CHAPTER V

THEOREM 1: PROOF FOR ABELIAN EXTENSIONS

Let $G = G(\mathbf{K} : \mathbf{k})$ be the Galois group of finite abelian extension $\mathbf{K}/\mathbf{k}$. G is a direct product of cyclic groups.

$$G = G_1 \times \cdots \times G_r$$

Put $\overline{G_i} = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_r$. Let $\mathbf{K}_i$ be the fixed field of $\overline{G_i}$. Then $G(\mathbf{K}_i : \mathbf{k}) = G/\hat{G}_i$ is cyclic. The fixed field of $\mathbf{K}_1\mathbf{K}_2, \ldots \mathbf{K}_j$ is $\overline{G}_1 \cap \cdots \cap \overline{G}_j$, and

$$\overline{G}_1 \cap \cdots \cap \overline{G}_j = \{(\sigma_1, \ldots, \sigma_r) \in G_1 \times \cdots \times G_r \mid \sigma_1 = \cdots = \sigma_j = 1\}$$

The fixed field of $\mathbf{K}_1 \ldots \mathbf{K}_j \cap \mathbf{K}_{j+1}$ is $\left(\overline{G}_1 \cap \cdots \cap \overline{G}_j\right) \overline{G}_{j+1} = G$. Therefore

$$\mathbf{K}_1 \ldots \mathbf{K}_j \cap \mathbf{K}_{j+1} = \mathbf{k}$$

so

$$G(\mathbf{K}_1 \ldots \mathbf{K}_j \mathbf{K}_{j+1}) \simeq G(\mathbf{K}_1 \ldots \mathbf{K}_j : \mathbf{k}) \times G(\mathbf{K}_{j+1} : \mathbf{k})$$

Arguing by induction, we arrive at

$$G(\mathbf{K}_1 \ldots \mathbf{K}_r : \mathbf{k}) \simeq G(\mathbf{K}_1 : \mathbf{k}) \times \cdots \times G(\mathbf{K}_r : \mathbf{k})$$

This isomophism maps σ to $(\sigma_1, \ldots, \sigma_r)$ where σ_i is the restriction of σ to $\mathbf{K}_i$.

Let E be the set of primes of $\mathbf{k}$ containing all infinite primes and all primes which are ramified in $\mathbf{K}$. For $p \notin E$ the Artin symbols are defined and we have

$$\left(\frac{\mathbf{K} : \mathbf{k}}{p}\right) = \left(\left(\frac{\mathbf{K}_1 : \mathbf{k}}{p}\right), \ldots, \left(\frac{\mathbf{K}_r : \mathbf{k}}{p}\right)\right),$$

and for $\mathbf{i} \in \mathbf{I}_{\mathbf{k}}\{E\}$ we have

$$\begin{aligned} \phi_{\mathbf{K}/\mathbf{k}}(\mathbf{i}) &= \prod_{p \notin E} \left(\frac{\mathbf{K} : \mathbf{k}}{p}\right)^{u_p} \qquad \text{where } |\mathbf{i}|_p = \mathrm{N}p^{-u_p} \\ &= \left(\left(\frac{\mathbf{K}_1 : \mathbf{k}}{p}\right)^{u_p}, \ldots, \left(\frac{\mathbf{K}_r : \mathbf{k}}{p}\right)^{u_p}\right), \end{aligned}$$

or

$$\phi_{\mathbf{K}/\mathbf{k}}(\mathbf{i}) = \left(\phi_{\mathbf{K}_1\mathbf{k}}(\mathbf{i}), \ldots, \phi_{\mathbf{K}_r/\mathbf{k}}(\mathbf{i})\right). \tag{5.1}$$

The right side of (5.1) agrees with (2.1) on $\mathbf{I}_{\mathbf{k}}\{E\}$, is defined for all $\mathbf{i}$ in $\mathbf{I}_{\mathbf{k}}$, and the kernel contains $\mathbf{k}^*$. Define $\phi_{\mathbf{K}/\mathbf{k}}(\mathbf{i})$ on $\mathbf{I}_{\mathbf{k}}$ by (5.1). Except for the proofs of the first and second fundamental inequalities, this completes the proof of theorem 1 for finite abelian extensions.