound the difference between estimator and exact gradient:

$$\begin{aligned} \mathbb{E}_k [\|g^k - \nabla f(x^k)\|^2] &= \mathbb{E}_k \left[ \left\| \nabla f_{i_k}(x^k) - \nabla f_{i_k}(w^k) + \frac{1}{n} \sum_{j=1}^n \nabla f_j(w^k) - \nabla f(x^k) \right\|^2 \right] \\ &\stackrel{(B.2)}{\leq} \frac{1}{n} \sum_{j=1}^n \left\| \nabla f_j(x^k) - \nabla f_j(w^k) - \left( \frac{1}{n} \sum_{i=1}^n [\nabla f_i(x^k) - \nabla f_i(w^k)] \right) \right\|^2 \\ &\leq \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(w^k)\|^2 \\ &\leq \frac{1}{n} (1 + \alpha) \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(x^{k-1})\|^2 + \frac{1}{n} \left( 1 + \frac{1}{\alpha} \right) \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - \nabla f_j(w^k)\|^2 \\ &\leq \tilde{L}^2 (1 + \alpha) \eta_{k-1}^2 D^2 + \left( 1 + \frac{1}{\alpha} \right) \sigma_{k-1}^2 \end{aligned}$$

for  $\forall \alpha > 0$ . The second inequation holds, since  $\frac{1}{n} \sum_{i=1}^n$  can be described, as an expected value. And  $\mathbb{E}\|x - \mathbb{E}x\|^2 \leq \mathbb{E}\|x\|^2$ . Then we need to bound the second term:

$$\begin{aligned} \mathbb{E}_k[\sigma_k^2] &= \mathbb{E}_k \left[ \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(w^{k+1})\|^2 \right] = (1-p) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(w^k)\|^2 \\ &= (1-p) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^k) - \nabla f_j(x^{k-1}) + \nabla f_j(x^{k-1}) - \nabla f_j(w^k)\|^2 \end{aligned}$$



$$\leq (1-p)(1+\beta) \frac{1}{n} \sum_{j=1}^n \|\nabla f_j(x^{k-1}) - \nabla f_j(w^k)\|^2 + (1-p) \left(1 + \frac{1}{\beta}\right) \tilde{L}^2 \|x^k - x^{k-1}\|^2.$$

With  $\beta = \frac{p}{2}$  we have:

$$\mathbb{E}_k[\sigma_k^2] \leq \left(1 - \frac{p}{2}\right) \sigma_{k-1}^2 + \frac{2}{p} \tilde{L}^2 \eta_{k-1}^2 D^2.$$

Taking  $\alpha = \frac{2}{p}$ , we obtain the needed constants. □

**Corollary B.35** *Suppose that Assumption 1.1 holds. For Algorithm 13 in the convex and non-convex cases the following convergences take place:*

$$\begin{aligned} \mathbb{E}[r_{K+1}] &= \mathcal{O} \left( \left( f(x^0) - f(x^*) \right) \exp \left( -\frac{Kp}{8} \right) + \frac{LD^2}{K + \frac{1}{p}} \left[ 1 + \frac{\tilde{L}}{L} \frac{1}{p} \right] \right). \\ \mathbb{E} \left[ \min_{0 \leq k \leq K-1} \mathbf{gap}(x^k) \right] &= \mathcal{O} \left( \frac{f(x^0) - f(x^*)}{\sqrt{K}} + \frac{LD^2}{\sqrt{K}} \left[ 1 + \frac{\tilde{L}}{L} \frac{1}{p} \right] \right). \end{aligned}$$

**Proof:**

It suffices to plug parameters from Lemma B.34 into Theorems 2.2 and 2.3. □

## C ADDITIONAL EXPERIMENTS

In this section, we provide additional experiments comparing convergence results for several methods from Tables 1, 2, and 3. Since one of the main goals of our experiments is to justify the theoretical findings of the paper, in the experiments, we use the stepsizes from the corresponding theoretical results for our methods. As already described in Section 4, we assume the particular case of finite-sum constrained optimization problem 5 with  $f(x)$  specified as:

$$f(x) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-b_i \cdot x^T a_i)),$$

where  $\{a_i, b_i\}_{i=1}^n$  is  $i$ -th data-label pair with  $a_i \in \mathbb{R}^d$  and  $b_i \in \{-1, 1\}$ . We choose  $\mathcal{X}$  as the  $l_1$  norm ball with radius  $r = 2 \cdot 10^3$ . One can show that for given  $\mathcal{X}$  the linear minimization oracle, i.e.,  $\arg \min_{s \in \mathcal{X}} \langle g, s \rangle$ , can be computed as:

$$s^* = -\text{sign}(g_i)e_i, \text{ with } i = \arg \max_j |g_j|.$$

The data and labels are obtained from LibSVM datasets `w1a`, `mushrooms` and `rcv1`.

### C.1 Point projection

Here we introduced missing experiment of comparison L-SVRG FW and SAGA SARAH FW with effective  $O(n)$  time point Euclidean projection method Duchi et al. (2008): SGD, as well as SVRG:

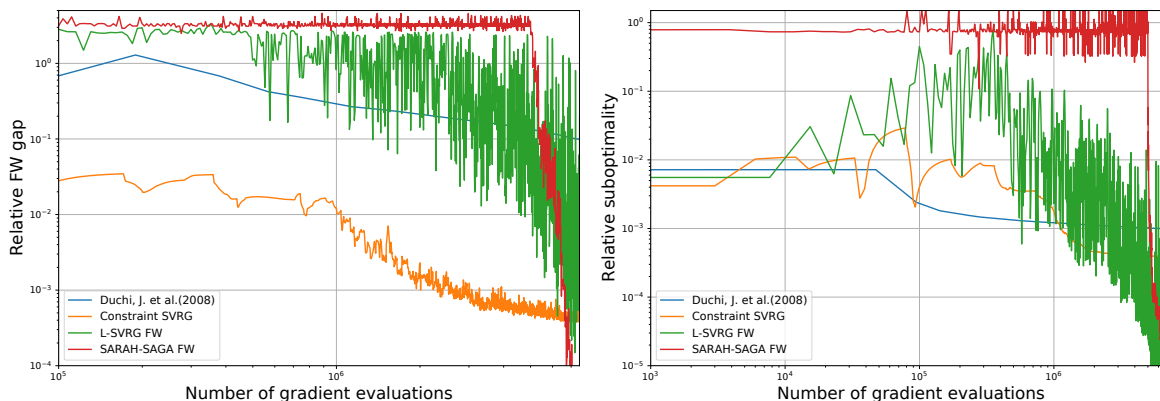


Figure 5: Comparison of Frank-Wolfe-based algorithms and methods with projection for the stochastic problem. L-SVRG FW, SAGA SARAH FW as well as SGD and SVRG with projection are considered. The comparison is made on LibSVM dataset `mushrooms`.