The Ramsey Number $R(3,t)$ has Order of Magnitude $t^2/\log t$

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Abstract

The Ramsey number $R(s,t)$ for positive integers s and t is the minimum integer n for which every red-blue coloring of the edges of a complete n-vertex graph induces either a red complete graph of order s or a blue complete graph of order t. This paper proves that $R(3,t)$ is bounded below by $(1-o(1))t^2/\log t$ times a positive constant. Together with the known upper bound of $(1+o(1))t^2/\log t$, it follows that $R(3,t)$ has asymptotic order of magnitude $t^2/\log t$.

1 Introduction

Throughout this paper, logarithms are natural logarithms, c denotes a positive constant, s,t and n are positive integers, K_n and $G_n^{(3)}$ denote respectively the complete graph and a triangle-free (K_3-free) graph on n vertices, and $\alpha(G)$ and $\chi(G)$ are respectively the independence number and the chromatic number of graph G. Our graph theory terminology follows [8], [5].

The Ramsey number $R(s,t)$ is the minimum n such that every red-blue coloring of the edges of K_n induces either a red K_s or a blue K_t. Equivalently, $R(s,t)$ is the smallest n such that every n-vertex graph has either an s-vertex clique or a t-vertex independent set. We focus on

$$R(3,t) := \min\{n : \alpha(G_n^{(3)}) \geq t \ \text{ for every } G_n^{(3)}\} \ .$$

Our main result is an upper bound on independence numbers of triangle-free graphs.

Theorem 1.1 *Every sufficiently large n has a $G_n^{(3)}$ for which*

$$\alpha(G_n^{(3)}) \leq 9\sqrt{n \log n} \ .$$

Since $\chi(G) \geq n/\alpha(G)$ for every graph G on n vertices, we have the following corollary.

Corollary 1.2 *Every sufficiently large n has a $G_n^{(3)}$ for which*

$$\chi(G_n^{(3)}) \geq \frac{1}{9}\sqrt{\frac{n}{\log n}} \quad .$$

An easy consequence of Theorem 1.1 is

$$c(1-o(1))\frac{t^2}{\log t} \leq R(3,t)$$

with $c = 1/162 = 1/(2 \cdot 9^2)$, where $o(1)$ goes to 0 as t goes to infinity. (We make no attempt here to find the tightest possible constants.) Because it is known [1], [2], [32] also that

$$R(3,t) \;\leq\; (1+o(1))\frac{t^2}{\log t} \;, \tag{1}$$

we now know that $t^2/\log t$ is the correct asymptotic order of magnitude of $R(3,t)$. Also, (1) easily gives an upper bound of $\chi(G_n^{(3)})$ which, together with Corollary 1.2, yields

$$(1-o(1))\frac{1}{9}\sqrt{\frac{n}{\log n}} \;\leq\; \max_{G_n^{(3)}} \chi(G_n^{(3)}) \;\leq\; (1+o(1))2\sqrt{2}\sqrt{\frac{n}{\log n}} \;.$$

The asymptotic behavior of $R(3,t)$ has been a major open problem in Ramsey theory for many years (see e.g [15], Appendix B of [3]). In 1961, Erdős [10] obtained a lower bound from a lovely probabilistic argument that has become a cornerstone of probabilistic methods in Combinatorics (see e.g. [3] or [6]). Graver and Yackel [16] found an upper bound in 1968 which, in conjunction with Erdős's bound, gave

$$c_1\frac{t^2}{(\log t)^2} \leq R(3,t) \leq c_2\frac{t^2 \log\log t}{\log t} \;.$$

Ajtai, Komlós and Szemerédi [1], [2] removed the $\log\log t$ factor in the upper bound, and Shearer [32] (see also [33]) reduced the constant and simplified the proof to obtain (1).

Meanwhile, the lower bound $c_1(t/\log t)^2$ defied improvement although Spencer [34], Bollobás [6], Erdős, Suen and Winkler [12] and Krivelevich [25] simplified its proof and/or increased c_1 through refined probabilistic arguments. We consider parts of their arguments later. More to the point of the present paper, Spencer [37] showed very recently that c_1 can be arbitrarily large, i.e.

$$\lim_{t\to\infty} \frac{R(3,t)}{(t/\log t)^2} = \infty \;.$$

He introduced also a differential equation which first suggested $c\,t^2/\log t$ as a lower bound and which inspired the present contribution. We consider this in Section 2.

Our approach uses the so-called "semirandom method" or "Rödl's nibble method", a version of which may have been used first in [2], the paper that removed the $\log\log t$ factor in the upper bound. More refined applications of the semirandom method were subsequently used to settle intriguing conjectures on hypergraph packings, colorings and list colorings, see e.g. [30],

[13], [28], [14], [29], [18], [19]. The present author [22] used a similar method to make progress on Vizing's old problem [38] of upper bounds for the chromatic number of a triangle-free graph of maximum degree D. We refer [18] and [22] for more on the history of the method.

The next section describes our random block construction of a triangle free graph along with Spencer's differential equation. Section 3 introduces basic parameters and proves Theorem 1.1 modulo the proof of our main lemma (Lemma 2.1) on the behavior of those parameters under the block construction. Section 4 presents tools used in the proof of the main lemma, including Azuma-Hoeffding type martingale inequalities that lead to proofs of the high concentrations of our parameters near their expected values. The main lemma is then proved in Section 5.

1.1 Block Construction

One of the most natural ways to construct a random $G_n^{(3)}$ is the following one-by-one construction. We assume henceforth that every graph has vertex set $V := \{v_1, ..., v_n\}$ and describe a graph G by its edge set $\mathcal{E}(G)$.

One-by-One Construction

(OC 1) Set $m := \binom{n}{2}$ and choose a random permutation $\pi := (e_1, ..., e_m)$ from the uniform distribution on the $m!$ permutations of the edges of K_n;

(OC 2) Let $G_0 = \emptyset$, so $\mathcal{E}(G_0)$ is empty;

(OC 3) Suppose we have G_i. For $j > i$ we say that edge e_j "survives" (see [36]) if the graph $\mathcal{E}(G_i) \cup \{e_j\}$ has no triangle. Define

$$\mathcal{E}(G_{i+1}) := \begin{cases} \mathcal{E}(G_i) \cup \{e_{i+1}\} & \text{if } e_{i+1} \text{ survives} \\ \mathcal{E}(G_i) & \text{otherwise.} \end{cases}$$

It is obvious that G_m has no triangle. However, it seems hard to find any tight upper bound on $\alpha(G_m)$. The main obstacle is the fact that the event "e_{i+1} survives" depends highly on the ordering $e_1, ..., e_i$. There is no apparent property that all surviving edges share and this makes it difficult to find a small upper bound on the variance of the random variable $|\mathcal{E}(G_{i+1})| - |\mathcal{E}(G_i)|$.

Erdős, Suen and Winkler [12] modified the preceding notion of surviving so that an edge e_{i+1} survives only if the graph $\{e_1, ..., e_{i+1}\}$ has no triangle. Their new notion and a real time version of the above construction enable them to prove the existence of a $G_n^{(3)}$ such that

$$\alpha(G_n^{(3)}) \leq (3/2 + o(1))\sqrt{n}\log n \ ,$$

which automatically yields

$$R(3,t) \geq (1 - o(1))\frac{t^2}{9(\log t)^2} \ .$$

Our block construction uses a combination of the above two notions of surviving, but is closer to the original. For our new construction, we write e_{vw} for edge $\{v, w\} \in \mathcal{E}(K_n)$. With $e, f, g \in \mathcal{E}(K_n)$, ef and efg denote the sets $\{e, f\}$ and $\{e, f, g\}$ respectively.

Block Construction with Parameter θ

(BC 1) Let $G_0 = \emptyset$ and $\mathcal{E}_0 = \emptyset$.

(BC 2) Suppose we have a set $\mathcal{E}_i$ of edges and a triangle-free graph G_i with $\mathcal{E}(G_i) \subseteq \mathcal{E}_i$. We now say that edge $e \in \mathcal{E}(K_n) \setminus \mathcal{E}_i$ survives (after stage i) if e cannot be extended to a triangle using edges from $\mathcal{E}_i$. For small $\theta > 0$ (our θ will be $(\log n)^{-2}$) the random set X_{i+1} is defined by

$$\Pr(e \in X_{i+1}) := \begin{cases} \theta/\sqrt{n} & \text{if } e \text{ survives} \\ 0 & \text{otherwise} \end{cases}$$

where all events "$e \in X_{i+1}$" are mutually independent.

(BC 3) We set $\mathcal{E}_{i+1} := \mathcal{E}_i \cup X_{i+1}$ and define the set of forbidden pairs of edges in $X_{i+1} = \mathcal{E}_{i+1} \setminus \mathcal{E}_i$ by

$$\Lambda(X_{i+1}) := \{e_{uv}e_{vw} : e_{uv}e_{vw} \subseteq X_{i+1}, e_{wu} \in \mathcal{E}_i\} \ . \tag{2}$$

Thus $ef \in \Lambda(X_{i+1})$ means that the pair ef makes a triangle with an edge g in $\mathcal{E}_i$. So we do not want both e and f in G_{i+1} regardless whether g is actually in G_i. The set of forbidden triples of edges is

$$\Delta(X_{i+1}) := \{e_{uv}e_{vw}e_{wu} : e_{uv}e_{vw}e_{wu} \subseteq X_{i+1}\} \ . \tag{3}$$

We now take a maximal disjoint collection $\mathcal{F}_{i+1}$ of elements of $\Lambda(X_{i+1}) \cup \Delta(X_{i+1})$ and define

$$\mathcal{E}(G_{i+1}) := \mathcal{E}(G_i) \cup (X_{i+1} \setminus \mathcal{E}(\mathcal{F}_{i+1})),$$

where $\mathcal{E}(\mathcal{F}_{i+1}) := \bigcup_{F \in \mathcal{F}_{i+1}} F$.

This removes all edges which are parts of elements of the maximal collection $\mathcal{F}_{i+1}$. Because of maximality, G_{i+1} has no triangles. This method, used by Krivelevich [25], will make the analysis simpler than the usual one that deletes an edge from each element of $\Lambda(X_{i+1}) \cup \Delta(X_{i+1})$.

We would like our block construction to do the following. When θ is small enough, the number of edges in $\cup_{j=0}^{i}\mathcal{E}(\mathcal{F}_j)$ should be small relative to $|\mathcal{E}_i|$ so that G_i is approximately $\mathcal{E}_i$. This is a big advantage because $\mathcal{E}_i$ has more random structure than G_i. Moreover, the notion of surviving in our block construction is now almost as loose as possible since we can not use any edge that makes a triangle with two edges in $G_i (\approx \mathcal{E}_i)$. A similar idea was first used in [2], [24].

1.2 Differential Equation

As mentioned earlier, Spencer [37] introduced a differential equation to analyze the behavior of a graph constructed by one-by-one construction. His differential equation also nicely explains block construction, which suggests that our constructions are similar. We use "$\approx$" to mean approximately equal and emphasize that what follows is only an aid to our intuition.

Suppose G_i has $\dfrac{\Psi(i\theta)n^{3/2}}{2}$ edges, where Ψ is an unknown function. This might occur because

$$\Pr\left(e \in G_i\right) \approx \frac{\Psi(i\theta)n^{3/2}}{2\binom{n}{2}} \approx \frac{\Psi(i\theta)}{\sqrt{n}} \quad \text{for all} \quad e \in \mathcal{E}(K_n)\ . \tag{4}$$

Recall that "e survives" ($e \notin \mathcal{E}_i$) if and only if there is no pair $f, g \in \mathcal{E}_i$ which together with e makes a triangle. Since $n-2$ triangles in $\mathcal{E}(K_n)$ contain e, if $G_i \approx \mathcal{E}_i$ and all events are independent, then (4) would yield

$$\Pr\left(\text{ ``}e \text{ survives after stage } i\text{''}\ \right) \approx \Big(1 - \Big(\frac{\Psi(i\theta)}{\sqrt{n}}\Big)^2\Big)^{n-2} \approx \exp(-\Psi^2(i\theta))\ , \tag{5}$$

where, as usual, $\Psi^2(i\theta) := (\Psi(i\theta))^2$. So we expect that the number of surviving edges after stage i is about $\binom{n}{2}\exp(-\Psi^2(i\theta))$.

Therefore, in expectation,

$$\begin{aligned}
|\mathcal{E}(G_{i+1})| &\approx |\mathcal{E}_{i+1}| = |\mathcal{E}_i| + |X_{i+1}| \\
&\approx \frac{\Psi(i\theta)n^{3/2}}{2} + \frac{\theta\exp(-\Psi^2(i\theta))}{\sqrt{n}}\binom{n}{2} \\
&\approx \Big(\Psi(i\theta) + \theta\exp(-\Psi^2(i\theta))\Big)\frac{n^{3/2}}{2}\ ,
\end{aligned}$$

which means that

$$\Psi((i+1)\theta) \approx \Psi(i\theta) + \theta\exp(-\Psi^2(i\theta))\ .$$

When $i\theta$ remains constant, we might obtain

$$\Psi'(i\theta) = \lim_{\theta\to 0}\frac{\Psi((i+1)\theta) - \Psi(i\theta)}{\theta} \approx \exp(-\Psi^2(i\theta))\ .$$

This yields the differential equation

$$\Psi'(x) = \exp(-\Psi^2(x))\ .$$

Since $\Psi(0)$ must be 0, $\Psi(x)$ may be defined implicitly by

$$x = \int_0^{\Psi(x)} e^{\xi^2}\ d\xi\ . \tag{6}$$

Thus $\Psi(x)$ is very close to $\sqrt{\log n}$ for sufficiently large x.

To see where this leads, set $a_0 := 0$, $b_0 := 1$ and

$$b_i := \Psi'(i\theta) = \exp(-\Psi^2(i\theta)) \ , \quad a_i := \sum_{j=0}^{i-1} b_j\theta = \sum_{j=0}^{i-1} \Psi'(j\theta)\theta \approx \Psi(i\theta) \quad \text{for} \ \ i = 1, 2, \cdots \ . \tag{7}$$

We might expect from (5) that every big subset T of V contains $b_i\binom{|T|}{2} \approx b_i|T|^2/2$ surviving edges. To be more precise, let

$$t := \lceil 9\sqrt{n \log n}\ \rceil \ , \quad \Gamma_i(T) :\approx \{e_{vw} \subseteq T : e_{vw} \text{ survives after stage } i\}$$

(cf. (9) in Section 3). In particular, $\Gamma_0(T) :\approx \{e_{vw} : e_{vw} \subseteq T\}$. What we expect is

$$|\Gamma_i(T)| \approx b_i t^2/2 \quad \text{for all } i \text{ and } T \text{ with } |T| = t. \tag{8}$$

Suppose $G_i \approx \mathcal{E}_i$. Then (8) and sufficient independence might give that

$$\Pr\left(\mathcal{E}(G_i) \cap \Gamma_0(T) = \emptyset\right) \approx \prod_{j=0}^{i-1}\Big(1 - \frac{\theta}{\sqrt{n}}\Big)^{b_j t^2/2} \le \exp\Big(-\sum_{j=0}^{i-1}\frac{b_j\theta t^2}{2\sqrt{n}}\Big) = \exp\Big(-\frac{a_i t^2}{2\sqrt{n}}\Big)$$

for each fixed t-subset T. Let $\mathcal{T}_i := \{T : |T| = t, \Gamma_0(T) \cap \mathcal{E}(G_i) = \emptyset\}$. Then, in expectation,

$$|\mathcal{T}_i| \lesssim \binom{n}{t}\exp\Big(-\frac{a_i t^2}{2\sqrt{n}}\Big) \le \exp\Big(t\log n - \frac{a_i t^2}{2\sqrt{n}}\Big) = \exp\Big(9\sqrt{n}\ (\log n)^{3/2} - (81/2)a_i\sqrt{n}\ \log n\Big) \ .$$

On the other hand, for $i_o := n^\delta/\theta$,

$$a_{i_o} = \sum_{j=0}^{i_o-1}\ \Psi'(j\theta)\theta \approx \Psi(i_o\theta) = \Psi(n^\delta) = (1+o(1))\sqrt{\delta\log n}\ .$$

If δ is not too small, then the expectation of $|\mathcal{T}_{i_o}|$ is almost 0, which implies $\alpha(G_{i_o}) \le t$ with probability almost 1. Because G_{i_o} is triangle-free, we might be done.

This suggests that Theorem 1.1 could follow easily from (8). Our main goal, in fact, is to derive a property (see Property 7 in Section 3) that is essentially the same as (8). To achieve it we define a subset Γ_i of the set of all surviving edges after stage i and adjoin fixed $\mathcal{E}_i$ and G_i to satisfy properties that allows us to continue to the next stage. The triple $(\mathcal{E}_i, \Gamma_i, G_i)$ is no longer random when we begin stage $i+1$. A small modification in Γ_i that gives up a few surviving edges to gain more regularity is noted early in Section 5.

2 Parameters and Main Lemma

This section presents our parameters and their desired properties. Throughout, $(\mathcal{E}, \Gamma, G)$ denotes a triple of two subsets of $\mathcal{E}(K_n)$ and a triangle-free graph. We set $\mathcal{E}_0 = G_0 = \emptyset$ and

$\Gamma_0 := \mathcal{E}(K_n)$. In addition, $\Delta_0 := \{efg \subseteq \mathcal{E}(K_n) : efg \text{ makes a triangle}\}$. Triple $(\mathcal{E}_i, \Gamma_i, G_i)$ for each i is required to satisfy

$$\mathcal{E}(G_i) \subseteq \mathcal{E}_i \quad \text{and} \quad \Gamma_i \subseteq \{e \in \mathcal{E}(K_n) \setminus \mathcal{E}_i : efg \notin \Delta_0 \ \text{ for all } f, g \in \mathcal{E}_i\} \ . \tag{9}$$

Our parameters depend only on $(\mathcal{E}_i, \Gamma_i, G_i)$. We denote the forbidden pairs and triples of edges in Γ_i (cf. (2) and (3)) by

$$\begin{aligned} \Lambda_i &:= \{ef \subseteq \Gamma_i : \exists\ g \in \mathcal{E}_i \ \cdot\!\ni\!\cdot \ efg \in \Delta_0\} \\ \Delta_i &:= \{efg \subseteq \Gamma_i : efg \in \Delta_0\} \ . \end{aligned}$$

Given $v \in V$, denote its neighborhood and degree in $\mathcal{E}_i$ by

$$N_{\mathcal{E}_i}(v) := \{w \in V : e_{vw} \in \mathcal{E}_i\} \ , \quad d_{\mathcal{E}_i}(v) := |N_{\mathcal{E}_i}(v)| \ ,$$

and the set of edges in Γ_i containing v as well as its neighborhood and degree in Γ_i by

$$\begin{aligned} \mathcal{N}_{\Gamma_i}(v) &:= \{e_{vw} : e_{vw} \in \Gamma_i\} \\ N_{\Gamma_i}(v) &:= \{w \in V : e_{vw} \in \Gamma_i\} = \{w \in V : e_{vw} \in \mathcal{N}_{\Gamma_i}(v)\} \\ d_{\Gamma_i}(v) &:= |N_{\Gamma_i}(v)| = |\mathcal{N}_{\Gamma_i}(v)| \ . \end{aligned}$$

For $v \in V$ and $e_{vw} \in \Gamma_i$, we denote the set of incident edges e_{uv} of v which together with e_{vw} form forbidden pairs in Γ_i, and the set of such vertices u, and their (same) size by

$$\begin{aligned} \mathcal{N}_{\Lambda_i}(e_{vw}, v) &:= \{e_{uv} \in \Gamma_i : e_{uv}e_{vw} \in \Lambda_i\} \\ N_{\Lambda_i}(e_{vw}, v) &:= \{u \in V : e_{uv}e_{vw} \in \Lambda_i\} = \{u \in V : e_{uv} \in \mathcal{N}_{\Lambda_i}(e_{vw}, v)\} \\ d_{\Lambda_i}(e_{vw}, v) &:= |N_{\Lambda_i}(e_{vw}, v)| = |\mathcal{N}_{\Lambda_i}(e_{vw}, v)| \ . \end{aligned}$$

Set

$$\begin{aligned} \mathcal{N}_{\Lambda_i}(e_{vw}) &:= \mathcal{N}_{\Lambda_i}(e_{vw}, v) \cup \mathcal{N}_{\Lambda_i}(e_{vw}, w) \\ N_{\Lambda_i}(e_{vw}) &:= N_{\Lambda_i}(e_{vw}, v) \cup N_{\Lambda_i}(e_{vw}, w) \ . \end{aligned}$$

Finally, for $e_{vw} \in \Gamma_i$, denote the set of vertices each of which together with v, w forms a triangle in Γ_i by

$$N_{\Delta_i}(e_{vw}) := \{u \in V : e_{uv}e_{vw}e_{wu} \subseteq \Gamma_i, \quad e_{uv}e_{vw}e_{wu} \in \Delta_0\} \ .$$

Our desired properties use the definitions of a_i and b_i in (7), i.e.,

$$b_i := \Psi'(i\theta) = \exp(-\Psi^2(i\theta)), \quad a_i := \sum_{j=0}^{i-1} b_j\theta = \sum_{j=0}^{i-1} \Psi'(j\theta)\theta = a_{i-1} + b_{i-1}\theta,$$

where $\theta := (\log n)^{-2}$ and Ψ is the function defined in (6). Also, for $A, B \subseteq V$, let

$$\Gamma_i(A, B) := \{e_{vw} \in \Gamma_i : v \in A, w \in B\}, \quad \text{and} \quad \Gamma_i(A) := \Gamma_i(A, A) \ .$$

We define eight properties based on the preceding concepts.

Property 1. For all $v \in V$, $d_{\mathcal{E}_i}(v) \le a_i\sqrt{n} \; + i n^{1/4}\log n$.

Property 2. For all $v \in V$, $d_{\Gamma_i}(v) \le b_i n$.

Property 3. For all $v \ne w$ in V, $|N_{\mathcal{E}_i}(v) \cap N_{\mathcal{E}_i}(w)| \le 3i\log n$.

Property 4. For all $e_{vw} \in \Gamma_i$, $d_{\Lambda_i}(e_{vw}, v) \le b_i(a_i + 5\theta)\sqrt{n}$.

Property 5. For all $e \in \Gamma_i$, $d_{\Delta_i}(e) \le b_i^2 n$.

Property 6. For all disjoint subsets A, B of V with $|A|, |B| \ge \theta^2 b_i^2\sqrt{n}$,

$$|\Gamma_i(A,B)| \le b_i|A||B| \;.$$

Similarly, for all $A \subseteq V$ with $|A| \ge \theta^2 b_i^2\sqrt{n}$

$$|\Gamma_i(A)| \le b_i\binom{|A|}{2} \;.$$

The final two properties use $t := \lceil 9\sqrt{n\log n}\;\rceil$,

$$\mu_i := 1 - 18a_i\theta - \frac{a_i}{3\sqrt{\log n}} = \mu_{i-1} - 18b_{i-1}\theta^2 - \frac{b_{i-1}\theta}{3\sqrt{\log n}}$$

and

$$\mathcal{T}_i := \{T \subseteq V : |T| = t,\ \mathcal{E}(G_i) \cap \Gamma_0(T) = \emptyset\} \;.$$

Property 7. (cf. (8)) For all $T \in \mathcal{T}_i$,

$$|\Gamma_i(T)| \ge b_i\mu_i\binom{t}{2} \;.$$

Property 8.

$$|\mathcal{T}_i| \le n^i\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{i-1}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big) \;,$$

where $\epsilon := (\log\log n)^{-1/4}$.

If (4) and (5) held (recall $a_i \approx \Psi(i\theta)$) and all events were independent, then all properties would seem quite natural except for terms such as $in^{1/4}\log n$ and $5\theta b_i\sqrt{n}$, which are basically error term estimates. For example, we would expect

$$\begin{aligned}
d_{\Lambda_i}(e_{vw}, v) &= \sum_{u \in V\setminus e_{vw}} 1(e_{wu} \in \mathcal{E}_i \text{ and } e_{uv} \text{ survives}) \\
&\approx n\Pr(e_{wu} \in \mathcal{E}_i \text{ and } e_{uv} \text{ survives}) && \text{in expectation} \\
&\approx n\Pr(e_{wu} \in \mathcal{E}_i)\Pr(e_{wu} \text{ survives}) && \text{assuming independence} \\
&\approx n(a_i/\sqrt{n})b_i = a_i b_i\sqrt{n} && \text{assuming (4) and (5).}
\end{aligned}$$

We note also that all properties are automatic for $i = 0$, and that Properties 1-6 for i are needed to guarantee Property 7 for $i+1$. Property 8 is a consequence of Property 7, as we roughly saw in the previous section.

Theorem 1.1 easily follows from our main lemma.

Lemma 2.1 (Main Lemma) *Let $\delta := 1/17 - 10^{-5}$ and $0 \le k \le \lfloor n^\delta/\theta \rfloor$. Suppose $(\mathcal{E}_k, \Gamma_k, G_k)$ satisfies (9) and Properties 1 through 8 for $i = k$. Then some triple $(\mathcal{E}_{k+1}, \Gamma_{k+1}, G_{k+1})$ satisfies (9) and Properties 1 through 8 for $i = k+1$.*

The rest of this section examines parameters a_i, b_i and μ_i, and proves Theorem 1.1 using Lemma 2.1. The following lemma presents upper and/or lower bounds of various terms involving the parameters which, while not best possible, are ideally suited to their frequent use in the rest of the paper.

Lemma 2.2 *The following inequalities hold for $i\theta \le n^\delta$ and sufficiently large n:*

$$0 \le a_i - \Psi(i\theta) \le \theta, \quad b_i > b_{i+1}, \quad b_i(a_i + 5\theta) < 1/2 \quad \textit{and} \quad b_i(a_i+5\theta)^2 < 1/2 \ ; \tag{10}$$

$$\sqrt{\log(i\theta)} - 1 \le a_i \le \sqrt{\log(i\theta)} + 1 + \theta \ , \quad \textit{for all} \quad i\theta \ge 1 \ ; \tag{11}$$

$$n^{-1/17}(\log n)^3 \le b_i \le 1, \quad 1/2 \le \mu_i \le 1 \ ; \tag{12}$$

$$|(1 - 2a_i b_i \theta) - \frac{b_{i+1}}{b_i}| \le 4 b_i \theta^2 \ ; \tag{13}$$

$$b_i^2 \le b_{i+1} b_i + \min\{b_{i+1} b_i \theta, \ 2 b_i^2 \theta (a_i + 5\theta)\} \ ; \tag{14}$$

$$\theta \sum_{j=0}^{i-1} a_j b_j \le (1/2 + \theta^{1/5}) \log(i\theta) \quad \textit{for all} \quad i \ge n^{10^{-4}}/\theta \ . \tag{15}$$

Proof. Observe that

$$\Psi(i\theta) = \int_0^{i\theta} \Psi'(\xi)\, d\xi = \sum_{j=0}^{i-1} \int_{j\theta}^{(j+1)\theta} \Psi'(\xi)\, d\xi \ .$$

Since $\Psi'(\xi) = \exp(-\Psi^2(\xi))$ is strictly decreasing and at most 1, clearly $b_{i+1} < b_i$ and

$$a_i - \theta \le \theta \sum_{j=0}^{i-1} \Psi'((j+1)\theta) \le \Psi(i\theta) = \sum_{j=0}^{i-1} \int_{j\theta}^{(j+1)\theta} \Psi'(\xi)\, d\xi \le \theta \sum_{j=0}^{i-1} \Psi'(j\theta) = a_i \ .$$

This verifies the first part of (10). The other parts of (10) follow easily from the first part and the fact that both $y \exp(-y^2)$ and $y^2 \exp(-y^2)$ are less than 0.43.

For (11) and (12) it is enough to point out that

$$\sqrt{\log x} - 1 \le \Psi(x) \le \sqrt{\log x} + 1 \quad \text{for } x \ge 1 \tag{16}$$

since this implies

$$\Psi'(x) = \exp(-\Psi^2(x)) = \exp(-(1+o(1))\log x) = x^{-1+o(1)} \ .$$

We apply Taylor's theorem to Ψ' to obtain

$$\Psi'((i+1)\theta) = \Psi'(i\theta) + \theta\Psi''(i\theta) + (\theta^2/2)\Psi'''(\xi) \ ,$$

for some ξ with $i\theta \le \xi \le (i+1)\theta$. Inequalities (13) and (14) follow routinely from this.

For (15) it is enough to consider

$$\theta \sum_{j=0}^{i-1} a_j b_j \le (1+2\theta)\int_0^{i\theta} \Psi(\xi)\Psi'(\xi)\, d\xi + \theta^2 \sum_{j=0}^{i-1} b_j \le (1/2+2\theta)\Psi^2(i\theta) \le (1/2+\theta^{1/5})\log(i\theta) \ ,$$

where (10) is used in the first inequality, and (16) and $\theta = (\log n)^{-2}$ in the last inequality.

□

Proof of Theorem 1.1 modulo Lemma 2.1. Let $k_o := \lfloor n^\delta/\theta \rfloor + 1$. Then by Lemma 2.1 we have a triangle-free graph G_{k_o} that satisfies Property 8. Thus it is enough to show that

$$n^{k_o}\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{k_o-1}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big) < 1 \tag{17}$$

(recall $t = 9\sqrt{n\log n}$.) Inequalities (15) and (11) give

$$\begin{aligned}
\theta\sum_{j=0}^{k_o-1} b_j\mu_j &= \theta\sum_{j=0}^{k_o-1} b_j - \Big(18\theta + \frac{1}{3\sqrt{\log n}}\Big)\theta\sum_{j=0}^{k_o-1} a_j b_j \\
&\ge a_{k_o} - (1/6+\theta^{1/5})\frac{\log(n^\delta+\theta)}{\sqrt{\log n}} \\
&\ge \sqrt{\delta\log n} \ -1 \ -\delta(1/6+2\theta^{1/5})\sqrt{\log n} \\
&\ge 0.23\sqrt{\log n} \ .
\end{aligned}$$

Hence

$$\begin{aligned}
\log\Big(n^{k_o}\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{k_o-1}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big)\Big) &\le k_o\log n + t\log n - 0.23(1-\epsilon)\sqrt{\frac{\log n}{n}}\binom{t}{2} \\
&\le n^{2\delta} + (9 - (0.23\cdot 81/2)(1-2\epsilon))\sqrt{n(\log n)^3} \\
&\le -0.3\sqrt{n(\log n)^3} \ < 0 \ ,
\end{aligned}$$

and (17) follows.

□

3 Tools

3.1 Azuma-Hoeffding type martingale inequalities

Most applications of the semirandom method involve Azuma-Hoeffding type martingale inequalities (from [17], [4]), which are very useful in showing that many events can happen simultaneously. Indeed, Azuma-Hoeffding type martingale inequalities, followed by Lovász's local lemma, have become the most popular way to prove the existence of certain packings, colorings and list colorings mentioned in Section 1. (See [9] for general treatment of probability, [35] for Lovász's local lemma, [27], [7], [26], [21], [19], [22] for more on Azuma-Hoeffding type inequalities, and [31], [20], [23] for simple applications.)

We need a simple version of the Azuma-Hoeffding type martingale inequality. Let $\Phi = \Phi(\tau_1, ..., \tau_m)$, where $\tau_1, ..., \tau_m$ are independent identically distributed (i.i.d) Bernoulli random variables with probability p:

$$\tau_i \in \{0,1\} \quad \text{and} \quad \Pr(\tau_i = 1) = p \ .$$

For $\tau = (\tau_1, ..., \tau_m)$, $\tau' = (\tau'_1, ..., \tau'_m)$ and $j = 1, 2, ..., m$ we write

$$\tau \equiv_j \tau' \quad \text{if} \ \ \tau_l = \tau'_l \quad \text{for all } l \neq j.$$

The following lemma is due to Kahn [19] (Proposition 3.8). We present here a simpler version, a proof of which can be found in the Appendix.

Lemma 3.1 *Suppose*

$$|\Phi(\tau) - \Phi(\tau')| \leq c_j \quad \textit{for} \ \ j = 1, 2, ..., m \quad \textit{and} \ \ \tau \equiv_j \tau' \ . \tag{18}$$

Then for all $\rho > 0$

$$\Pr(|\Phi - E[\Phi]| \geq \lambda) \leq 2 \exp\Big(- \rho\lambda + (\rho^2/2) p(1-p) \sum_{j=1}^{m} c_j^2 \exp(\rho c_j) \Big) \ .$$

Proof. A proof is given in the Appendix.

□

We have the following corollaries.

Corollary 3.2 *Suppose the hypotheses of Lemma 3.1 hold and* $\rho > 0$ *satisfies*

$$\rho \max\{c_j : j = 1, ..., m\} \leq \log 2 \ .$$

Then

$$\Pr(|\Phi - E[\Phi]| \geq \lambda) \leq 2 \exp\Big(- \rho\lambda + \rho^2 p \sum_{j=1}^{m} c_j^2 \Big) \ .$$

Proof. Since $(1-p)\exp(\rho c_j) \le 2$, for all j, we are done.

□

Corollary 3.3 *Let* $\Phi = \sum_{j=1}^{m} 1_{A_j}$ *where each* A_j *is an event that depends only on* τ_j*. Then*

$$\Pr(|\Phi - E[\Phi]| \ge \lambda) \le 2\exp\Big(-\rho\lambda + (\rho^2/2)E[\Phi]\exp(\rho)\Big)\ .$$

Proof. It is easy to see for all $\tau \equiv_j \tau'$ that

$$|\Phi(\tau) - \Phi(\tau')| \le c_j := \begin{cases} 0 & \text{if } \Pr(A_j)(1-\Pr(A_j)) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.1 says that

$$\Pr(|\Phi - E[\Phi]| \ge \lambda) \le 2\exp(-\rho\lambda + (\rho^2/2)p(1-p)\exp(\rho)\sum_{j=1}^{m} c_j^2)\ .$$

Thus it is enough to show

$$p(1-p)\sum_{j=1}^{m} c_j^2 \le E[\Phi]\ .$$

But since $\Pr(A_j) \in \{0, p, 1-p, 1\}$,

$$E[\Phi] = \sum_{j=1}^{m} \Pr(A_j) \ge \sum_{j=1}^{m} p(1-p)c_j = p(1-p)\sum_{j=1}^{m} c_j^2\ .$$

□

3.2 Disjointness Lemma

Erdős and Tetali introduced a useful lemma in [11]. We will use their proof idea rather than the lemma itself in Section 5.9, so we simply state the lemma without proof.

Let $\mathcal{A}$ be a collection of events in a probability space. We are interested in the simultaneous occurrence of many independent events in $\mathcal{A}$. Let

$$In(\mathcal{A}) := \{\mathcal{B} \subseteq \mathcal{A} : \text{all events } \{A\}_{A\in\mathcal{B}} \text{ are mutually independent}\}\ .$$

We want to have a nice upper bound on

$$\Pr\Big(\bigcup_{\substack{\mathcal{B}\in In(\mathcal{A})\\ |\mathcal{B}|=l}} \bigcap_{A\in\mathcal{B}} A\Big)\ .$$

Lemma 3.4 (Disjointness Lemma). *If* $\sum_{A\in\mathcal{A}} \Pr(A) \le \eta$, *then*

$$\Pr\Big(\bigcup_{\substack{\mathcal{B}\in In(\mathcal{A})\\ |\mathcal{B}|=l}} \bigcap_{A\in\mathcal{B}} A\Big) \le \frac{\eta^l}{l!} \quad \textit{for} \quad l = 1, 2, \cdots\ .$$

□

3.3 Almost Disjoint Covering Lemma

Suppose $A_1, ..., A_l$ are subsets of a set B. In the proof of the Main Lemma (see Sections 5.7 and 5.8), we need good upper bounds on l and $\sum_j |A_j|$ when the sets A_j are not too small but their pairwise intersections $A_j \cap A_{j'}$ are small enough. Notice that when $|A_j| \geq |B|^{1/2}$ and $A_j \cap A_{j'} = \emptyset$ for all $j \neq j'$, we easily have $l \leq |B|^{1/2}$ and $\sum_j |A_j| \leq |B|$. The following lemma considers more general cases.

Lemma 3.5 *Let B be a set of size at least 4, $A_1, ..., A_l \subseteq B$, and $1 \leq \beta, \gamma \leq |B|^{1/2}/2$. If*

$$|A_j| \geq 2\beta\gamma|B|^{1/2} \quad \textit{and} \quad |A_j \cap A_{j'}| \leq \beta^2 \quad \textit{for all} \quad j \neq j' \in [l] := \{1, ..., l\}$$

then

$$l \leq \beta^{-1}\gamma^{-1}|B|^{1/2} \quad \textit{and} \quad \sum_{j=1}^{l} |A_j| \leq \Big(1 - \frac{1}{2\gamma^2}\Big)^{-1} |B| \; .$$

Proof. Suppose to the contrary of the first part that $l \geq l_0 := \lfloor \beta^{-1}\gamma^{-1}|B|^{1/2} \rfloor + 1$. Then

$$\begin{aligned} |B| \geq |\bigcup_{j=1}^{l_0} A_j| &\geq \sum_{j=1}^{l_0} |A_j| - \sum_{1 \leq j < j' \leq l_0} |A_j \cap A_{j'}| \\ &\geq 2\beta\gamma|B|^{1/2} l_0 - (l_0)^2 \beta^2/2 \\ &\geq 2|B| - |B|/2 > |B| \; . \end{aligned}$$

This is a contradiction.

For the second part, set

$$A'_j := A_j \setminus \bigcup_{j' \neq j} A_j \quad \text{for all} \quad j \in [l].$$

Then $A'_1, ..., A'_l$ are mutually disjoint and

$$|A'_j| \geq |A_j| - \sum_{j' \neq j} |A_j \cap A_{j'}| \geq |A_j| - \beta^{-1}\gamma^{-1}|B|^{1/2}\beta^2 \geq \Big(1 - \frac{1}{2\gamma^2}\Big)|A_j| \; .$$

Therefore,

$$\sum_{j=1}^{l} |A_j| \leq \Big(1 - \frac{1}{2\gamma^2}\Big)^{-1} \sum_{j=1}^{l} |A'_j| = \Big(1 - \frac{1}{2\gamma^2}\Big)^{-1} |\bigcup_{j=1}^{l} A'_j| \leq \Big(1 - \frac{1}{2\gamma^2}\Big)^{-1} |B| \; .$$

□

4 Proof of the Main Lemma

4.1 Preliminaries

Suppose $(\mathcal{E}_k, \Gamma_k, G_k)$ as stated in Lemma 2.1. We first modify Γ_k and Λ_k to obtain exact equality in Property 4.

Modification. For each pair (e_{vw}, v) with $e_{vw} \in \Gamma_i$, let $U(e_{vw}, v)$ be a set of $\lfloor b_k(a_k + 5\theta)\sqrt{n} \rfloor - d_{\Lambda_k}(e_{vw}, v)$ new vertices so that $U(e_{vw}, v) \cap V = \emptyset$ and all $U(e_{vw}, v)$ are mutually disjoint. Define

$$\begin{aligned} V^* &:= V \cup \bigcup_{(e_{vw},v)} U(e_{vw}, v) \\ \Gamma_k^* &:= \Gamma_k \cup \bigcup_{(e_{vw},v)} \{e_{uv} : u \in U(e_{vw}, v)\} \end{aligned}$$

and

$$\Lambda_k^* := \Lambda_k \cup \bigcup_{(e_{vw},v)} \{e_{uv}e_{vw} : u \in U(e_{vw}, v)\} \ .$$

We also define $N^*(e, v)$, $\mathcal{N}^*(e, v)$, $\cdots$ analogously as in Section 3. Notice that

$$d_{\Lambda_k^*}(e_{vw}, v) = \begin{cases} b_k(a_k + 5\theta)\sqrt{n} & \text{if } e_{vw} \in \Gamma_k \\ 1 & \text{if } e_{vw} \in \Gamma_k^* \setminus \Gamma_k. \end{cases} \tag{19}$$

Given $\theta := (\log n)^{-2}$ and $p := \theta/\sqrt{n}$, we now define a random subset X_{k+1}^* of Γ_k^* (cf. (BC 2) of Section 2):

$$\Pr(e \in X_{k+1}^*) = p \quad \text{for all} \quad e \in \Gamma_k^*$$

with all events "$e \in X_{k+1}^*$" mutually independent. Let

$$\begin{aligned} X_{k+1} &:= X_{k+1}^* \cap \Gamma_k \\ \mathcal{E}_{k+1} &:= \mathcal{E}_k \cup X_{k+1} \ . \end{aligned}$$

Notice that for all $e \in \Gamma_k$,

$$\Pr(e \in X_{k+1}) = \Pr(e \in X_{k+1}^*) = p = \theta/\sqrt{n} \ , \tag{20}$$

and all events "$e \in X_{k+1}$" are mutually independent. Using (BC 3) of Section 2 we may also define a triangle-free graph G_{k+1}. Finally, set

$$\begin{aligned} Y_{k+1} &:= \{e \in \Gamma_k : \exists\ f \in X_{k+1}^* \ \cdot\!\ni\!\cdot \ ef \in \Lambda_k^*\} \\ Z_{k+1} &:= \{e \in \Gamma_k : \exists\ f, g \in X_{k+1}^* \ \cdot\!\ni\!\cdot \ efg \in \Delta_k\} \end{aligned}$$

and

$$\Gamma_{k+1} := \Gamma_k \setminus (X_{k+1} \cup Y_{k+1} \cup Z_{k+1}) \ . \tag{21}$$

The (random) triple $(\mathcal{E}_{k+1}, \Gamma_{k+1}, G_{k+1})$ obviously satisfies (9). It remains to verify

$$\Pr((\mathcal{E}_{k+1}, \Gamma_{k+1}, G_{k+1}) \ \text{ satisfies Properties 1-8}) > 0 \ .$$

We prove this by showing that $(\mathcal{E}_{k+1}, \Gamma_{k+1}, G_{k+1})$ satisfies each property with probability at least $1 - 1/n$, that is,

$$\Pr\left((\mathcal{E}_{k+1}, \Gamma_{k+1}, G_{k+1}) \ \text{ does not satisfy Property } l\right) \le 1/n \quad \text{for} \quad l = 1, ..., 8 \ . \qquad (22)$$

Three preliminary lemmas will be needed. Henceforth, we fix $k \le \lfloor n^\delta/\theta \rfloor$, omit subscript k ($\Gamma = \Gamma_k, a = a_{_k}, d_{\Lambda^*}(v) = d_{\Lambda^*_k}(v), \cdots$) and write $\Gamma', a', \cdots$ for $\Gamma_{k+1}, a_{_{k+1}}, \cdots$. Also, we just write X' for $(X^*)'$ and generally omit the asterisk if there is another superscript.

Lemma 4.1 *For all* $e \in \Gamma$

$$-14\, b\, \theta^2 \le \Pr\left(e \notin Y'\right) - \frac{b'}{b} \le -5b\theta^2 \ .$$

Proof. By (19) and $\mathcal{N}_{\Lambda^*}(e_{vw}, v) \cap \mathcal{N}_{\Lambda^*}(e_{vw}, w) = \emptyset$

$$\Pr\left(e \notin Y'\right) = \prod_{f \in \mathcal{N}_{\Lambda^*}(e)} \Pr\left(f \notin X^*\right) = (1-p)^{2b(a+5\theta)\sqrt{n}} \qquad \text{for all} \quad e \in \Gamma \ .$$

Also, since $p = \theta/\sqrt{n}$ and

$$1 - hx \le (1-x)^h \le 1 - hx + h^2x^2/2 \quad \text{for all} \quad 0 \le x \le 1, \ h \ge 2 \ ,$$

(10) gives

$$1 - 2ab\theta - 10b\theta^2 \le \Pr\left(e \notin Y'\right) \le 1 - 2b\theta(a + 5\theta) + 2\theta^2 b^2 (a + 5\theta)^2 \le 1 - 2ab\theta - 9b\theta^2 \ .$$

The upper and lower bounds of the lemma follow from (13).

□

Lemma 4.2 *Let* $\mathcal{A} \subseteq \Gamma$. *Then*

$$\sum_{e \in \Gamma^*} |\mathcal{N}_{\Lambda^*}(e) \cap \mathcal{A}| = 2b(a + 5\theta)\sqrt{n}\ |\mathcal{A}| \ .$$

Proof. By (19),

$$\begin{aligned}
\sum_{e \in \Gamma^*} |\mathcal{N}_{\Lambda^*}(e) \cap \mathcal{A}| &= \sum_{e \in \Gamma^*} \sum_{f \in \mathcal{A}} 1(ef \in \Lambda^*) \\
&= \sum_{f \in \mathcal{A}} \sum_{e \in \Gamma^*} 1(ef \in \Lambda^*) \\
&= \sum_{f \in \mathcal{A}} d_{\Lambda^*}(f) \\
&= 2b(a + 5\theta)\sqrt{n}\ |\mathcal{A}| \ .
\end{aligned}$$

□

We will prove (22) one property at a time. In most cases we first compute the expectations of corresponding random variables, then derive good concentration results using the Azuma-Hoeffding type inequalities of Section 4. Unless specified otherwise, our i.i.d Bernoulli random variables are $\{\tau_e\}_{e\in\Gamma^*}$ with $\tau_e = 1$ if and only if $e \in X^*$:

$$\Pr(\tau_e = 1) = p = \theta/\sqrt{n}\ .$$

Applications of Corollary 3.2 will simply note the c_e in the hypotheses of Lemma 3.1. If we do not mention c_e for some edges, then those edges are irrelevant.

The following lemma completes our preliminaries. Let

$$N_{X'}(v) := \{w \in V : e_{vw} \in X'\} \quad \text{for } v \in V.$$

Lemma 4.3 *The following three conditions hold simultaneously with probability at least* $1 - 3/n^2$*:*

(i) For all $v \in V$*,* $|N_{X'}(v)| \le b\theta\sqrt{n} + n^{1/4}\log n$*;*
(ii) For all $v \ne w \in V$*,* $|N_{\mathcal{E}}(v) \cap N_{X'}(w)| \le \log n$*;*
(iii) For all $v \ne w \in V$*,* $|N_{X'}(v) \cap N_{X'}(w)| \le \log n$*.*

Proof. We show that each of (i), (ii), and (iii) occurs with probability at least $1 - 1/n^2$.

For (i), let

$$\Phi_v := |N_{X'}(v)| = \sum_{w\in N_\Gamma(v)} 1(e_{vw} \in X')\ .$$

Property 2 (for $i = k$, of course) and (20) give

$$E[\Phi_v] = \sum_{w\in N_\Gamma(v)} \Pr(e_{vw} \in X') \le pbn = b\theta\sqrt{n}\ .$$

Thus it is enough to show

$$\Pr\left(\Phi_v - E[\Phi_v] \ge n^{1/4}\log n\right) \ \le\ 1/n^3 \quad \text{for all} \quad v \in V\ ,$$

which gives

$$\Pr\left(\exists\ v \in V \ \cdot\!\ni\!\cdot\ \Phi_v - E[\Phi_v] \ge n^{1/4}\log n\right) \le n(1/n^3) = 1/n^2\ .$$

Corollary 3.3 with $\rho = 4n^{-1/4}$ yields

$$\begin{aligned}\Pr\left(\Phi_v - E[\Phi_v] \ge n^{1/4}\log n\right) &\le 2\exp(-\rho n^{1/4}\log n + \rho^2 b\theta\sqrt{n}\,)\\ &\le \exp(-3\log n) = 1/n^3\ .\end{aligned}$$

For (ii), let

$$\Phi^{(1)}_{v,w} := |N_{\mathcal{E}}(v) \cap N_{X'}(w)| = \sum_{u\in N_{\mathcal{E}}(v)} 1(e_{wu} \in X')\ .$$

Property 1 and (11) give

$$E[\Phi_{v,w}^{(1)}] \le p|N_{\mathcal{E}}(v)| \le \theta^{1/2} \le 1 \ ,$$

and Corollary 3.3 with $\rho = 5$ yields

$$\begin{aligned} \Pr(\Phi_{v,w}^{(1)} \ge \log n) &\le \Pr(\Phi_{v,w}^{(1)} - E[\Phi_{v,w}^{(1)}] \ge \log n - 1) \\ &\le 2\exp(-\rho(\log n - 1) + \rho^2) \\ &\le \exp(-4\log n) = 1/n^4 \ . \end{aligned}$$

Because there are at most n^2 pairs of v, w, we are done.

For (iii), let

$$\Phi_{v,w}^{(2)} := |N_{X'}(v) \cap N_{X'}(w)| = \sum_{u \in V} 1(e_{uv}e_{wu} \subseteq X') \ .$$

Our i.i.d Bernoulli random variables in this case are $\tau_u := \tau_{e_{uv}}\tau_{e_{wu}}$, indexed by $u \in V$. Because

$$\Pr(\tau_u = 1) \le p^2 = \theta^2/n \ ,$$

we have

$$E[\Phi_{v,w}^{(2)}] \le p^2 n = \theta^2 < 1 \ .$$

Let $\rho = 5$. Then

$$\begin{aligned} \Pr(\Phi_{v,w}^{(2)} \ge \log n) &\le \Pr(\Phi_{v,w}^{(2)} - E[\Phi_{v,w}^{(2)}] \ge \log n - 1) \\ &\le 2\exp(-\rho(\log n - 1) + \rho^2) \\ &\le \exp(-4\log n) = 1/n^4 \ . \end{aligned}$$

This completes the proof.

□

We now consider each property separately to prove (22). The definitions of $\Phi_v, \Phi_{v,w}, \cdots$ vary from property to property and therefore hold only within subsections. Also, to prove (22) for the properties 2,4 and 5, we show that each vertex or pair of vertices violates the properties with probability at most $1/n^2$ or $1/n^3$, respectively. This is enough because there are at most n vertices and n^2 pairs.

4.2 Property 1

Since

$$N_{\mathcal{E}'}(v) = N_{\mathcal{E}}(v) \cup N_{X'}(v) \ ,$$

Property 1 and (i) of Lemma 4.3 give

$$\begin{aligned} d_{\mathcal{E}'}(v) = d_{\mathcal{E}}(v) + |N_{X'}(v)| &\le a\sqrt{n} + kn^{1/4}\log n + b\theta\sqrt{n} + n^{1/4}\log n \\ &= a'\sqrt{n} + (k+1)n^{1/4}\log n \end{aligned}$$

with probability at least $1 - 3/n^2 > 1 - 1/n$.

4.3 Property 2

Let

$$\Phi_v := \sum_{e \in \mathcal{N}_\Gamma(v)} 1(e \notin Y') \ .$$

Then $d_{\Gamma'}(v) \le \Phi_v$ by (21). Property 2 and Lemma 4.1 imply

$$\begin{aligned} E[\Phi_v] &= \sum_{e \in \mathcal{N}_\Gamma(v)} \Pr(e \notin Y') \\ &\le bn(b'/b - 5b\theta^2) \le b'n - b^2\theta^2 n \ . \end{aligned}$$

We take

$$c_e = |\mathcal{N}_{\Lambda^*}(e) \cap \mathcal{N}_\Gamma(v)| \le 2b(a+5\theta)\sqrt{n}$$

for all $e \in \Gamma^*$. Lemma 4.2 and Property 2 then give

$$\sum_e c_e^2 \le 2b(a+5\theta)\sqrt{n} \ \sum_e c_e \le 4b^2(a+5\theta)^2 n \cdot bn \le n^2 \ .$$

Together with (12) and Corollary 3.2 with $\rho = n^{-3/4}$, this yields

$$\begin{aligned} \Pr(\Phi_v - E[\Phi_v] \ge b^2\theta^2 n) &\le 2\exp(-b^2\theta^2 n^{1/4} + (\theta/\sqrt{n}\ )n^{1/2}) \\ &\le \exp(-n^{1/4-2/17}) \le 1/n^2 \ . \end{aligned}$$

4.4 Property 3

Since

$$|N_{\mathcal{E}'}(v) \cap N_{\mathcal{E}'}(w)| \le |N_{\mathcal{E}}(v) \cap N_{\mathcal{E}}(w)| + |N_{X'}(v) \cap N_{\mathcal{E}}(w)| + |N_{\mathcal{E}}(v) \cap N_{X'}(w)| + |N_{X'}(v) \cap N_{X'}(w)| \ ,$$

we are done by Property 3 and (ii), (iii) of Lemma 4.3.

4.5 Property 4

For $e_{vw} \in \Gamma \setminus X'$ let

$$\Phi^{(1)}_{v,w} := \sum_{e \in \mathcal{N}_\Lambda(e_{vw}, v)} 1(e \notin Y')$$

and

$$\Phi^{(2)}_{v,w} := \sum_{u \in N_\Delta(e_{vw})} 1(e_{wu} \in X') \ .$$

Note that $\Phi^{(1)}_{v,w}$ and $\Phi^{(2)}_{v,w}$ are considered under the condition $e_{vw} \notin X'$. Clearly,

$$d_{\Lambda'}(e_{vw}, v) \le \Phi^{(1)}_{v,w} + \Phi^{(2)}_{v,w} \quad \text{for all } e_{vw} \in \Gamma'. \tag{23}$$

Because events "$e \in X'$" are mutually independent, Property 5 yields

$$E[\Phi^{(2)}_{v,w} | e_{vw} \notin X'] = E[\Phi^{(2)}_{v,w}] = p d_\Delta(e_{vw}) \le b^2\theta\sqrt{n} \ .$$

Thus (12) and Corollary 3.3 with $\rho = n^{-1/4}$ imply that

$$\begin{aligned}
\Pr(\Phi^{(2)}_{v,w} - b^2\theta\sqrt{n} \ \geq b^2\theta^2(a+5\theta)\sqrt{n}\ ) &\leq \Pr(\Phi^{(2)}_{v,w} - E[\Phi^{(2)}_{v,w}] \geq b^2\theta^2(a+5\theta)\sqrt{n}\ ) \\
&\leq 2\exp(-b^2\theta^2(a+5\theta)n^{1/4} + b^2\theta) \\
&\leq \exp(-n^{1/4-2/17}) \leq 1/n^4 \ . \qquad (24)
\end{aligned}$$

We now consider $\Phi^{(1)}_{v,w}$. Since

$$\Pr(e \notin Y' | e_{vw} \notin X') = \Pr(e \notin Y')(1-p)^{-1} \quad \text{for all} \quad e \in \mathcal{N}_\Lambda(e_{vw}, v)$$

Lemma 4.1 and Property 4 give

$$\begin{aligned}
E[\Phi^{(1)}_{v,w}] &\leq (1+2p)\Pr(e \notin Y')d_\Lambda(e_{vw}, v) \\
&\leq (1+2\theta/\sqrt{n}\ )(b'/b - 5b\theta^2)\, b(a+5\theta)\sqrt{n} \\
&\leq b'(a+5\theta)\sqrt{n} \ - 4b^2\theta^2(a+5\theta)\sqrt{n}\ . \qquad (25)
\end{aligned}$$

Let

$$c_e := |\mathcal{N}_\Lambda(e_{vw}, v) \cap \mathcal{N}_{\Lambda^*}(e)| \quad \text{for } e \neq e_{vw}.$$

Since

$$N_\Lambda(e_{uv}, v) \subseteq N_{\mathcal{E}}(u) \quad \text{for all} \quad e_{uv} \in \Gamma, \qquad (26)$$

Property 3 implies for $e_{uv} \in \Gamma$ that

$$\begin{aligned}
|\mathcal{N}_\Lambda(e_{vw}, v) \cap \mathcal{N}_{\Lambda^*}(e_{uv}, v)| &= |N_\Lambda(e_{vw}, v) \cap N_\Lambda(e_{uv}, v)| \\
&\leq |N_{\mathcal{E}}(w) \cap N_{\mathcal{E}}(u)| \leq 3k\log n\ .
\end{aligned}$$

Hence, for all $e \in \Gamma^* \setminus \{e_{vw}\}$,

$$c_e \leq 1 + 3k\log n\ .$$

Thus Lemma 4.2 and Property 4 yield

$$\sum_{e\in\Gamma^*\setminus\{e_{vw}\}} c_e^2 \leq (1+3k\log n) \sum_{e\in\Gamma^*\setminus\{e_{vw}\}} c_e = 2(1+3k\log n)b^2(a+5\theta)^2 n\ .$$

Let $\rho = n^{-1/4}$. Then

$$\begin{aligned}
&\Pr(\Phi^{(1)}_{v,w} - E[\Phi^{(1)}_{v,w}] \geq b^2\theta^2(a+5\theta)\sqrt{n}\ ) \\
\leq\ & 2\exp(-b^2\theta^2(a+5\theta)n^{1/4} + 2(\theta/\sqrt{n}\ )(1+3k\log n)b^2(a+5\theta)^2 n^{1/2}) \\
\leq\ & \exp(-n^{1/4-2/17}) \leq 1/n^4\ . \qquad (27)
\end{aligned}$$

It now follows from (25), (27), (24), and (14) that, with probability at least $1 - 2/n^4$,

$$\Phi^{(1)}_{v,w} + \Phi^{(2)}_{v,w} \ \leq b'(a+5\theta)\sqrt{n} \ - 2b^2\theta^2(a+5\theta)\sqrt{n} \ + b^2\theta\sqrt{n} \ \leq b'(a'+5\theta)\sqrt{n}\ .$$

By (23) we are done.

4.6 Property 5

Let

$$\Phi_{vw} := \sum_{u \in N_\Delta(e_{vw})} 1(e_{uv}e_{wu} \cap Y' = \emptyset)$$

for each $e_{vw} \in \Gamma$. Then

$$d_{\Delta'}(e_{vw}) \leq \Phi_{vw} \ . \tag{28}$$

Because (26) yields

$$|N_\Lambda(e_{uv}, u) \cap N_\Lambda(e_{wu}, u)| \leq |N_\mathcal{E}(v) \cap N_\mathcal{E}(w)| \leq 3k \log n \ ,$$

Lemma 4.1 gives

$$\begin{aligned}
\Pr(e_{uv}e_{wu} \cap Y' = \emptyset) &= \Pr(e_{uv} \notin Y')\Pr(e_{wu} \notin Y' | e_{uv} \notin Y') \\
&\leq \Pr(e_{uv} \notin Y')\Pr(e_{wu} \notin Y')(1-p)^{-3k \log n} \\
&\leq (b'/b - 5b\theta^2)^2(1 + 4\theta k (\log n)/\sqrt{n}\ ) \\
&\leq (b'/b)^2 - b\theta^2 \ .
\end{aligned}$$

Property 5 easily yields

$$\begin{aligned}
E[\Phi_{vw}] &= \sum_{u \in N_\Delta(e_{vw})} \Pr(e_{uv}e_{wu} \cap Y' = \emptyset) \\
&\leq ((b'/b)^2 - b\theta^2)b^2 n \leq (b')^2 n - b^3\theta^2 n \ .
\end{aligned} \tag{29}$$

Let

$$c_e := \begin{cases} b(a+5\theta)\sqrt{n} & \text{if } N_{\Lambda^*}(e) \cap N_\Delta(e_{vw}) \neq \emptyset \text{ and } e_{vw} \cap e \neq \emptyset \\ 2 & \text{if } N_{\Lambda^*}(e) \cap (N_\Delta(e_{vw}) \cup e_{vw}) \neq \emptyset \text{ and } e_{vw} \cap e = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $e \in \Gamma^*$. (Actually, we may take $c_e = 1$ for the second case.) Clearly, $c_e \geq 2$ for at most $2b(a+5\theta)\sqrt{n}\ \cdot 2b^2 n$ edges (cf. Lemma 4.2), and $c_e > 2$ for at most $2n$ edges. Hence

$$\sum c_e^2 \leq 4 \cdot 4b^3(a+5\theta)n^{3/2} + 2nb^2(a+5\theta)^2 n \leq 3b^2(a+5\theta)^2 n^2 \ .$$

Then (28), (29), (12), and Corollary 3.2 with $\rho = n^{-5/8}$ imply that

$$\begin{aligned}
\Pr(d_{\Delta'}(e_{vw}) \geq (b')^2 n) &\leq \Pr(\Phi_{vw} - E[\Phi_{vw}] \geq b^3\theta^2 n) \\
&\leq 2\exp(-b^3\theta^2 n^{3/8} + 3(\theta/\sqrt{n}\ )b^2(a+5\theta)^2 n^{3/4}) \\
&\leq \exp(-n^{3/8-3/17}) \leq 1/n^3 \ .
\end{aligned}$$

4.7 Property 6

We prove only the first part. An analogous proof holds for the other part.

It is enough to prove the property for all A, B with $|A| = |B| = b^2\theta^2\sqrt{n}$. (Of course, we should really write $\lfloor b^2\theta^2\sqrt{n}\rfloor$ here.) Let

$$\mathcal{L} := \{(e, v) \in \Gamma^* \times V^* : v \in e\}$$

and for $A, B \subseteq V$ set

$$\mathcal{L}^{(1)}(A, B) := \{(e, v) \in \mathcal{L} : |\mathcal{N}_{\Lambda^*}(e, v) \cap \Gamma(A, B)| < 4((k+1)\log n)^{1/2}|A \cup B|^{1/2}\} . \tag{30}$$

Because

$$Y' = \bigcup_{\substack{e \in X^* \\ (e,v) \in \mathcal{L}}} \mathcal{N}_{\Lambda^*}(e, v) \cap \Gamma , \tag{31}$$

we further define

$$Y^{(1)}(A, B) := \bigcup_{\substack{e \in X^* \\ (e,v) \in \mathcal{L}^{(1)}(A,B)}} \mathcal{N}_{\Lambda^*}(e, v) \cap \Gamma(A, B) \quad \text{and} \quad \Phi_{A,B} := \sum_{e \in \Gamma(A,B)} 1(e \notin Y^{(1)}(A, B)) .$$

Then

$$|\Gamma'(A, B)| \leq \Phi_{A,B} . \tag{32}$$

In regard to the expectation of $\Phi_{A,B}$, we claim that, for all $e_{vw} \in \Gamma(A, B)$,

$$\Pr(e_{vw} \notin Y^{(1)}(A, B)) \leq \Pr(e_{vw} \notin Y')(1-p)^{-2n^{1/4}} . \tag{33}$$

This holds if

$$|\{u \in N_{\Lambda^*}(e_{vw}, v) : (e_{uv}, v) \notin \mathcal{L}^{(1)}(A, B)\}| \leq n^{1/4} \tag{34}$$

and

$$|\{u \in N_{\Lambda^*}(e_{vw}, w) : (e_{wu}, w) \notin \mathcal{L}^{(1)}(A, B)\}| \leq n^{1/4} . \tag{35}$$

By (26), $(e_{uv}, v) \notin \mathcal{L}^{(1)}(A, B)$ implies that

$$\begin{aligned} 4((k+1)\log n)^{1/2}|A \cup B|^{1/2} &\leq |\mathcal{N}_{\Lambda^*}(e_{uv}, v) \cap \Gamma(A, B)| \\ &\leq |N_{\Lambda}(e_{uv}, v) \cap (A \cup B)| \leq |N_{\mathcal{E}}(u) \cap (A \cup B)| . \end{aligned}$$

This inequality, Property 3, and Lemma 3.5 with $\beta = (3(k+1)\log n)^{1/2}, \gamma = 1$ then imply that there are at most $(3(k+1)\log n)^{-1/2}|(A \cup B)|^{1/2}$ such $u \in V^*$. Since

$$(3(k+1)\log n)^{-1/2}|(A \cup B)|^{1/2} \leq n^{1/4} ,$$

(34) follows. By the same method (35) also holds.

Lemma 4.1, Property 6, (33), and our condition of $|A| = |B| = b^2\theta^2\sqrt{n}$ imply

$$\Pr\left(e_{vw} \notin Y^{(1)}(A,B)\right) \le (b'/b - 5b\theta^2)(1 + n^{-1/4}) \le b'/b - b\theta^2$$

and

$$E[\Phi_{A,B}] \le (b'/b - b\theta^2)|\Gamma(A,B)| \le b'|A||B| - b^6\theta^6 n \ . \tag{36}$$

Let

$$c_e := \min\{4((k+1)\log n)^{1/2}|A\cup B|^{1/2}, |\mathcal{N}_{\Lambda^*}(e) \cap \Gamma(A,B)|\} \le 6((k+1)\log n)^{1/2} b\theta n^{1/4}$$

for all $e \in \Gamma^*$. Then Lemma 4.2 and Property 6 give

$$\begin{aligned}\sum c_e^2 &\le 6((k+1)\log n)^{1/2} b\theta n^{1/4} \sum c_e \\ &\le 6((k+1)\log n)^{1/2} b\theta n^{1/4} \cdot 2b(a+5\theta)n^{1/2}|\Gamma(A,B)| \\ &\le 12((k+1)\log n)^{1/2} b^7\theta^5(a+5\theta)n^{7/4} \ .\end{aligned}$$

It follows from (12) and Corollary 3.2 with $\rho = n^{-1/4-\varepsilon_o}$, where $\varepsilon_o := 1/4 - 4/17 = 1/68$, that

$$\begin{aligned}&\Pr\left(\Phi_{A,B} - E[\Phi_{A,B}] \ge b^6\theta^6 n\right) \\ \le\ & 2\exp\Big(-b^6\theta^6 n^{3/4-\varepsilon_o} + 12(\theta/\sqrt{n}\,)((k+1)\log n)^{1/2} b^7\theta^5(a+5\theta)n^{5/4-2\varepsilon_o}\Big) \\ \le\ & \exp(-b^2\theta^2(\log n)^4 n^{3/4-4/17-\varepsilon_o}/2) \\ \le\ & \exp(-b^2\theta^2(\log n)^2 n^{1/2}) \ .\end{aligned}$$

This result along with (32) and (36) then gives

$$\begin{aligned}&\Pr\Big(\exists \text{ disjoint } A,B\subseteq V \text{ with } |A|=|B|=b^2\theta^2\sqrt{n} \ \cdot\ni\cdot\ |\Gamma'(A,B)| > b'|A||B|\Big) \\ \le\ & \Pr\Big(\exists \text{ disjoint } A,B\subseteq V \text{ with } |A|=|B|=b^2\theta^2\sqrt{n} \ \cdot\ni\cdot\ \Phi_{A,B} - E[\Phi_{A,B}] \ge b^6\theta^6 n\Big) \\ \le\ & \binom{n}{b^2\theta^2 n^{1/2}}^2 \exp(-b^2\theta^2 n^{1/2}(\log n)^2) \\ \le\ & \exp(2b^2\theta^2 n^{1/2}\log n - b^2\theta^2 n^{1/2}(\log n)^2) \le 1/n \ .\end{aligned}$$

4.8 Property 7

Recall for $T \in \mathcal{T}$ that $|T| = t = \lceil 9\sqrt{n\log n}\,\rceil$. Let $T \in \mathcal{T}$. (Actually, our proof works for all T of size t.) We know by (21) that

$$|\Gamma'(T)| \ge |\Gamma(T)| - |X' \cap \Gamma(T)| - |Y' \cap \Gamma(T)| - |Z' \cap \Gamma(T)| \ . \tag{37}$$

We verify Property 7 by first proving

$$\Pr\left(\exists\, T \in \mathcal{T} \ \cdot\ni\cdot\ |X' \cap \Gamma(T)| \ge b'\theta^2|\Gamma(T)|/2\ \right) \ \le\ 1/n^2 \ . \tag{38}$$

Proof of (38). Set

$$\Phi_T^{(0)} := |X' \cap \Gamma(T)| = \sum_{e \in \Gamma(T)} 1(e \in X') \ .$$

Then

$$E[\Phi_T^{(0)}] = |\Gamma(T)| \Pr(e \in X') = (\theta/\sqrt{n}\ )|\Gamma(T)| \ ,$$

and Corollary 3.3 with $\rho = n^{-5/17}$, Property 7 and (12) give

$$\begin{aligned}
\Pr(\Phi_T^{(0)} \geq b'\theta^2|\Gamma(T)|/2) &\leq \Pr(\Phi_T^{(0)} - E[\Phi_T^{(0)}] \geq b'\theta^2|\Gamma(T)|/4) \\
&\leq 2\exp(-\rho b'\theta^2|\Gamma(T)|/4 + \rho^2(\theta/\sqrt{n}\ )|\Gamma(T)|) \\
&\leq \exp(-\rho b b'\theta^2 n) \\
&\leq \exp(-n^{1-5/17-2/17}) = \exp(-n^{10/17}) \ .
\end{aligned}$$

Thus

$$\begin{aligned}
\Pr(\exists\ T \in \mathcal{T} \quad \cdot\ni\cdot\ \ \Phi_T^{(0)} \geq b'\theta^2|\Gamma(T)|/2\ ) &\leq \binom{n}{t}\exp(-n^{10/17}) \\
&\leq \exp(9\sqrt{n}\ (\log n)^{3/2} - n^{10/17}) \leq 1/n^2 \ .
\end{aligned}$$

□

We now divide $|Y' \cap \Gamma(T)|$ into two parts (cf. (30)). Let

$$\begin{aligned}
\mathcal{L}^{(1)}(T) &:= \{(e,v) \in \mathcal{L} : |\mathcal{N}_{\Lambda^*}(e,v) \cap \Gamma(T)| < 4((k+1)\log n)^{1/2}|T|^{1/2}\}, \\
\mathcal{L}^{(2)}(T) &:= \mathcal{L} \setminus \mathcal{L}^{(1)}(T) \ ,
\end{aligned}$$

and

$$\begin{aligned}
Y^{(1)}(T) &:= \bigcup_{\substack{e \in X^* \\ (e,v) \in \mathcal{L}^{(1)}(T)}} \mathcal{N}_{\Lambda^*}(e,v) \cap \Gamma(T) \ , \\
Y^{(2)}(T) &:= \bigcup_{\substack{e \in X^* \\ (e,v) \in \mathcal{L}^{(2)}(T)}} \mathcal{N}_{\Lambda^*}(e,v) \cap \Gamma(T) \ .
\end{aligned}$$

The corresponding random variables are

$$\Phi_T^{(1)} := |\Gamma(T)| - |Y^{(1)}(T)| = \sum_{e \in \Gamma(T)} 1(e \notin Y^{(1)}(T)) \quad \text{and} \quad \Phi_T^{(2)} := |Y^{(2)}(T)| \ .$$

By (31),

$$|\Gamma(T)| - |Y' \cap \Gamma(T)| \geq \Phi_T^{(1)} - \Phi_T^{(2)} \ .$$

We claim that

$$\Pr(\exists\ T \in \mathcal{T} \ \cdot\ni\cdot\ \ \Phi_T^{(1)} \leq (b'/b - 15b'\theta^2)|\Gamma(T)|\ ) \leq 1/n^2 \tag{39}$$

and that

$$\Pr\Big(\exists\ T \in \mathcal{T} \ \ \cdot\ni\cdot\ \ \Phi_T^{(2)} > \frac{2b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2}\ \Big) \le 3/n^2\ , \tag{40}$$

which imply directly that, with probability at least $1-4/n^2$,

$$|\Gamma(T)| - |Y' \cap \Gamma(T)| \ge (b'/b - 15b'\theta^2)|\Gamma(T)| - \frac{2b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2}\ . \tag{41}$$

Proof of (39). Lemma 4.1 (the lower bound) gives

$$E[\Phi_T^{(1)}] = \sum_{e\in\Gamma(T)} \Pr(e \notin Y^{(1)}(T)) \ge \sum_{e\in\Gamma(T)} \Pr(e \notin Y') \ge (b'/b - 14b\theta^2)|\Gamma(T)|\ .$$

Since (13) yields $b = (1+O(\theta))b'$, we have

$$\Pr(\Phi_T^{(1)} \le (b'/b - 15b'\theta^2)|\Gamma(T)|) \le \Pr(\Phi_T^{(1)} - E[\Phi_T^{(1)}] \le -b'\theta^2|\Gamma(T)|/2)\ . \tag{42}$$

Set

$$c_e := \min\{|\mathcal{N}_{\Lambda^*}(e) \cap \Gamma(T)|, 8((k+1)\log n)^{1/2}t^{1/2}\}\ .$$

Then Lemma 4.2 yields

$$\begin{aligned}\sum c_e^2 &\le 8((k+1)\log n)^{1/2}t^{1/2}\sum c_e \\ &\le 16((k+1)\log n)^{1/2}t^{1/2}b(a+5\theta)\sqrt{n}\ |\Gamma(T)|\ .\end{aligned}$$

Let $\rho = n^{-1/4-1/17}$. Then Property 7 and (12) imply that

$$\begin{aligned}&\Pr(\Phi_T^{(1)} - E[\Phi_T^{(1)}] \le -b'\theta^2|\Gamma(T)|/2) \\ \le\ & 2\exp\Big(-\rho b'\theta^2|\Gamma(T)|/2 + 16\rho^2(\theta/\sqrt{n}\ )((k+1)\log n)^{1/2}t^{1/2}b(a+5\theta)\sqrt{n}\ |\Gamma(T)|\Big) \\ \le\ & \exp(-n^{1-1/4-3/17}) \le \exp(-\sqrt{n}\ (\log n)^2)\ .\end{aligned}$$

We combine this with (42) and the inequality

$$\exp(-\sqrt{n}\ (\log n)^2)\binom{n}{t} \le 1/n^2$$

to obtain (39).

□

Proof of (40). It is enough to show that (i), (ii), and (iii) of Lemma 4.3 imply (40).

Take $N_{\mathcal{E}}(w,T) := N_{\mathcal{E}}(w) \cap T$ and let

$$W_T := \{w \in V : |N_{\mathcal{E}}(w,T)| \ge 4((k+1)\log n)^{1/2}t^{1/2}\}\ .$$

We show first that

$$Y^{(2)}(T) \subseteq \bigcup_{w \in W_T} \Gamma(N_{\mathcal{E}}(w,T), N_{X'}(w,T)) \tag{43}$$

where, as usual, $N_{X'}(w,T) := N_{X'}(w) \cap T$.

Suppose $f \in Y^{(2)}(T) \subseteq \Gamma(T)$. Then there exists $(e,v) \in \mathcal{L}^{(2)}(T)$, say $e = e_{vw}$, such that $f \in \mathcal{N}_{\Lambda^*}(e_{vw}, v)$ and $e_{vw} \in X'$. In particular, $v \in f \subseteq T$, and the other vertex u of f is in $N_{\mathcal{E}}(w,T)$. Moreover, since $e_{vw} \in X'$, we have $v \in N_{X'}(w,T)$. Since $(e_{vw}, v) \in \mathcal{L}^{(2)}(T)$, we know also by (26) that

$$4((k+1)\log n)^{1/2} t^{1/2} \le |\mathcal{N}_{\Lambda^*}(e_{vw}, v) \cap \Gamma(T)| = |N_{\Lambda}(e_{vw}, v) \cap T| \le |N_{\mathcal{E}}(w,T)| \ .$$

Thus $w \in W_T$, and the proof of (43) is complete.

Let

$$A(w,T) := \begin{cases} N_{\mathcal{E}}(w,T) & \text{if } |N_{\mathcal{E}}(w,T)| \ge |N_{X'}(w,T)| \\ N_{X'}(w,T) & \text{otherwise} \end{cases}$$

and for summations let

$$\sum{}' := \sum_{\substack{w \in W_T \\ \min\{|N_{\mathcal{E}}(w,T)|, |N_{X'}(w,T)|\} \le b^2\theta^2\sqrt{n}}} \quad \text{and} \quad \sum{}'' := \sum_{\substack{w \in W_T \\ \min\{|N_{\mathcal{E}}(w,T)|, |N_{X'}(w,T)|\} > b^2\theta^2\sqrt{n}}} .$$

Note that

$$|A(w,T)| \ge |N_{\mathcal{E}}(w,T)| \ge 4((k+1)\log n)^{1/2} t^{1/2} \quad \text{for all} \quad w \in W_T \ . \tag{44}$$

Property 6, (i) of Lemma 4.3, and (43) yield

$$\begin{aligned} |Y^{(2)}(T)| &\le \sum_{w \in W_T} |\Gamma(N_{\mathcal{E}}(w,T), N_{X'}(w,T))| \\ &\le \sum{}' |N_{\mathcal{E}}(w,T)||N_{X'}(w,T)| + b\sum{}'' |N_{\mathcal{E}}(w,T)||N_{X'}(w,T)| \\ &\le b^2\theta^2\sqrt{n} \sum{}' |A(w,T)| + (1+\theta)b^2\theta\sqrt{n} \sum{}'' |N_{\mathcal{E}}(w,T)| \ . \end{aligned}$$

Moreover, Lemma 3.5 (with $\beta = \sqrt{3(k+1)\log n}$ for both of the following inequalities, $\gamma = 1$ for the first inequality, and $\gamma = \log n$ for the second), Property 3, (ii) and (iii) of Lemma 4.3, and (44) imply

$$\sum{}' |A(w,T)| \le 2t \quad \text{and} \quad \sum{}'' |N_{\mathcal{E}}(w,T)| \le (1+\theta)t \ .$$

Therefore, by (14),

$$|Y^{(2)}(T)| \le 2b^2\theta^2\sqrt{n}\ t + (1+\theta)^2 b^2\theta\sqrt{n}\ t \le \frac{2b'b\theta(1+6\theta)}{9\sqrt{\log n}} \binom{t}{2} \ .$$

□

We return to (37) and divide $|Z' \cap \Gamma(T)|$ into two parts. We recall that $N_{X'}(v,T) = N_{X'}(v) \cap T$ and note that

$$Z' \cap \Gamma(T) = \bigcup_{v \in V} \Gamma(N_{X'}(v)) \cap \Gamma(T) = \bigcup_{v \in V} \Gamma(N_{X'}(v) \cap T) = \bigcup_{v \in V} \Gamma(N_{X'}(v,T)) \; .$$

Let $h := \lceil 4((k+1)\log n)^{1/2} t^{1/2} \rceil$. Given $N_{X'}(v,T) = \{v_{j_1}, ..., v_{j_r}\}$ with $j_1 < \cdots < j_r$, let

$$\hat{N}_{X'}(v,T) := \begin{cases} N_{X'}(v,T) & \text{if } r \le h \\ \{v_{j_1}, ..., v_{j_h}\} & \text{if } r > h. \end{cases}$$

Then

$$Z' \cap \Gamma(T) = \bigcup_{v \in V} \Gamma(\hat{N}_{X'}(v,T)) \quad \cup \bigcup_{\substack{v \in V \\ |N_{X'}(v)| \ge h}} \Gamma(N_{X'}(v,T)) \; .$$

Let

$$\Phi_T^{(3)} := \sum_{v \in V} |\Gamma(\hat{N}_{X'}(v,T))|, \quad \text{and} \quad \Phi_T^{(4)} := \sum_{\substack{v \in V \\ |N_{X'}(v,T)| \ge h}} |\Gamma(N_{X'}(v,T))| \; .$$

Clearly

$$|Z' \cap \Gamma(T)| \le \Phi_T^{(3)} + \Phi_T^{(4)} \; . \tag{45}$$

We now claim that

$$\Pr(\Phi_T^{(3)} \ge 2b'\theta^2 |\Gamma(T)| \,) \;\le\; 2\exp(-\sqrt{n}\,(\log n)^2) \quad \text{for all } T \in \mathcal{T}, \tag{46}$$

which implies

$$\Pr(\exists T \in \mathcal{T} \;\cdot\!\ni\!\cdot\; \Phi_T^{(3)} \ge 2b'\theta^2 |\Gamma(T)| \,) \le 1/n^2. \tag{47}$$

Proof of (46). Property 5 gives, for expectation,

$$\begin{aligned} E[\Phi_T^{(3)}] &\le E[\sum_{v \in V} |\Gamma(N_{X'}(v,T))| \,] \\ &= \sum_{v \in V} \sum_{\substack{e_{wu} \in \Gamma(T) \\ v \in N_\Delta(e_{wu})}} \Pr(e_{uv} e_{vw} \subseteq X') \\ &= p^2 \sum_{e_{wu} \in \Gamma(T)} \sum_{v \in N_\Delta(e_{wu})} 1 \\ &\le p^2 b^2 n |\Gamma(T)| = b^2 \theta^2 |\Gamma(T)| \le (3/2) b' \theta^2 |\Gamma(T)| \; . \end{aligned}$$

Thus

$$\Pr(\Phi_T^{(3)} \ge 2b'\theta^2 |\Gamma(T)|\,) \;\le\; \Pr(\Phi_T^{(3)} - E[\Phi_T^{(3)}] \ge b'\theta^2 |\Gamma(T)|/2) \; .$$

We obtain a concentration result by taking

$$\Phi_T^{(5)} := \sum_{v \in T} |\Gamma(\hat{N}_{X'}(v,T))| \quad \text{and} \quad \Phi_T^{(6)} := \sum_{v \in V \setminus T} |\Gamma(\hat{N}_{X'}(v,T))| \; .$$

Since

$$\Phi_T^{(3)} = \Phi_T^{(5)} + \Phi_T^{(6)} \ ,$$

it is enough to show that

$$\Pr\left(\Phi_T^{(5)} - E[\Phi_T^{(5)}] \ge b'\theta^2|\Gamma(T)|/4\right) \le \exp(-\sqrt{n}\ (\log n)^2) \tag{48}$$

and

$$\Pr\left(\Phi_T^{(6)} - E[\Phi_T^{(6)}] \ge b'\theta^2|\Gamma(T)|/4\right) \le \exp(-\sqrt{n}\ (\log n)^2)\ . \tag{49}$$

Consider (48). Let

$$c_e := \begin{cases} 2h & \text{if } e \in \Gamma(T) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum c_e^2 = 4h^2|\Gamma(T)|\ .$$

We take $\rho = n^{-1/4-1/17}$ to obtain

$$\begin{aligned} \Pr\left(\Phi_T^{(5)} - E[\Phi_T^{(5)}] \ge b'\theta^2|\Gamma(T)|/4\right) &\le 2\exp(-\rho b'\theta^2|\Gamma(T)|/4 + 4\rho^2(\theta/\sqrt{n}\ )h^2|\Gamma(T)|) \\ &\le \exp(-n^{1-1/4-3/17}) \ \le \ \exp(-\sqrt{n}\ (\log n)^2)\ . \end{aligned}$$

Consider (49). All random variables $|\Gamma(\hat{N}_{X'}(v,T))|$ for $v \in V \setminus T$ are mutually independent, so

$$E[\exp(\rho\Phi_T^{(6)})] = \prod_{v \in V\setminus T} E[\exp(\rho|\Gamma(\hat{N}_{X'}(v,T))|)]\ . \tag{50}$$

Our aim is to find a good upper bound of $E[\exp(\rho|\Gamma(\hat{N}_{X'}(v,T))|)]$ so that we may apply Markov's inequality with the function $\exp(\rho x)$ (see (53)). Let

$$\phi_v(\rho) := E[\ \exp(\rho|\Gamma(\hat{N}_{X'}(v,T))|)\ ]\ .$$

Then there is ρ^* with $0 \le \rho^* \le \rho$ such that

$$\phi_v(\rho) = 1 + \rho\, E[\ |\Gamma(\hat{N}_{X'}(v,T))|\ ] + (\rho^2/2)\phi_v''(\rho^*)\ .$$

We prove the following claim.

Claim.

$$\phi_v''(\rho^*) \le 1 \quad \text{if } \rho^* \le h^{-1}\ . \tag{51}$$

Proof. Let $B = B(v,T) := N_\Gamma(v) \cap T$. Then, since $\rho^* \le h^{-1}$,

$$\begin{aligned} \phi_v''(\rho^*) &= E[\ |\Gamma(\hat{N}_{X'}(v,T))|^2 \exp(\rho^*|\Gamma(\hat{N}_{X'}(v,T))|)\ ] \\ &\le \sum_{l=0}^{h-1}\binom{|B|}{l}\binom{l}{2}^2 \exp\left(\rho^*\binom{l}{2}\right)p^l(1-p)^{|B|-l} + \sum_{l=h}^{|B|}\binom{|B|}{l}\binom{h}{2}^2 \exp\left(\rho^*\binom{h}{2}\right)p^l(1-p)^{|B|-l} \\ &\le \frac{1}{4}\sum_{l=2}^{|B|}\binom{|B|}{l}(l+2)(l+1)l(l-1)\exp(l/2)p^l(1-p)^{|B|-l}\ . \end{aligned} \tag{52}$$

Let

$$\omega(x) := x^2(1-p+x)^{|B|} = \sum_{l=0}^{|B|} \binom{|B|}{l} x^{l+2}(1-p)^{|B|-l} .$$

Then it is clear that the last term of (52) is exactly $(p^2 \exp(1)/4)\omega^{(4)}(p\exp(1/2))$, where $\omega^{(4)}$ is the fourth derivative of ω. Therefore, with

$$p|B| \le pt \le (\log n)^{-1} ,$$

we easily have

$$(p^2/4)\omega^{(4)}(p\exp(1/2)) \le 1 .$$

□

We use 51 and set $\rho = n^{-1/4-1/17}$ (notice that $\rho \le h^{-1}$) to obtain

$$\phi_v(\rho) \le \exp(E[\ |\Gamma(\hat{N}_{X'}(v,T))|\ ]\,\rho + \rho^2/2) .$$

Then, by (50),

$$E[\exp(\rho\Phi_T^{(6)})] \le \exp(\rho \sum_{v\in V\setminus T} E[\ |\Gamma(\hat{N}_{X'}(v,T))|\ ] + \rho^2 n/2) = \exp(\rho E[\Phi_T^{(6)}] + \rho^2 n/2) .$$

Thus Markov's inequality gives

$$\begin{aligned}
\Pr(\Phi_T^{(6)} - E[\Phi_T^{(6)}] \ge b'\theta^2|\Gamma(T)|/4\ ) &= \Pr(\exp(\rho(\Phi_T^{(6)} - E[\Phi_T^{(6)}])) \ge \exp(\rho b'\theta^2|\Gamma(T)|/4)\ ) \\
&\le \exp(-\rho b'\theta^2|\Gamma(T)|/4)E[\ \exp(\rho(\Phi_T^{(6)} - E[\Phi_T^{(6)}]))\ ] \\
&\le \exp(-\rho b'\theta^2|\Gamma(T)|/4 + \rho^2 n/2) \\
&\le \exp(-n^{1-1/4-3/17}) \le \exp(-\sqrt{n}\ (\log n)^2) , \qquad (53)
\end{aligned}$$

which completes the proof of (49) and therefore of (46).

□

Finally, we claim that

$$\Pr\Big(\ \exists\ T \in \mathcal{T}\ \ \cdot\!\ni\!\cdot\ \ \Phi_T^{(4)} > \frac{b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2}\ \Big) \le 3/n^2 . \qquad (54)$$

Proof of (54). It is enough to show that (i) and (iii) in Lemma 4.3 imply

$$\Phi_T^{(4)} \le \frac{b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2} \quad \text{for all} \quad T \in \mathcal{T} .$$

Let

$$\sum{}^{*} := \sum_{\substack{v \in V \\ h \le |N_{X'}(v,T)| < b^2\theta^2\sqrt{n}}} \quad \text{and} \quad \sum{}^{**} := \sum_{\substack{v \in V \\ |N_{X'}(v,T)| \ge b^2\theta^2\sqrt{n}}} .$$

Then Property 6 and (i) of Lemma 4.3 give

$$\begin{aligned}
\Phi_T^{(4)} &= \sum_{\substack{v \in V \\ |N_{X'}(v,T)| \ge h}} \Gamma(N_{X'}(v,T)) \\
&\le \frac{1}{2}\sum{}^{*}|N_{X'}(v,T)|^2 + \frac{b}{2}\sum{}^{**}|N_{X'}(v,T)|^2 \\
&\le \frac{b^2\theta^2\sqrt{n}}{2}\sum{}^{*}|N_{X'}(v,T)| + \frac{(1+\theta)b^2\theta\sqrt{n}}{2}\sum{}^{**}|N_{X'}(v,T)| .
\end{aligned}$$

Also, (iii) of Lemma 4.3 and Lemma 3.5 (with $\beta = \sqrt{\log n}$ for both of the following inequalities, $\gamma = 1$ for the first inequality, and $\gamma = 2\log n$ for the second) give

$$\sum{}^{*}|N_{X'}(v,T)| \le 2t \quad \text{and} \quad \sum{}^{**}|N_{X'}(v,T)| \le (1+\theta/2)t .$$

Thus (14) yields

$$\Phi_T^{(4)} \le b^2\theta^2\sqrt{n}\; t + (1/2+\theta)b^2\theta\sqrt{n}\; t \le \frac{b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2} .$$

□

Therefore, combining (37), (38), (41), (45), (47), (54) and Property 7, we have that, with probability at least $1-1/n$,

$$\begin{aligned}
|\Gamma'(T)| &\ge (b'/b - 35b'\theta^2/2)|\Gamma(T)| - \frac{3b'b\theta(1+6\theta)}{9\sqrt{\log n}}\binom{t}{2} \\
&\ge b'\left(\mu - 35b\mu\theta^2/2 - \frac{b\theta(1+6\theta)}{3\sqrt{\log n}}\right)\binom{t}{2} \\
&\ge b'\left(\mu - 18b\theta^2 - \frac{b\theta}{3\sqrt{\log n}}\right)\binom{t}{2} = b'\mu'\binom{t}{2} .
\end{aligned}$$

4.9 Property 8

We will show that

$$E[\,|\mathcal{T}'|\,] \le n^k\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{k}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big) , \tag{55}$$

which together with Markov's inequality implies

$$\begin{aligned}
&\Pr\left(|\mathcal{T}'| \ge n^{k+1}\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{k}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big)\right) \\
&\le \left(n^{k+1}\binom{n}{t}\exp\Big(-(1-\epsilon)\sum_{j=0}^{k}\frac{b_j\mu_j\theta}{\sqrt{n}}\binom{t}{2}\Big)\right)^{-1}E[\,|\mathcal{T}'|\,] \le 1/n .
\end{aligned}$$

Proof of (55). Since

$$E[\ |\mathcal{T}'|\ ] = \sum_{T\in\mathcal{T}} \Pr\left(\mathcal{E}(G')\cap\Gamma(T)=\emptyset\right)\ ,$$

it is enough to show that for each $T \in \mathcal{T}$

$$\Pr\left(\mathcal{E}(G')\cap\Gamma(T)=\emptyset\right) \le \exp\Big(-\frac{(1-\epsilon)b\mu\theta}{\sqrt{n}}\binom{t}{2}\Big) \tag{56}$$

(recall that b and μ actually mean b_k and μ_k). But notice that for

$$\mathcal{F}'(T) := \{F\in\mathcal{F}' : F\cap\Gamma(T)\neq\emptyset\}$$

(again recall that $\mathcal{F}'(=\mathcal{F}_{k+1})$ is a maximal disjoint collection of forbidden pairs and triples in X': see (BC 3) in Section 2.1), we clearly have

$$\begin{aligned}
\Pr\left(\mathcal{E}(G')\cap\Gamma(T)=\emptyset\right) &\le \Pr\left(|X'\cap\Gamma(T)|\le 3|\mathcal{F}'(T)|\right)\\
&\le \Pr\Big(|X'\cap\Gamma(T)|\le \frac{3\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)| \quad\text{or}\quad |\mathcal{F}'(T)|\ge\frac{\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big)\\
&\le \Pr\Big(|X'\cap\Gamma(T)|\le \frac{3\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big)+\Pr\Big(|\mathcal{F}'(T)|\ge\frac{\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big). \quad (57)
\end{aligned}$$

Set

$$\Phi_T := |X'\cap\Gamma(T)| = \sum_{e\in\Gamma(T)} 1(e\in X')\ .$$

Then, since all events "$e\in X'$" are mutually independent and $\Pr(e\in X')=p$, we have

$$E[\exp(\rho\Phi_T)] = \prod_{e\in\Gamma(T)}\Big(1-p(1-\exp(\rho))\Big) \le \exp\Big(-\frac{\theta(1-\exp(\rho))}{\sqrt{n}}|\Gamma(T)|\Big)\ .$$

Markov's inequality with $\rho=-\epsilon^{-1}/4$ then gives

$$\begin{aligned}
\Pr\Big(\Phi_T\le\frac{3\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big) &= \Pr\Big(\exp(\rho\Phi_T)\ge\exp\Big(\frac{3\rho\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big)\Big)\\
&\le \exp\Big(-\frac{3\rho\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big)E[\exp(\rho\Phi_T)]\\
&\le \exp\Big(-\theta\Big(\frac{1-\exp(\rho)}{\sqrt{n}}+\frac{3\rho\epsilon^2}{\sqrt{n}}\Big)|\Gamma(T)|\Big)\\
&\le \frac{1}{2}\exp\Big(-\frac{(1-\epsilon)b\mu\theta}{\sqrt{n}}\binom{t}{2}\Big)\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\Pr\Big(|X'\cap\Gamma(T)|\le\frac{3\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big) &= \Pr\Big(\Phi_T\le\frac{3\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\Big)\\
&\le \frac{1}{2}\exp\Big(-\frac{(1-\epsilon)b\mu\theta}{\sqrt{n}}\binom{t}{2}\Big)\ . \quad (58)
\end{aligned}$$

On the other hand, with $l := \lfloor \epsilon^2\theta|\Gamma(T)|/\sqrt{n}\ \rfloor$, we know that

$$\begin{aligned}\Pr\left(|\mathcal{F}'(T)| \geq \frac{\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\right) \ \leq \ & \Pr\Big(\exists\ F_1,...,F_l \in \Lambda\cup\Delta \ \cdot\ni\cdot \\ & F_j \subseteq X'\ ,\quad F_j\cap\Gamma(T)\neq\emptyset \ \text{ and } \ F_j\cap F_{j'}=\emptyset \quad \forall\ j\neq j'\Big)\ .\end{aligned}$$

Let

$$\sum{}^{(l)} := \sum_{\substack{\{F_1,...,F_l\}\subseteq\Lambda\cup\Delta \\ F_j\cap\Gamma(T)\neq\emptyset \ \ \forall j\in[l] \\ F_j\cap F_{j'}=\emptyset \ \ \forall j\neq j'}} .$$

Then

$$\begin{aligned}\Pr\left(|\mathcal{F}'(T)| \geq \frac{\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\right) \ & \leq \ \sum{}^{(l)}\Pr\left(F_j\subseteq X' \ \ \forall j\in[l]\right) \\ & = \ \sum{}^{(l)}\prod_{j=1}^{l}\Pr\left(F_j\subseteq X'\right) \\ & \leq \ \frac{1}{l!}\Big(\sum_{\substack{F\in\Lambda\cup\Delta \\ F\cap\Gamma(T)\neq\emptyset}}\Pr\left(F\subseteq X'\right)\Big)^l\end{aligned}$$

(cf. Lemma 3.4). Properties 4 and 5, and (10) yield

$$\begin{aligned}\sum_{\substack{F\in\Lambda\cup\Delta \\ F\cap\Gamma(T)\neq\emptyset}}\Pr\left(F\subseteq X'\right) \ & = \ \sum_{\substack{F\in\Lambda \\ F\cap\Gamma(T)\neq\emptyset}}\Pr\left(F\subseteq X'\right) + \sum_{\substack{F\in\Delta \\ F\cap\Gamma(T)\neq\emptyset}}\Pr\left(F\subseteq X'\right) \\ & \leq \ 2b(a+5\theta)\sqrt{n}\ |\Gamma(T)|\,p^2 + b^2n|\Gamma(T)|\,p^3 \\ & \leq \ \frac{\theta^2}{\sqrt{n}}|\Gamma(T)| + \frac{\theta^3}{\sqrt{n}}|\Gamma(T)| \\ & \leq \ \frac{2\theta^2}{\sqrt{n}}|\Gamma(T)|\ .\end{aligned}$$

Thus with $\eta := \dfrac{2\theta^2}{\sqrt{n}}|\Gamma(T)|$, we have

$$\Pr\left(|\mathcal{F}'(T)| \geq \frac{\epsilon^2\theta}{\sqrt{n}}|\Gamma(T)|\right) \leq \frac{\eta^l}{l!} \ \leq \frac{1}{2}\exp\Big(-\frac{\theta|\Gamma(T)|}{\sqrt{n}}\Big) \leq \frac{1}{2}\exp\Big(-\frac{b\mu\theta}{\sqrt{n}}\binom{t}{2}\Big)\ , \tag{59}$$

where the second inequality uses $\dfrac{\eta^l}{l!} \ \leq \Big(\dfrac{\eta\exp(1)}{l}\Big)^l$. Hence (57), (58) and (59) yield (56).

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Appendix

To prove Lemma 3.1, set

$$\Omega_j := E[\Phi|\tau_1, ..., \tau_j] - E[\Phi|\tau_1, ..., \tau_{j-1}] \quad \text{for } j = 1, 2, ...m.$$

We first verify two more lemmas. The first is from [31].

Lemma A.1 *The hypotheses of Lemma 3.1 imply*

$$|\Omega_j| \le c_j \quad \textit{for} \quad j = 1, 2, ..., m \ .$$

Proof. Note first for fixed $\kappa = (\kappa_1, ..., \kappa_m)$ that

$$E[\Phi|\tau_1, ..., \tau_{j-1}](\kappa) = \sum_{\gamma_j, \cdots, \gamma_m} \Phi(\kappa_1, \cdots, \kappa_{j-1}, \gamma_j, \cdots, \gamma_m) \mathrm{Pr}\,(\tau_j = \gamma_j, \cdots, \tau_m = \gamma_m),$$

and $\sum_{\gamma_j} \mathrm{Pr}\,(\tau_j = \gamma_j \cdots, \tau_m = \gamma_m) = \mathrm{Pr}\,(\tau_{j+1} = \gamma_{j+1}, \cdots, \tau_m = \gamma_m)$ yields

$$\begin{aligned} E[\Phi|\tau_1, ..., \tau_j](\kappa) &= \sum_{\gamma_{j+1}, \cdots, \gamma_m} \Phi(\kappa_1, \cdots, \kappa_j, \gamma_{j+1}, \cdots, \gamma_m) \mathrm{Pr}\,(\tau_{j+1} = \gamma_{j+1}, \cdots, \tau_m = \gamma_m) \\ &= \sum_{\gamma_j, \cdots, \gamma_m} \Phi(\kappa_1, \cdots, \kappa_j, \gamma_{j+1}, \cdots, \gamma_n) \mathrm{Pr}\,(\tau_j = \gamma_j, \cdots, \tau_m = \gamma_m) \ . \end{aligned}$$

Thus (18) implies

$$\begin{aligned} |\Omega_j(\kappa)| &= |(E[\Phi|\tau_1, ..., \tau_j] - E[\Phi|\tau_1, ..., \tau_{j-1}])(\kappa)| \\ &\le \sum_{\gamma_j, \cdots, \gamma_m} |\Phi(\kappa_1, \cdots, \kappa_j, \gamma_{j+1}, \cdots, \gamma_m) - \Phi(\kappa_1, \cdots, \kappa_{j-1}, \gamma_j, \cdots, \gamma_m)| \\ &\qquad\qquad \times \ \mathrm{Pr}\,(\tau_j = \gamma_j, \cdots, \tau_m = \gamma_m) \\ &\le \sum_{\gamma_j, \cdots, \gamma_m} c_j \mathrm{Pr}\,(\tau_j = \gamma_j, \cdots, \tau_m = \gamma_m) = c_j \ . \end{aligned}$$

□

Lemma A.2 *The hypotheses of Lemma 3.1 imply*

$$E[(\Omega_j)^2|\tau_1, ..., \tau_{j-1}] \le p(1-p)c_j^2 \quad \textit{for} \quad j = 1, 2, ..., m \ .$$

Proof. Jensen's inequality gives

$$\begin{aligned} (\Omega_j)^2 &= (E[\Phi|\tau_1, ..., \tau_j] - E[\Phi|\tau_1, ..., \tau_{j-1}])^2 \\ &= (E[\Phi - E[\Phi|\tau_1, ..., \tau_{j-1}, \tau_{j+1}, ..., \tau_m]|\tau_1, ..., \tau_j])^2 \\ &\le E[(\Phi - E[\Phi|\tau_1, ..., \tau_{j-1}, \tau_{j+1}, ..., \tau_m])^2|\tau_1, ..., \tau_j] \ . \end{aligned}$$

Hence

$$\begin{aligned}
& E[(\Omega_j)^2|\tau_1,...,\tau_{j-1}] \\
\leq \;& E[\,(\Phi - E[\Phi|\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m])^2\,|\tau_1,...,\tau_{j-1}] \\
= \;& E[\,E[\,(\Phi - E[\Phi|\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m])^2\,|\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m\,]\,|\,\tau_1,...,\tau_{j-1}\,]\ .
\end{aligned}$$

Thus it is enough to show that

$$E[\,(\Phi - E[\Phi|\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m])^2\,|\,\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m] \leq p(1-p)c_j^2\ .$$

For fixed $\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m$ let

$$x := \Phi(\tau_1,...,\tau_{j-1},0,\tau_{j+1},...,\tau_m) \quad \text{and} \quad y := \Phi(\tau_1,...,\tau_{j-1},1,\tau_{j+1},...,\tau_m)$$

Then (18) gives

$$|x-y| \leq c_j\ .$$

Since $\Pr(\tau_j = 1) = p$, this implies

$$\begin{aligned}
& E[\,(\Phi - E[\Phi|\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m])^2\,|\,\tau_1,...,\tau_{j-1},\tau_{j+1},...,\tau_m] \\
= \;& (1-p)(x-((1-p)x+py))^2 + p(y-((1-p)x+py))^2 \\
= \;& p(1-p)(x-y)^2 \leq p(1-p)c_j^2\ .
\end{aligned}$$

□

Proof of Lemma 3.1. (cf. Lemma 5.3 of [22]) It is enough to show that

$$\Pr(\Phi - E[\Phi] \geq \lambda) \leq \exp(-\rho\lambda + (\rho^2/2)\sum_{j=1}^{m} c_j \exp(\rho c_j))\ ,$$

because the same argument gives

$$\Pr(-\Phi - E[-\Phi] \geq \lambda) \leq \exp(-\rho\lambda + (\rho^2/2)\sum_{j=1}^{m} c_j \exp(\rho c_j))\ .$$

We claim first that

$$E[\exp(\rho\Omega_j)|\tau_1,...,\tau_{j-1}] \leq \exp((\rho^2/2)p(1-p)c_j^2\exp(\rho c_j))\ . \tag{60}$$

For fixed $\tau_1,...,\tau_{j-1}$ let

$$\phi(\rho) := E[\exp(\rho\Omega_j)|\tau_1,...,\tau_{j-1}]\ .$$

Then Taylor's theorem and Lemma A.1 imply for some $0 \leq \rho^* \leq \rho$ that

$$\begin{aligned}
\phi(\rho) \;&=\; \phi(0) + \phi'(0)\rho + \frac{\phi''(\rho^*)}{2}\rho^2 \\
&=\; 1 + (\rho^2/2)E[(\Omega_j)^2\exp(\rho^*\Omega_j)|\tau_1,...,\tau_{j-1}] \\
&\leq\; 1 + (\rho^2/2)p(1-p)c_j^2\exp(\rho c_j) \leq \exp((\rho^2/2)p(1-p)c_j^2\exp(\rho c_j))\ ,
\end{aligned}$$

where the second equality uses $E[\Omega_j] = 0$.

Next we show that

$$E[\exp(\rho(\Phi - E[\Phi]))] \leq \exp((\rho^2/2)p(1-p)\sum_{j=1}^{m} c_j^2 \exp(\rho c_j)) \ . \qquad (61)$$

Since $\Phi - E[\Phi] = \sum_{j=1}^{m} \Omega_j$, it is enough to show that

$$E[\exp(\rho \sum_{j=1}^{l} \Omega_j)] \leq \exp((\rho^2/2)p(1-p)\sum_{j=1}^{l} c_j^2 \exp(\rho c_j))$$

by induction on $l = 1, 2, ..., m$. For $l = 1$, (60) gives

$$E[\exp(\rho\Omega_1)] \leq \exp((\rho^2/2)p(1-p)c_1^2 \exp(\rho c_1)) \ .$$

For $l > 1$, (60) and the induction hypothesis yield

$$\begin{aligned}
E[\exp(\rho \sum_{j=1}^{l} \Omega_j)] &= E[E[\exp(\rho \sum_{j=1}^{l} \Omega_j)|\tau_1, ..., \tau_{l-1}]] \\
&= E[\exp(\rho \sum_{j=1}^{l-1} \Omega_j) E[\exp(\rho\Omega_l)|\tau_1, ..., \tau_{l-1}]] \\
&\leq E[\exp(\rho \sum_{j=1}^{l-1} \Omega_j)] \exp((\rho^2/2)p(1-p)c_l \exp(\rho c_l)) \\
&\leq \exp((\rho^2/2)p(1-p)\sum_{j=1}^{l} c_j^2 \exp(\rho c_j)) \ .
\end{aligned}$$

Finally, Markov's inequality and (61) give

$$\begin{aligned}
\Pr(\Phi - E[\Phi] \geq \lambda) &= \Pr(\exp(\rho(\Phi - E[\Phi])) > \exp(\rho\lambda)) \\
&\leq \exp(-\rho\lambda) E[\exp(\rho(\Phi - E[\Phi]))] \\
&\leq \exp(-\rho\lambda + (\rho^2/2)p(1-p)\sum_{j=1}^{m} c_j^2 \exp(\rho c_j)) \ .
\end{aligned}$$

□

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