A tight (non-combinatorial) conditional lower bound for Klee's Measure Problem in 3D

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***Abstract*—We revisit the classic geometric problem of computing the volume of the union of n 3-dimensional axis-parallel boxes (*Klee's measure problem in 3D*). It is well known that the problem can be solved in time $O(n^{3/2})$ (Overmars, Yap SICOMP'91; Chan FOCS'13). Can we justify this 30-year old barrier of $n^{3/2\pm o(1)}$ under plausible fine-grained complexity assumptions? The only previous conditional lower bound (Chan Comp. Geom.'10) shows that this barrier holds for *purely combinatorial* algorithms, i.e., algorithms avoiding algebraic techniques for fast matrix multiplication. This leaves open an algorithmic improvement exploiting algebraic techniques, and does not give any superlinear bound if the matrix multiplication exponent ω turns out to be equal to 2.**

We resolve this issue by giving a tight conditional lower bound for general algorithms, based on the 3-uniform hyperclique hypothesis. Specifically, we prove that an $O(n^{3/2-\epsilon})$ algorithm for Klee's measure problem in 3D would give a $O(n^{k-\epsilon'})$-time algorithm for counting k-cliques in 3-uniform hypergraphs – this in turn would give a novel $O((2-\epsilon'')^n)$-algorithm for Max-3SAT.

Our lower bound can be generalized to $n^{\frac{d}{3-3/d}-o(1)}$, which matches the upper bound up to a factor of $n^{\frac{d-3}{6-6/d}+o(1)}$ and separates the general problem from popular special cases: For all $d \geq 3$, known $\tilde{O}(n^{\frac{d+1}{3}})$ algorithms (Bringmann Comp. Geom.'12; Chan FOCS'13) compute the problem for arbitrary hypercubes polynomially faster than our lower bound for the general problem.

***Index Terms*—fine-grained complexity theory, geometric algorithms, hyperclique detection, (non-)combinatorial algorithms**

I. Introduction

Klee's measure problem [1] (KMP) is one of the most natural problems that you may ask about axis-aligned boxes: Given n axis-parallel boxes in d-dimensional space, compute the volume of their union. Besides significant interest as a fundamental problem of its own, the problem is related to other interesting geometric problems, such as the depth problem for an arrangement of boxes[1], finding a cluster of k points with small L_∞-diameter [2], [3], Hausdorff distance under translation [4], discrepancy of boxes [5], the largest empty box problem [6], and many more.

For $d \leq 2$, the problem is essentially resolved: a simple sweep-line approach due to Bentley [7] solves the 2-dimensional problem in time $O(n \log n)$, and a tight lower bound in the decision tree model already holds for $d = 1$ [8]. However, already the case $d = 3$ remains an interesting open problem: Bentley's approach yields a $O(n^2 \log n)$-time solution, from which van Leeuwen and Wood [9] could shave off a logarithmic factor. A remarkable improvement to $O(n^{3/2} \log n)$ was given by Overmars and Yap [10] which was subsequently improved to $O(n^{3/2})$ *and* simplified in works by Chan [11], [12]. This running time (which generalizes to $O(n^{d/2})$ in d dimensions) remains the state of the art since, even despite interesting special cases such as a restriction to cubes being solvable in near-linear time [13]; we review the extensive work on special cases of Klee's measure problem in more detail below.

Consequently, it is no surprise that researchers have asked whether the exponent $3/2$ might be optimal, at the latest since the late 1990s[2]. Notable evidence for optimality was presented by Chan [11] who reduces the problem of finding a triangle in an n-node graph G to the 3-dimensional KMP on $O(n^2)$ boxes. For this, consider the bounding box $B = [0, n]^3$, where we think of the subcube

$$A_{v_1,v_2,v_3} = [v_1, v_1 + 1) \times [v_2, v_2 + 1) \times [v_3, v_3 + 1)$$

for $v_1, v_2, v_3 \in \{0, \ldots, n-1\}$ as representing a potential triangle (v_1, v_2, v_3). The task is to define a set of boxes $\mathcal{B}$ that cover A_{v_1,v_2,v_3} if and only if $\{v_1, v_2, v_3\}$ do *not* form a triangle in G. We can do so by adding a box $[v_1, v_1+1) \times [v_2, v_2+1) \times [0, n)$ whenever $\{v_1, v_2\}$ does not form an edge[3] in G as well as analogous boxes for non-edges of the form $\{v_2, v_3\}$ and $\{v_1, v_3\}$. This way, we obtain a set of $O(n^2)$ boxes with the desired property – in particular, the volume of the union of $\mathcal{B}$ is n^3 if and only if there is no triangle in G.

This elegant reduction yields several consequences based on the current state of the art for triangle detection algorithms. Specifically, the only known way to beat $n^{3\pm o(1)}$ time for triangle detection uses fast matrix multiplication based on algebraic techniques; this results in a time bound of $O(n^\omega)$, where ω denotes the matrix multiplication constant with a current bound of $\omega < 2.37286$ [14]. As a consequence, we can rule out a *"purely combinatorial"* $O(n^{3/2-\epsilon})$-time algorithm for Klee's measure problem in 3D if we are willing to assume

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[1]Given a set of axis-parallel boxes $\mathcal{B}$, determine the point contained in the maximum number of $B \in \mathcal{B}$.

[2]For an early explicit account, see Jeff Erickson's notes from 1998: https://jeffe.cs.illinois.edu/open/klee.html.

[3]This includes the case that $v_1 = v_2$.

that there is no *"purely combinatorial"* subcubic algorithm for triangle detection. However, this interpretation is only partially satisfying since (1) exploiting algebraic techniques for matrix multiplication appears a reasonable possibility for algorithmic improvements and (2) there is no universally accepted formal definition of *"purely combinatorial"* algorithms. The first objection might be the most severe: There are numerous instances of problems with cubic-time lower bounds for combinatorial algorithms that can be broken by sophisticated uses of fast matrix multiplication, including various "intermediate" matrix products (e.g., [15], [16]), node-weighted triangle [17], [18], RNA folding [19], and more.

Alternatively, we may view Chan's reduction as a lower bound for non-combinatorial algorithms: any $O(n^{\omega/2-\epsilon})$-time algorithm for Klee's measure problem in 3D would give a surprising breakthrough for triangle detection, circumventing fast matrix multiplication. This last statement tacitly assumes $\omega > 2$ to be meaningful; if the current matrix multiplication exponent of $\omega \approx 2.37286$ was optimal, we would obtain a lower bound of roughly $\Omega(n^{1.18643})$. Given that the long history of algorithmic progress for this problem has stalled, we thus ask whether we can close the gap by improved lower bounds:

Can we prove a tight $n^{1.5-o(1)}$ conditional lower bound for Klee's measure problem in 3D, ideally independent of the assumption that $\omega > 2$?

This question is representative of a diverse family of problems with a similar fine-grained complexity status: Sliding Window Hamming Distance admits a classic $\tilde{O}(n^{1.5})$ algorithm [20], which is known to be conditionally tight for *combinatorial* algorithms, by a folklore reduction from Boolean Matrix Multiplication that is credited to Indyk, see [21]. Likewise, cubic-time algorithms for context-free language reachability are known to be conditionally optimal for combinatorial algorithms [22], but improved non-combinatorial algorithms or non-combinatorial lower bounds remain elusive; recent work [23] gives evidence against certain non-combinatorial conditional lower bounds (namely, based on the Strong Exponential Time Hypothesis). For many other problems, sophisticated uses of matrix multiplication were able to break a combinatorial cubic barrier.

Our Result: We give an affirmative answer to the above question by proving a tight $n^{3/2-o(1)}$ lower bound under the plausible fine-grained complexity assumption that counting all k-cliques in a 3-uniform hypergraphs requires essentially exhaustive search time $n^{k-o(1)}$. This assumption is implied by the *3-uniform hyperclique hypothesis* that receives increasing interest for uncovering hardness barriers in P: among others, it has been used to show conditional lower bounds for basic graph problems in sparse graphs [24], the orthogonal vectors problem [25], for model-checking first-order properties [26], [27], for query enumeration [28], for Boolean constrainst satisfaction parameterized by solution size [29], for the d-dimensional maximum subarray problem [30] and subgraph isomorphism parameterized by treewidth [31]. A refutation of our assumption, in fact, already a $O(n^{k-\epsilon})$-time algorithm for *detecting* k-cliques in 3-uniform hypergraphs, would have notable consequences: It would give a $O((2-\epsilon)^n)$-time algorithm for Max 3-SAT [24] (non-existence of such an algorithm has recently been used as a working hypothesis in [32]), and would lead to substantial progress for model-checking certain first-order properties [26] and for finding size-k solutions to certain Boolean constraint satisfaction problems like 3-SAT [29]. We refer to Section II and [24] for further discussion on the plausibility of this conjecture, in particular, why the algebraic techniques for clique detection in graphs fail in hypergraphs.

Our lower bound thus gives an explanation for the barrier observed for the 3-dimensional Klee's measure problem: Any polynomial improvement over $O(n^{3/2})$ would have significant consequences beyond its purely geometric setting.

a) Higher Dimensions and Special Cases: Addressing larger dimensions, we generalize our result to an $n^{\frac{d}{3-3/d}-o(1)}$ lower bound – the gap to the upper bound is thus smallest for small values of d. However, even for larger dimension, the lower bound separates the general problem from several notable special cases[4]. Such special cases, including the hypervolume indicator, (unit) hypercubes/fat boxes, orthants and k-grounded boxes, have been addressed by a large body of works, see, e.g., [12], [13], [33]–[37]; a detailed account of their relationships is given in [38]. Notably, almost all of these special cases can be solved in time $O(n^{\frac{d+1}{3}})$ [12], [36], while our lower bound of $n^{\frac{d}{3-3/d}-o(1)}$ for the general problem is always strictly larger. This truly seperates the special cases of hypervolume indicator, orthants, hypercubes and fat boxes from the general problem, independent of the value of ω. We refer to Table I for an illustration of our results.

b) Related problem: unweighted depth: With the same techniques as for Klee's measure problem, one can compute the depth of an arrangment of boxes in time $n^{3/2\pm o(1)}$, even with weights [10], [12]. For the weighted variant, the natural generalization of Chan's reduction to weighted clique yields a lower bound of $n^{d/2-o(1)}$ under the weighted d-clique conjecture (which is equivalent to APSP for $d = 3$), see [40]. Our lower bounds transfer to the *unweighted* depth problem, and thus we obtain optimality of $n^{3/2\pm o(1)}$ in 3D already for the unweighted case, assuming the 3-uniform hyperclique conjecture.

II. Preliminaries

Throughout the paper, we let $[n] = \{1, \dots, n\}$. We use $\binom{V}{k}$ to denote all k-element subsets of V.

The 3-uniform k-hyperclique problem is the following problem: We are given a k-partite[5] 3-uniform hypergraph $G = (V, E)$, i.e., V is the disjoint union of k sets $V_1, \dots, V_k$ of size n each, and E is a set of edges of the form $\{v_a, v_b, v_c\}$ for $v_a \in V_a, v_b \in V_b, v_c \in V_c$ where a, b, c are all different. For notational convenience, we think of each V_i as a disjoint

[4]I.e., even without assuming $\omega > 2$, but of course conditional on the 3-uniform hyperclique hypothesis.

[5]The assumption that G is k-partite is equivalent to the general problem using a standard color-coding argument.

d	Unit hypercubes (incl. Hypervolume Indicator)	Arbitrary hypercubes	General problem	
			LB	UB
3	$\tilde{O}(n)$ [13], [39]	$\tilde{O}(n)$ [13]	$n^{1.5-o(1)}$**[here]**	$O(n^{1.5})$ [10], [12]
4	$\tilde{O}(n^{1.33\dots})$ [12]	$\tilde{O}(n^{1.66\dots})$ [12]	$n^{1.77\dots-o(1)}$**[here]**	$O(n^2)$ [10], [12]
5	$\tilde{O}(n^{1.66\dots})$ [12]	$\tilde{O}(n^2)$ [12]	$n^{2.083\dots-o(1)}$**[here]**	$O(n^{2.5})$ [10], [12]
$d \geq 5$	$\tilde{O}(n^{d/3})$ [12]	$\tilde{O}(n^{\frac{d+1}{3}})$ [12]	$n^{\frac{d}{3-3/d}-o(1)}$**[here]**	$O(n^{\frac{d}{2}})$ [10], [12]

TABLE I

UPPER AND LOWER BOUNDS FOR KLEE'S MEASURE PROBLEM (KMP) AND ITS SPECIAL CASES.

copy of $\{0, \dots, n-1\}$. The task is to detect whether there are $v_1 \in V_1, \dots, v_k \in V_k$ that form a clique, i.e., for all $a, b, c \in \binom{[k]}{3}$, we have $\{v_a, v_b, v_c\} \in E$.

Exhaustive search solves the problem in time $O(n^k)$, over which no polynomial improvement is known. This has lead to the following hypothesis:

Hypothesis 1 (3-Uniform Hyperclique Conjecture)**.** *For no $k \geq 4$ and $\epsilon > 0$, there is an $O(n^{k-\epsilon})$-time algorithm for the 3-uniform k-hyperclique problem.*

The perhaps most important argument for the plausibility of this hypothesis is the following: The only known way to polynomially beat exhaustive search for the clique problem in graphs (rather than hypergraphs) use fast matrix multiplication, giving an $O(n^{\omega k/3})$ algorithm whenever k is a multiple of 3. However, such approaches do not transfer to clique detection in *hyper*graphs, as the corresponding tensor has full rank, see [24] for a thorough discussion.

Besides this failure of current techniques, refuting the 3-uniform hyperclique conjecture would have several significant consequences: by the split-and-list technique of Williams (see [24]), it would give an exponential improvement over current $2^{n-o(n)}$ algorithms for Max 3-SAT (see [41], [42] for an overview). Furthermore, it would lead to polynomial improvements for the best known model-checking algorithm for certain first-order formulas [26], and polynomially beat exhaustive search for finding size-k solutions of a class of Boolean CSPs that includes 3-SAT [29].

It turns out that our lower bounds already follow from the weaker assumption that *counting* all k-cliques in a given 3-uniform hypergraph G requires time $n^{k-o(1)}$. Note that while the fastest known algorithm for detecting k-cliques also readily returns the number of k-cliques, there is currently little evidence that these two problems should indeed be equivalent. We refer to the natural counting variant of Hypothesis 1 as the *3-uniform hyperclique counting conjecture*.

III. TECHNICAL OVERVIEW

The natural starting point is an attempt to adapt Chan's reduction from clique finding in graphs to clique finding in 3-uniform hypergraphs. Indeed, a straightforward adaptation results in a lower bound of $n^{\frac{d}{3}-o(1)}$ for the d-dimensional problem, where the factor $1/3$ in the exponent results from creating up to $\Theta(n^3)$ boxes due to the increased arity of the (non-)edges. Without somehow compressing the size of the given 3-uniform hypergraph[6] to $O(n^{3-\epsilon})$, this loss appears unavoidable, and fails to give any superlinear lower bound for $d = 3$. Unfortunately, a general-purpose sparsification for hyperclique detection appears rather unlikely.

However, we may make use of the following surprising (at least to the author) insight: we can encode a choice of 4 rather than 3 vertices[7] by *encoding a fourth vertex redundantly* in the three dimensions, requiring only $O(n^3)$ boxes to ensure consistency of the redundant encoding and $O(n^3)$ boxes to check whether the chosen vertices form a clique.

Specifically, we consider a bounding box $B = [0, n^2)^3$ where in each dimension $i \in \{1, 2, 3\}$, we will encode the vertices (v_i, v_4) using a lexicographic ordering: the choice $(v_i, v_4) \in \{0, \dots, n-1\}^2$ is encoded as the length-1 interval $[v_i n + v_4, v_i n + v_4 + 1)$. An important observation is that the order of v_i and v_4 in the lexicographic ordering is decisive: We can represent the set $\{(v_i, 0), \dots, (v_i, n-1)\}$ as a single interval $[v_i n, (v_i+1)n)$ – note that the same is not true for the set $\{(0, v_4), \dots, (n-1, v_4)\}$. Thus, we can efficiently cover all boxes representing a *forbidden* choice $(v_1, \cdot), (v_2, \cdot), (v_3, \cdot)$: for every non-edge $\{v_1, v_2, v_3\}$, introduce a single box $[v_1 n, (v_1+1)n) \times [v_2 n, (v_2+1)n) \times [v_3 n, (v_3+1)n)$. It is not difficult to cover non-edges involving v_4 using similar boxes. Thus, it remains to ensure that we cover all parts of the bounding box representing *inconsistent* choices, i.e., $(v_1, v_1'), (v_2, v_2'), (v_3, v_3')$ such that v_1', v_2', v_3' are not all equal. It turns out that we can cover configurations with $v_1' \neq v_2'$ by "guessing" v_1, v_1', v_2 and including the two boxes

$$[v_1 n + v_1', v_1 n + v_1' + 1) \times [v_2 n, v_2 n + v_1') \times [0, n^2) \text{ and}$$
$$[v_1 n + v_1', v_1 n + v_1' + 1) \times [v_2 n + v_1' + 1, (v_2+1)n) \times [0, n^2).$$

This covers all configurations except those with $v_1' = v_2'$, since the first box covers configurations with $v_2' < v_1'$ and the second covers configurations with $v_2' > v_1'$. Since we only have to guess three vertices (not four!), we obtain $O(n^3)$ boxes for covering all configurations with $v_1' \neq v_2'$. We give the details of the reduction, generalized to an $n^{\frac{d+1}{3}-o(1)}$ lower bound in arbitrary dimension $d \geq 3$, in Section IV.

Tight lower bound for $d = 3$: One might hope that the above ideas can be used to also reduce 3-uniform 6-hyperclique to an instance of 3D-KMP with $O(n^4)$ boxes, which would yield a tight lower bound. However, natural

[6] More precisely, we would need to compress its complement graph.

[7] Thus reducing from 4-clique detection in hypergraphs, rather than from triangle detection.

adaptions of this idea fail. Instead, we are able to give a tight conditional lower using the full power of the 3-uniform hyperclique conjecture, by reducing from 3-uniform k-hyperclique for arbitrarily large k. We may assume that G is k-partite with parts $V_1 \cup \cdots \cup V_k$ and the task is to find $v_1 \in V_1, \ldots, v_k \in V_k$ forming a clique. The crucial idea is to consider $k = 3g$ for large enough g, divide the k parts into 3 *groups* $(V_1^{(1)}, \ldots, V_g^{(1)}), (V_1^{(2)}, \ldots, V_g^{(2)})$ and $(V_1^{(3)}, \ldots, V_g^{(3)})$, and encode each group $i \in [3]$ in *two different* ways: once in the i-th dimension using the order $V_1^{(i)}, \ldots, V_g^{(i)}$, and a redundant second time in the $(i+1)$-st dimension (in circular order of $[3]$) using the *reverse ordering* $V_g^{(i)}, \ldots, V_1^{(i)}$. Surprisingly, introducing this group hierarchy with reverse redundant encodings enables us to encode both invalid and non-clique encodings efficiently enough to obtain $n^{3/2}$-hardness.

Let us elaborate on the corresponding ideas: Generalizing the above lexicographic encoding, for any $v_1, \ldots, v_g, v'_g, \ldots, v'_1 \in \{0, \ldots, n-1\}$, we define an index $\mathrm{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_1)$ by interpreting $v_1 \ldots v_g v'_g \ldots v'_1$ as a base-n number with $2g$ digits, i.e.,

$$\begin{aligned}&\mathrm{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_1)\\ &= v_1 n^{2g-1} + \cdots + v_g n^g + v'_g n^{g-1} + \cdots + v'_1.\end{aligned}$$

Then a candidate solution $(v_1^{(1)}, \ldots, v_g^{(1)}) \in V_1^{(1)} \times \cdots \times V_g^{(1)}$, $(v_1^{(2)}, \ldots, v_g^{(2)}) \in V_1^{(2)} \times \cdots \times V_g^{(2)}$,$(v_1^{(3)}, \ldots, v_g^{(3)}) \in V_1^{(3)} \times \cdots \times V_g^{(3)}$ will be encoded as the cube

$$\begin{aligned}&[\mathrm{ind}(v_1^{(1)}, \ldots, v_g^{(1)}, v_g^{(2)}, \ldots, v_1^{(2)}),\\ &\quad \mathrm{ind}(v_1^{(1)}, \ldots, v_g^{(1)}, v_g^{(2)}, \ldots, v_1^{(2)}) + 1)\\ &\times[\mathrm{ind}(v_1^{(2)}, \ldots, v_g^{(2)}, v_g^{(3)}, \ldots, v_1^{(3)}),\\ &\quad \mathrm{ind}(v_1^{(2)}, \ldots, v_g^{(2)}, v_g^{(3)}, \ldots, v_1^{(3)}) + 1)\\ &\times[\mathrm{ind}(v_1^{(3)}, \ldots, v_g^{(3)}, v_g^{(1)}, \ldots, v_1^{(1)}),\\ &\quad \mathrm{ind}(v_1^{(3)}, \ldots, v_g^{(3)}, v_g^{(1)}, \ldots, v_1^{(1)}) + 1).\end{aligned}$$

Note that the bounding box $B = [0, n^{2g})^3$ is the disjoint union of (1) the volume-1 cubes given by the valid encoding of each candidate solution and (2) volume-1 cubes representing invalid encodings, i.e., cubes differing from the form above.

Let us sketch the ideas for **ensuring valid encodings** (i.e., covering all inconsistent encodings) and **detecting solutions** (i.e., covering all non-cliques): Consider some cube $Q = [q_1, q_1{+}1) \times [q_2, q_2{+}1) \times [q_3, q_3{+}1)$ with $q_i \in \{0, \ldots, n^{2g}{-}1\}$ that represents an invalid encoding, i.e., it encodes vertices $v_1^{(i)}, \ldots, v_g^{(i)}, \bar{v}_g^{(i+1)}, \ldots, \bar{v}_1^{(i+1)}$ in some dimension i, and vertices $v_1^{(i+1)}, \ldots, v_g^{(i+1)}$ (as well as some additional vertices $\bar{v}_g^{(i+2)}, \ldots, \bar{v}_1^{(i+2)}$) in dimension $i+1$ such that there is an inconsistency, namely $v_j^{(i+1)} \neq \bar{v}_j^{(i+1)}$ for some j. Crucially, we can catch this inconsistency using two boxes that only depend on $v_1^{(i)}, \ldots, v_g^{(i)}$ and $v_1^{(i+1)}, \ldots, v_j^{(i+1)}, \bar{v}_{j+1}^{(i+1)}, \ldots, \bar{v}_g^{(i+1)}$: For the case $\bar{v}_j^{(i+1)} > v_j^{(i+1)}$, we use a box that covers all encodings with a prefix $(v_1^{(i)}, \ldots, v_g^{(i)}, \bar{v}_g^{(i+1)}, \ldots, \bar{v}_{j+1}^{(i+1)}, \bar{v})$ with $\bar{v} > v_j^{(i+1)}$ in dimension i, and prefix $v_1^{(i+1)}, \ldots, v_j^{(i+1)}$ in dimension $i+1$, and the full range $[0, n^{2g})$ in the remaining dimension. Likewise, for the case $\bar{v}_j^{(i+1)} < v_j^{(i+1)}$, we use almost the same box, except that we change the prefixes covered in dimension i to $(v_1^{(i)}, \ldots, v_g^{(i)}, \bar{v}_g^{(i+1)}, \ldots, \bar{v}_{j+1}^{(i+1)}, \bar{v})$ with $\bar{v} < v_j^{(i+1)}$. Note that our lexicographic encoding nicely allows us to cover such prefixes, see Section V for details. Finally, we observe that each box only depends on $i \in [3], j \in [g]$ and the $2g$ vertex choices $v_1^{(i)}, \ldots, v_g^{(i)}, v_1^{(i+1)}, \ldots, v_j^{(i+1)}, \bar{v}_{j+1}^{(i+1)}, \ldots, \bar{v}_g^{(i+1)}$, thus we need at most $O(gn^{2g}) = O(n^{2g})$ such boxes.

For detecting whether a candidate solution S indeed forms a hyperclique, we need to rule out that there are three vertices $v_i^{(a)}, v_j^{(b)}, v_j^{(c)} \in S$ that do not form an edge. To do so, for all $a, b, c \in [3], i, j, k \in [g]$ and $\{v_i^{(a)}, v_j^{(b)}, v_k^{(c)}\} \notin E$, we cover all valid encodings of candidate solutions containing $v_i^{(a)}, v_j^{(b)}, v_k^{(c)}$. The case that a, b, c are not all different is straightforward to handle, so we focus on the case that $a = 1, b = 2, c = 3$. The natural approach would be to "guess" all $v_1^{(1)}, \ldots, v_{i-1}^{(i)}$, $v_1^{(2)}, \ldots, v_{j-1}^{(2)}$ and $v_1^{(3)}, \ldots, v_{k-1}^{(3)}$ and cover all encodings with prefixes $v_1^{(1)}, \ldots, v_i^{(1)}$ in dimension 1, $v_1^{(2)}, \ldots, v_j^{(2)}$ in dimension 2 and $v_1^{(3)}, \ldots, v_k^{(3)}$ in dimension 3. However, this leads to $O(n^{3g})$ boxes in the worst case, which fails to give any superlinear lower bound. Instead, we crucially exploit the redundant reverse encodings, by observing that it suffices to cover all cubes with prefixes according to one of the following options:

1) prefix $(v_1^{(1)}, \ldots, v_g^{(1)}, v_g^{(2)}, \ldots, v_j^{(2)})$ in dimension 1 and prefix $(v_1^{(3)}, \ldots, v_k^{(3)})$ in dimension 3, or
2) prefix $(v_1^{(2)}, \ldots, v_g^{(2)}, v_g^{(3)}, \ldots, v_k^{(3)})$ in dimension 2 and prefix $(v_1^{(1)}, \ldots, v_i^{(1)})$ in dimension 1, or
3) prefix $(v_1^{(3)}, \ldots, v_g^{(3)}, v_g^{(1)}, \ldots, v_i^{(1)})$ in dimension 3 and prefix $(v_1^{(2)}, \ldots, v_j^{(2)})$ in dimension 2,

where we guess all listed vertices except $v_i^{(1)}, v_j^{(2)}, v_k^{(3)}$. Note that for these options we have to guess $2g - j + k - 2$ vertices for Option 1, $2g - k + i - 2$ vertices for Option 2, and $2g - i + j - 2$ vertices for Option 3. We will decide between these options depending on which value among i, j, k is smallest: Option 1 if k is smallest, Option 2 if i is smallest and Option 3 if j is smallest. By this choice we need to guess at most $2g - 2 + \min\{k - j, i - k, j - i\} \leq 2g - 2$ vertices. Thus, over all $i, j, k \in [g]$ and $v_i^{(1)} \in V_i^{(1)}, v_j^{(2)} \in V_j^{(2)}, v_k^{(3)} \in V_k^{(3)}$with $\{v_i^{(1)}, v_j^{(2)}, v_k^{(3)}\} \notin E$, we need at most $O(g^3 n^{2g+1}) = O(n^{2g+1})$ boxes.

Thus, by checking whether the union of the constructed boxes is equal to the full bounding box $[0, n^{2g})^3$, we can decide any given $3g$-hyperclique instance. In particular, a $O(n^{3/2-\epsilon})$ algorithm for Klee's measure problem in 3D would give an algorithm for $3g$-uniform hyperclique running in time $O((n^{2g+1})^{3/2-\epsilon}) = O(n^{3g+3/2-(2g+1)\epsilon})$, which polynomially beats running time $O(n^{3g})$ for any $g \geq 1/\epsilon$. We give the formal proof with all details in Section V.

Generalization to $d \geq 3$: A naive generalization of the above reduction to larger dimension $d \geq 3$ will quickly fail to give any lower bound beyond $n^{d/3-o(1)}$: If the redundant part encoded in each dimension consists of a single group, then already for $d = 6$, we will not be able to exploit this redundant encoding for all possible non-edges.[8] On the other hand, if we encode more than a single group in the redundant part, we require significantly more boxes to ensure consistent encodings.

In the remainder of this section, we give a high-level description how we resolve this tension: We notice that we may afford to encode, in the redundant part of each dimension $a \in [d]$, *short suffixes* $v_g^{(a')}, \dots, v_{(1-\alpha)g}^{(a')}$ of a carefully chosen number of groups $a' \neq a$, where $0 < \alpha < 1$. In particular, we will *interleave* these short suffixes, so that we first encode the g-th elements of all considered groups, then the $(g-1)$-st elements, up to the $(1-\alpha)g$-th elements.

More specifically, in each dimension a, we will encode $d-2$ other groups redundantly, namely all groups except for a and $a+1$. As it turns out, to ensure consistency of the encoding, we need roughly $O(n^{2g+(d-3)\alpha g})$ boxes: to check that group $b \notin \{a, a+1\}$ is consistently encoded in dimension a, we essentially need to guess the complete groups a and b (giving $O(n^{2g})$ vertices), as well as the suffixes of all $d-3$ other groups (different from b) that are redundantly encoded in dimension a (giving $O(n^{(d-3)\alpha g})$ vertices).

It remains to discuss how to cover all non-cliques: For any non-edge $\{v_i^{(a)}, v_j^{(b)}, v_k^{(c)}\} \notin E$, we need to cover all hypercubes representing a candidate solution S that contains $v_i^{(a)}, v_j^{(b)}, v_k^{(c)}$. Here, we focus on the most interesting case that a, b, c are all distinct. If i, j, k are all at most $(1-\alpha)g$, then we cannot exploit any redundant encoding, since we only redundantly encode the last αg elements of each group. However, in this case we also only need to guess $i-1+j-1+k-1 \leq 3(1-\alpha)g-3$ vertices to cover all prefixes containing $v_i^{(a)}, v_j^{(b)}, v_k^{(c)}$, leading to only $O(n^{3(1-\alpha)g})$ boxes to handle this case. It remains to consider that cases that the largest element of i, j, k, say i, is at least $(1-\alpha)g+1$. In this case, we observe that a is redundantly encoded in dimension b or dimension c (or both), say in dimension b. We can thus cover $v_i^{(a)}, v_j^{(b)}, v_k^{(c)}$ by covering the prefix of dimension b that contains the redundant encoding of $v_i^{(a)}$ (this prefix has length at most $g + (d-2)(g-i+1)$ and contains $v_j^{(b)}$) and the prefix of dimension c that contains the encoding of $v_k^{(c)}$ (this has length k). In total, since $k \leq i$ and $i \geq (1-\alpha)g+1$, the total length of prefixes is bounded by $g + (d-2)(g-i+1) + k \leq 2g + (d-3)\alpha g + 1$. Thus, by choosing $\alpha = 1/d$, we can balance the terms of $O(n^{3(1-\alpha)g})$ and $O(n^{2g+(d-3)\alpha g})$, which yields a total number of $O(n^{(3-3/d)g+1})$ boxes. From this, the claimed $n^{\frac{d}{3-3/d}-o(1)}$ lower bound for KMP in $\mathbb{R}^d$ follows. We give all details in Section VI.

Remark 1. *Implicit in our generalization to general $d \geq 3$ is a slightly different construction for the case $d = 3$, in which we only redundantly encode suffixes of length $\frac{g}{3}$ of each group, rather than the full group as done in Section V. This change in construction does not affect the lower bound in 3D, but helps the generalization to larger dimension.*

Remark 2. *In the below theorems, we formulate our lower bounds for Klee's measure problem under the 3-uniform hyperclique* counting *conjecture. The proofs all show the same lower bounds for the* ***coverage*** *problem*[9] *under the usual 3-uniform hyperclique conjecture – our constructed sets of boxes cover the full bounding box if and only if there is no desired hyperclique in G. It is known (cf. [11]) that the coverage problem in $\mathbb{R}^d$ reduces in linear time to the* ***depth*** *problem for an arrangement of axis-parallel boxes*[10] *in $\mathbb{R}^d$. Thus we obtain a $n^{\frac{d}{3-3/d}-o(1)}$ lower bound under the 3-uniform hyperclique conjecture also for the depth problem in $\mathbb{R}^d$.*

IV. Simple $n^{\frac{d+1}{3}-o(1)}$ lower bound

In this section, we give the details for a relatively simple $n^{\frac{d+1}{3}-o(1)}$ lower bound. Higher lower bounds are proven in Sections V (for $d = 3$) and VI (for general $d \geq 3$).

Theorem IV.1. *For no $d \geq 3$ and $\epsilon > 0$, there is an $O(n^{\frac{d+1}{3}-\epsilon})$-time algorithm for Klee's measure problem in $\mathbb{R}^d$ unless the 3-unifom $(d+1)$-hyperclique counting hypothesis fails.*

Throughout the reduction, we fix the bounding box $B = [0, U)^d$ with $U = n^2$ and construct boxes of the following form:

Definition IV.2. Let $i_1, \dots, i_k$ be distinct dimensions in $[d]$. Then for any choice of intervals $I_1, \dots, I_k$ in $[0, U)$, we define the corresponding *checking box* $B(i_1 : I_1, \dots, i_k : I_k) = J_1 \times \dots \times J_d$ *where $J_i = I_\ell$ if $i = i_\ell$ for some $\ell \in [k]$ and $J_i = [0, U)$ otherwise.*

Consider any given $(d+1)$-partite 3-uniform hypergraph $G = (V_1 \cup \dots \cup V_{d+1}, E)$ where we let each V_i be a disjoint copy of $\{0, \dots, n-1\}$. Specifically, given G, we construct three types of boxes:

1) **Edges among** $V_1, \dots, V_d$**:** For every $\{a, b, c\} \in \binom{[d]}{3}$ and any non-edge $\{v_a, v_b, v_c\} \notin E$ with $v_a \in V_a, v_b \in V_b, v_c \in V_c$, construct the edge-checking box

$$\begin{aligned} C_{v_a, v_b, v_c} = B(&a : [v_a n, (v_a+1)n), \\ &b : [v_b n, (v_b+1)n), \\ &c : [v_c n, (v_c+1)n)). \end{aligned}$$

[8] E.g., to cover any non-edge $v_g^{(1)} \in V_g^{(1)}, v_g^{(3)} \in V_g^{(3)}, v_g^{(5)} \in V_g^{(5)}$ using the previous redundant encoding, we cannot make use of any redundantly encoded information (since the redundant information in dimensions 1,3,5 encodes groups 2,4,6, which is useless for covering $v_g^{(1)}, v_g^{(3)}, v_g^{(5)}$). Instead, covering this non-edge would require $\Omega(n^{3g-3})$ boxes.

[9] Given a set of axis-parallel boxes $\mathcal{B}$ inside a bounding box B_0, check whether B_0 is fully covered, i.e., $\bigcup_{B \in \mathcal{B}} B = B_0$.

[10] Given a set of axis-parallel boxes $\mathcal{B}$, determine the point contained in the maximum number of $B \in \mathcal{B}$.

2) **Edges involving V_{d+1}:** For every $\{a,b\} \in \binom{[d]}{2}$ and any non-edge $\{v_a, v_b, v_{d+1}\} \notin E$ with $v_a \in V_a, v_b \in V_b, v_{d+1} \in V_{d+1}$, construct the edge-checking box

$$\begin{aligned} C_{v_a,v_b,v_{d+1}} = B(a &: [v_a n + v_{d+1}, v_a n + v_{d+1} + 1), \\ b &: [v_b n + v_{d+1}, v_b n + v_{d+1} + 1)). \end{aligned}$$

3) **Consistency boxes:** For every $\{a,b\} \in \binom{[d]}{2}$ and any vertices $v_a \in V_a, v_b \in V_b, v_{d+1} \in V_{d+1}$, construct the consistency-checking boxes

$$\begin{aligned} D^{\text{lower}}_{v_a,v_b,v_{d+1}} = B(a &: [v_a n + v_{d+1}, v_a n + v_{d+1} + 1), \\ b &: [v_b n, v_b n + v_{d+1})) \\ D^{\text{higher}}_{v_a,v_b,v_{d+1}} = B(a &: [v_a n + v_{d+1}, v_a n + v_{d+1} + 1), \\ b &: [v_b n + v_{d+1} + 1, (v_b + 1)n)) \end{aligned}$$

Let $\mathcal{B}$ denote the union of boxes. Observe that each box in $\mathcal{B}$ is of the form $\prod_{i=1}^d [c_i, c_i + d_i)$ for some $c_i \in \{0, \dots, U-1\}, d_i \in \{1, \dots, U\}$, and hence any unit cube $C = \prod_{i=1}^d [c_i, c_i + 1)$ with $c_i \in \{0, \dots, U-1\}$ either does not intersect any box in $\mathcal{B}$ or is fully contained in some box $B \in \mathcal{B}$. In the latter case, we say that C *is covered by* $\mathcal{B}$.

Lemma IV.3. *Let $v_1, \dots, v_d, \bar{v}_1, \dots, \bar{v}_d \in \{0, \dots, n-1\}$ and define $A \subseteq [0,U]^d$ as the axis-parallel cube whose i-th dimension is equal to $[v_i n + \bar{v}_i, v_i n + \bar{v}_i + 1)$. Then A is not covered by $\mathcal{B}$ if and only if there is some $v_{d+1} \in \{0, \dots, n-1\}$ such that $\bar{v}_1 = \dots = \bar{v}_d = v_{d+1}$ and $\{v_1, \dots, v_{d+1}\}$ forms a clique in G.*

Proof. Consider the case that A is not covered by $\mathcal{B}$. We observe that this implies $\bar{v}_1 = \dots = \bar{v}_d$: Otherwise, there would be $\{a,b\} \in \binom{[d]}{2}$ such that $\bar{v}_a \neq \bar{v}_b$. Observe that in this case, we would have $A \subseteq D^{\text{lower}}_{v_a,v_b,\bar{v}_a}$ if $\bar{v}_b < \bar{v}_a$ or $A \subseteq D^{\text{higher}}_{v_a,v_b,\bar{v}_a}$ if $\bar{v}_b > \bar{v}_a$, and thus A would be covered by $\mathcal{B}$. From now on, we may assume that $\bar{v}_1 = \dots = \bar{v}_d = v_{d+1}$ for some $v_{d+1} \in \{0, \dots, n-1\}$.

We observe that there cannot be any edge missing among $\{v_1, \dots, v_{d+1}\}$: Consider first the case that there is some $\{a,b,c\} \in \binom{[d]}{3}$ such that $\{v_a, v_b, v_c\} \notin E$. Then $A \subseteq C_{v_a,v_b,v_c}$ as $[v_i n + v_{d+1}, v_i n + v_{d+1} + 1) \subseteq [v_i n, (v_i+1)n)$ for $i \in \{a,b,c\}$. It remains to argue that there is no edge involving v_{d+1}, i.e., no $\{a,b\} \in \binom{[d]}{2}$ such that $\{v_a, v_b, v_{d+1}\} \notin E$. However, then we would have $A \subseteq C_{v_a,v_b,v_{d+1}}$. Thus, $\{v_1, \dots, v_{d+1}\}$ forms a clique in G.

Conversely, if $\{v_1, \dots, v_{d+1}\}$ forms a clique in G, then the corresponding cube A with i-th dimension $[v_i n + v_{d+1}, v_i n + v_{d+1} + 1)$ cannot be covered by $\mathcal{B}$: Observe that $C_{v'_a,v'_b,v'_c}$ boxes (which implies $\{v'_a, v'_b, v'_c\} \notin E$) can only cover A if $(v'_a, v'_b, v'_c) = (v_a, v_b, v_c)$; however, no such box C_{v_a,v_b,v_c} can exist, since $\{v_1, \dots, v_{d+1}\}$ is a clique. Furthermore, observe that A cannot be covered by any box $D^{\text{lower}}_{v'_a,v'_b,v'_{d+1}}$ or $D^{\text{higher}}_{v'_a,v'_b,v'_{d+1}}$ by definition of A (as it chooses v_{d+1} consistently in all dimensions). This concludes the claim. $\square$

With the above reduction, we can prove our lower bound.

Proof of Theorem IV.1. Assume that there is an $O(N^{\frac{d+1}{3}-\epsilon})$-algorithm for computing the volume of the union of N d-dimensional axis-parallel boxes. Then given a 3-uniform $(d+1)$-partite hypergraph G, we construct the set of boxes $\mathcal{B}$ as above. Observe that by definition, $\mathcal{B}$ consists of $N = O(\binom{d+1}{3} n^3) = O(n^3)$ boxes in $[0, n^2)^d$ that can be computed in time $O(N)$. By Lemma IV.3 (noting that the A's yield a set of disjoint cubes of volume 1 each), we obtain that there are $n^{2d} - \mathrm{vol}(\bigcup_{B \in \mathcal{B}} B)$ many $(d+1)$-cliques in G. Since we can compute $\mathrm{vol}(\bigcup_{B \in \mathcal{B}} B)$ in time $O(N^{\frac{d+1}{3}-\epsilon}) = O(n^{d+1-3\epsilon})$ by assumption, this would refute the 3-uniform $(d+1)$-hyperclique hypothesis. $\square$

V. Tight Bound for $d = 3$

In this section, we give the tight $n^{3/2-o(1)}$ lower bound for $\mathbb{R}^3$. To prepare the proof, we define for any tuple $t = (t_1, \dots, t_L) \in \{0, \dots, n-1\}^L$ (throughout this section, we will use $L = 2g$), a corresponding index

$$\mathrm{ind}(t) = t_1 \cdot n^{L-1} + t_2 \cdot n^{L-2} + \dots + t_L,$$

with the understanding that $[\mathrm{ind}(t), \mathrm{ind}(t)+1)$ represents the tuple t. Crucially, we use that we may represent all tuples t with a prefix $p = (p_1, \dots, p_\ell)$ with $\ell \leq L$ using the interval:

$$\begin{aligned} &[\mathrm{ind}(p_1, \dots, p_\ell, 0, \dots, 0), \mathrm{ind}(p_1, \dots, p_\ell, 0, \dots, 0) + n^{L-\ell}) \\ = &[\mathrm{ind}(p_1, \dots, p_{\ell-1}, \boldsymbol{p_\ell}, 0, \dots, 0), \\ &\ \mathrm{ind}(p_1, \dots p_{\ell-1}, \boldsymbol{p_\ell + 1}, 0, \dots, 0)); \end{aligned}$$

here the bold font highlights the single coordinate with changes. (When $p_\ell = n-1$, this is a slight abuse of notation, since we implicitly extend the definition of $\mathrm{ind}(t)$ to tuples $t \in \{0, \dots, n\}^L$ in the natural way.) Crucially, this interval is equal to the union of all $[\mathrm{ind}(t), \mathrm{ind}(t)+1)$ where p is a prefix of t, i.e., $t = (p_1, \dots, p_\ell, t_{\ell+1}, \dots, t_L)$.

We are ready to prove the result of this section.

Theorem V.1. *There is no $\epsilon > 0$ such that Klee's measure problem in 3D can be solved in time $O(n^{1.5-\epsilon})$ unless the 3-uniform hyperclique counting conjecture fails.*

Proof. Assume there exists some $\epsilon > 0$ such that Klee's measure problem in 3D can be solved in time $O(n^{1.5-\epsilon})$. Choose any $g \geq 1/\epsilon$ and consider an arbitrary 3-uniform $3g$-hyperclique instance $G = (V, E)$ with

$$V = V_1^{(1)} \cup \dots \cup V_g^{(1)} \cup V_1^{(2)} \cup \dots \cup V_g^{(2)} \cup V_1^{(3)} \cup \dots \cup V_g^{(3)}.$$

We define a corresponding 3D-KMP instance $\mathcal{B}$ inside the bounding box $[0,U]^3$ with $U = n^{2g}$. To this end, for any $a \in [3]$, $b \in [g]$ and $v = (v_1, \dots, v_g), v' = (v'_1, \dots, v'_g) \in \{0, \dots, n-1\}^g$, we define the consistency-checking boxes (using checking boxes $B(\cdot)$ as introduced in Definition IV.2) as follows:[11]

$$
\begin{aligned}
&D^{\text{lower}}_{a,b,v,v'}\\
&\quad = B(a : [\text{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_{b+1}, \mathbf{0}, 0, \ldots, 0),\\
&\qquad\qquad \text{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_{b+1}, \boldsymbol{v'_b}, 0, \ldots, 0)),\\
&\qquad a+1 : [\text{ind}(v'_1, \ldots, v'_{b-1}, \boldsymbol{v'_b}, 0, \ldots, 0),\\
&\qquad\qquad \text{ind}(v'_1, \ldots, v'_{b-1}, \boldsymbol{v'_b + 1}, 0, \ldots, 0))),\\
&D^{\text{upper}}_{a,b,v,v'}\\
&\quad = B(a : [\text{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_{b+1}, \boldsymbol{v'_b + 1}, 0, \ldots, 0),\\
&\qquad\qquad \text{ind}(v_1, \ldots, v_g, v'_g, \ldots, v'_{b+1}, \boldsymbol{n}, 0, \ldots, 0)),\\
&\qquad a+1 : [\text{ind}(v'_1, \ldots, v'_{b-1}, \boldsymbol{v'_b}, 0, \ldots, 0),\\
&\qquad\qquad \text{ind}(v'_1, \ldots, v'_{b-1}, \boldsymbol{v'_b + 1}, 0, \ldots, 0))).
\end{aligned}
$$

Furthermore, for any $0 \le i, j, k \le 2g$ with $i + j + k \le 2g + 1$ and $v^{(1)} = (v^{(1)}_1, \ldots, v^{(1)}_i) \in \{0, \ldots, n-1\}^i, v^{(2)} = (v^{(2)}_1, \ldots, v^{(2)}_j) \in \{0, \ldots, n-1\}^j, v^{(3)} = (v^{(3)}_1, \ldots, v^{(3)}_k) \in \{0, \ldots, n-1\}^k$, we do the following: For the first i sets of $(V^{(1)}_1, \ldots, V^{(1)}_g, V^{(2)}_g, \ldots, V^{(2)}_1)$, we choose vertices according to $v^{(1)}$, for the first j sets of $(V^{(2)}_1, \ldots, V^{(2)}_g, V^{(3)}_g, \ldots, V^{(3)}_1)$ we choose vertices according to $v^{(2)}$ and for the first k sets of $(V^{(3)}_1, \ldots, V^{(3)}_g, V^{(1)}_g, \ldots, V^{(1)}_1)$, we choose vertices according to $V^{(3)}$. If this yields a set S_{partial} of $i + j + k$ vertices from distinct sets (i.e., we disregard choices like $i = 2g, j = 1, k = 0$, as it would contain possibly two choices for $V^{(2)}_1$) *and* there exists a non-edge $\{v, v', v''\} \notin E$ among vertices $v, v', v'' \in S_{\text{partial}}$, we define the edge-checking box

$$
\begin{aligned}
C_{v^{(1)},v^{(2)},v^{(3)}} = B(1 &: [\text{ind}(v^{(1)}_1, \ldots, \boldsymbol{v^{(1)}_i}, 0, \ldots, 0),\\
&\quad \text{ind}(v^{(1)}_1, \ldots, \boldsymbol{v^{(1)}_i + 1}, 0, \ldots, 0)),\\
2 &: [\text{ind}(v^{(2)}_1, \ldots, \boldsymbol{v^{(2)}_j}, 0, \ldots, 0),\\
&\quad \text{ind}(v^{(2)}_1, \ldots, \boldsymbol{v^{(2)}_j + 1}, 0, \ldots, 0)),\\
3 &: [\text{ind}(v^{(3)}_1, \ldots, \boldsymbol{v^{(3)}_k}, 0, \ldots, 0),\\
&\quad \text{ind}(v^{(3)}_1, \ldots, \boldsymbol{v^{(3)}_k + 1}, 0, \ldots, 0))).
\end{aligned}
$$

We call S_{partial} the *partial solution* encoded by $v^{(1)}, v^{(2)}, v^{(3)}$.

We let $\mathcal{B}$ denote the set of all boxes $D^{\text{lower}}_{a,b,v,v'}, D^{\text{upper}}_{a,b,v,v'}, C_{v^{(1)},v^{(2)},v^{(3)}}$ constructed this way. Observe that there are at most $6gn^{2g} = O(n^{2g})$ consistency-checking boxes and at most $(2g+1)^3 n^{2g+1} = O(n^{2g+1})$ edge-checking boxes, and that $\mathcal{B}$ can be computed in time $O(n^{2g+1})$.

By the following claim, we have that $n^{6g} - \text{vol}(\bigcup_{B \in \mathcal{B}} B)$ is equal to the number of $3g$-cliques in G. Recall that we call a unit hypercube $H = \prod_i [h_i, h_i + 1)$ with $h_i \in \{0, \ldots, U-1\}$ *covered* by $\mathcal{B}$, if it is contained in some $B \in \mathcal{B}$. Otherwise, observe that it does not intersect any $B \in \mathcal{B}$.

Claim V.2. Let $H = [h_1, h_1 + 1) \times [h_2, h_2 + 1) \times [h_3, h_3 + 1)$ with $h_1, h_2, h_3 \in \{0, \ldots, n^{2g} - 1\}$. Then H is not covered by $\mathcal{B}$ if and only if there is $S = (v^{(1)}_1, \ldots, v^{(1)}_g, v^{(2)}_1, \ldots, v^{(2)}_g, v^{(3)}_1, \ldots, v^{(3)}_g) \in V^{(1)}_1 \times \cdots \times V^{(1)}_g \times V^{(2)}_1 \times \cdots \times V^{(2)}_g \times V^{(3)}_1 \times \cdots \times V^{(3)}_g$ such that $h_a = \text{ind}(v^{(a)}_1, \ldots, v^{(a)}_g, v^{(a+1)}_g, \ldots, v^{(a+1)}_1)$ for all $a \in [3]$ and S is a clique in G.

Proof. We start with the easy direction. For a clique $S = (v^{(1)}_1, \ldots, v^{(1)}_g, v^{(2)}_1, \ldots, v^{(2)}_g, v^{(3)}_1, \ldots, v^{(3)}_g) \in V^{(1)}_1 \times \cdots \times V^{(1)}_g \times V^{(2)}_1 \times \cdots \times V^{(2)}_g \times V^{(3)}_1 \times \cdots \times V^{(3)}_g$, we show that the corresponding cube $H = [h_1, h_1+1) \times [h_2, h_2+1) \times [h_3, h_3+1)$ with $h_a = \text{ind}(v^{(a)}_1, \ldots, v^{(a)}_g, v^{(a+1)}_g, \ldots, v^{(a+1)}_1)$ for $a \in [3]$ does not intersect any box in $\mathcal{B}$:

- Any $D^{\text{lower}}_{a,b,v,v'}$ can intersect $[h_{a+1}, h_{a+1} + 1)$ in dimension $a + 1$ only if $(v^{(a+1)}_1, \ldots, v^{(a+1)}_b) = (v'_1, \ldots, v'_b)$. Likewise, it can intersect $[h_a, h_a + 1)$ in dimension a only if we have $(v^{(a)}_1, \ldots, v^{(a)}_g) = (v_1, \ldots, v_g)$, $(v^{(a+1)}_g, \ldots, v^{(a+1)}_{b+1}) = (v'_g, \ldots, v'_{b+1})$ and $v^{(a+1)}_b < v'_b$. Thus, if $D^{\text{lower}}_{a,b,v,v'}$ intersects H, we obtain the contradiction $v'_b = v^{(a+1)}_b < v'_b$.
- Similarly, any $D^{\text{upper}}_{a,b,v,v'}$ intersecting H would give the contradiction $v'_b = v^{(a+1)}_b > v'_b$.
- Finally, any $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ with $\tilde{v}^{(1)} \in \{0, \ldots, n-1\}^{i_1}, \tilde{v}^{(2)} \in \{0, \ldots, n-1\}^{i_2}, \tilde{v}^{(3)} \in \{0, \ldots, n-1\}^{i_3}$ can intersect H only if for all $a \in [3]$, we have that the vertices chosen by S in the first i_a sets of $(V^{(a)}_1, \ldots, V^{(a)}_g, V^{(a+1)}_g, \ldots, V^{(a+1)}_1)$ agree with $v^{(a)}$. Note that $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ exists only if the partial solution S_{partial} encoded by $\tilde{v}^{(1)}, \tilde{v}^{(2)}, \tilde{v}^{(3)}$ contains vertices $w_1, w_2, w_3 \in S_{\text{partial}}$ forming a non-edge $\{w_1, w_2, w_3\} \notin E$. Thus, if $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ intersects H, we have $w_1, w_2, w_3 \in S_{\text{partial}} \subseteq S$, which yields the contradiction that S is not a clique.

We proceed with the more interesting direction. Let $H = [h_1, h_1 + 1) \times [h_2, h_2 + 1) \times [h_3, h_3 + 1)$ with $h_a \in \{0, \ldots, n^{2g} - 1\}$ be such that H is not covered by $\mathcal{B}$. We let $v^{(a)}_1, \ldots, v^{(a)}_g, \bar{v}^{(a)}_1, \ldots, \bar{v}^{(a)}_g$ for $a \in [3]$ be such that $h_a = \text{ind}(v^{(a)}_1, \ldots v^{(a)}_g, \bar{v}^{(a+1)}_g, \ldots, \bar{v}^{(a+1)}_1)$ for $a \in [3]$. We will show that H encodes a clique S as follows:

- Since H is not covered by any $D^{\text{lower}}_{a,b,v,v'}, D^{\text{upper}}_{a,b,v,v'}$, we claim that $(\bar{v}^{(a)}_1, \ldots, \bar{v}^{(a)}_g) = (v^{(a)}_1, \ldots, v^{(a)}_g)$ for all $a \in [3]$. Otherwise, consider $a \in [3], b \in [g]$ such that $v^{(a+1)}_b \neq \bar{v}^{(a+1)}_b$: If $\bar{v}^{(a+1)}_b < v^{(a+1)}_b$, then define $v = (v^{(a)}_1, \ldots, v^{(a)}_g)$ and $v' = (v^{(a+1)}_1, \ldots, v^{(a+1)}_b, \bar{v}^{(a+1)}_{b+1}, \ldots, \bar{v}^{(a+1)}_g)$ and observe that $D^{\text{lower}}_{a,b,v,v'}$ would contain H, yielding a contradiction. Symmetrically, if $\bar{v}^{(a+1)}_b > v^{(a+1)}_b$, then define $v = (v^{(a)}_1, \ldots, v^{(a)}_g)$ and $v' = (v^{(a+1)}_1, \ldots, v^{(a+1)}_b, \bar{v}^{(a+1)}_{b+1}, \ldots, \bar{v}^{(a+1)}_g)$ and observe that $D^{\text{upper}}_{a,b,v,v'}$ would contain H.
- Thus, this gives a set $S = (v^{(1)}_1, \ldots, v^{(1)}_g, v^{(2)}_1, \ldots, v^{(2)}_g, v^{(3)}_1, \ldots, v^{(3)}_g) \in V^{(1)}_1 \times \cdots \times V^{(1)}_g \times V^{(2)}_1 \times \cdots \times V^{(2)}_g \times V^{(3)}_1 \times \cdots \times V^{(3)}_g$ such that $h_a = \text{ind}(v^{(a)}_1, \ldots, v^{(a)}_g, v^{(a+1)}_g, \ldots, v^{(a+1)}_1)$

[11] Recall that we use a circular order for $a \in [3]$, so that whenever $a = 3$, we have that $a + 1$ equals 1.

for all $a \in [3]$. It remains to show that S is indeed a clique: Assume for contradiction that there are $w_1, w_2, w_3 \in S$ such that $\{w_1, w_2, w_3\} \notin E$. We show that then H would be covered by a corresponding box $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$, yielding a contradiction. First, if $\{w_1, w_2, w_3\}$ contains two vertices from the same group, i.e., $\{w_1, w_2, w_3\} = \{v_i^{(a)}, v_j^{(a)}, v_k^{(a')}\}$ for $a, a' \in [3]$, then define $\tilde{v}^{(1)}, \tilde{v}^{(2)}, \tilde{v}^{(3)}$ such that $\tilde{v}^{(a)} = (v_1^{(a)}, \dots, v_g^{(a)})$, $\tilde{v}^{(a')} = (v_1^{(a')}, \dots, v_g^{(a')})$ and $\tilde{v}^{(a'')}$ being empty for all $a'' \in [3] \setminus \{a, a'\}$ (observe that this is well-defined even if $a = a'$). Observe that $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ exists, since it chooses a partial solution S_{partial} of at most $2g$ vertices that contains the non-edge $\{w_1, w_2, w_3\}$; thus, H would be covered by $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$, since $\tilde{v}^{(1)}, \tilde{v}^{(2)}, \tilde{v}^{(3)}$ agree with S. It remains to consider the most interesting case, in which all vertices are from different groups, i.e., $\{w_1, w_2, w_3\} = \{v_i^{(1)}, v_j^{(2)}, v_k^{(3)}\}$. If k is the smallest index among i, j, k, then we claim that H is covered by $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ with

$$\begin{aligned}
\tilde{v}^{(1)} &= (v_1^{(1)}, \dots, v_g^{(1)}, v_g^{(2)}, \dots, v_j^{(2)}), \\
\tilde{v}^{(2)} &= (), \\
\tilde{v}^{(3)} &= (v_1^{(3)}, \dots, v_k^{(3)}).
\end{aligned}$$

Note that the prefixes $\tilde{v}^{(1)}, \tilde{v}^{(2)}, \tilde{v}^{(3)}$ encode a partial solution containing the non-edge vertices w_1, w_2, w_3 and that their lengths sum up to $g + (g - j + 1) + k = 2g + 1 + (k - j) \leq 2g + 1$ since $k \leq j$. Thus, $C_{\tilde{v}^{(1)},\tilde{v}^{(2)},\tilde{v}^{(3)}}$ exists and covers H, as $\tilde{v}^{(1)}, \tilde{v}^{(2)}, \tilde{v}^{(3)}$ agree with S. The remaining cases where i or j are smallest among i, j, k are symmetric: we use

$$\begin{aligned}
\tilde{v}^{(1)} &= (v_1^{(1)}, \dots, v_i^{(1)}), \\
\tilde{v}^{(2)} &= (v_1^{(2)}, \dots, v_g^{(2)}, v_g^{(3)}, \dots, v_k^{(3)}), \\
\tilde{v}^{(3)} &= (),
\end{aligned}$$

or

$$\begin{aligned}
\tilde{v}^{(1)} &= (), \\
\tilde{v}^{(2)} &= (v_1^{(2)}, \dots, v_j^{(2)}), \\
\tilde{v}^{(3)} &= (v_1^{(3)}, \dots, v_k^{(3)}, v_g^{(1)}, \dots, v_i^{(1)}),
\end{aligned}$$

respectively.

□

We are ready to conclude the proof of V.1: Given the 3-uniform $3g$-hyperclique instance G, we compute the set $\mathcal{B}$ with $|\mathcal{B}| = O(n^{2g+1})$ boxes in time $O(|\mathcal{B}|)$. By the previous claim, the number of $3g$-cliques in G is given by $n^{6g} - \mathrm{vol}(\bigcup_{B \in \mathcal{B}} B)$. Thus, using a $O(|\mathcal{B}|^{1.5-\epsilon})$-algorithm for Klee's measure problem in 3D, we can count the $3g$-cliques in time $O(n^{(2g+1)(1.5-\epsilon)}) = O(n^{3g+1.5-(2g+1)\epsilon})$. Since $g \geq 1/\epsilon$, we have $3g + 1.5 - (2g+1)\epsilon \leq 3g + 1.5 - 2 = 3g - 0.5$. Thus, we would obtain a 3-uniform $3g$-clique counting algorithm running in time $O(n^{3g-0.5})$, refuting the 3-uniform hyperclique counting conjecture. □

VI. Generalization to $d \geq 3$

In this section, we extend our tight lower bound for $d = 3$ to an $n^{\frac{d}{3-3/d} - o(1)}$ lower bound in general dimension $d \geq 3$.

Theorem VI.1. *Let $d \geq 3$. There is no $\epsilon > 0$ such that Klee's measure problem in $\mathbb{R}^d$ can be solved in time $O(n^{\frac{d}{3-3/d}-\epsilon}) = O(n^{\frac{d+1}{3} + \frac{1}{3(d-1)} - \epsilon})$ unless the 3-uniform hyperclique conjecture fails.*

Let g be large enough integer divisible by d. Let $G = (V, E)$ be a 3-uniform dg-partite hypergraph with

$$V = V_1^{(1)} \cup \dots \cup V_g^{(1)} \cup \dots \cup V_1^{(d)} \cup \dots \cup V_g^{(d)},$$

where each $V_i^{(a)}$ is a disjoint copy of $\{0, \dots, n-1\}$. We call $V_1^{(a)} \cup \dots \cup V_g^{(a)}$ the a-th *group*. When referring to groups, we will let $a + 1$ denote the group subsequent to a in circular order of $[d]$, i.e., we identify $d + 1$ with 1.

To a candidate solution $S = (v_1^{(1)}, \dots, v_g^{(1)}, \dots, v_1^{(d)}, \dots, v_g^{(d)}) \in V_1^{(1)} \times \dots \times V_g^{(1)} \times \dots \times V_1^{(d)} \times \dots \times V_g^{(d)}$, we will associate a corresponding unit hypercube $Q(S)$ in $[0, n^L]^d$ with $L = \left(2 - \frac{2}{d}\right) g$. The a-th dimension of $Q(S)$ consists of an encoding of the vertices $v_1^{(a)}, \dots, v_g^{(a)}$ chosen by S in the a-th group, as well as a *redundant encoding* part that encodes a certain choice of $(1 - \frac{2}{d})g$ vertices from other groups in a carefully chosen way.

To define the encoding, for any $a \in [d]$, we define the *set template* $T^{(a)}$ as

$$T^{(a)} := (V_1^{(a)}, \dots, V_g^{(a)}) \circ R^{(a)},$$

where $(V_1^{(a)}, \dots, V_g^{(a)})$ is the *main part* of the template, and $R^{(a)}$ denotes the *redundant part*, defined as a *reversed interleaving* of the last g/d sets of all groups except the a-th and $(a+1)$-st. Formally, we define

$$\begin{aligned}
R_S^{(a)} = & \bigcirc_{b=1}^{g/d} \bigcirc_{a' \in [d] \setminus \{a, a+1\}} V_{g-b+1}^{(a')} \\
= & (V_g^{(1)}, \dots, V_g^{(a-1)}, V_g^{(a+2)}, \dots, V_g^{(d)}, \\
& \dots, \\
& V_{g-g/d+1}^{(1)}, \dots, V_{g-g/d+1}^{(a-1)}, V_{g-g/d+1}^{(a+2)}, \dots, V_{g-g/d+1}^{(d)}).
\end{aligned}$$

Put differently, from the sequence $(V_g^{(1)}, \dots, V_g^{(d)}, \dots, V_{g-g/d+1}^{(1)}, \dots, V_{g-g/d+1}^{(d)})$ we leave out all sets of the form $V_b^{(a)}$ or $V_b^{(a+1)}$ to obtain $R^{(a)}$ (recall that for $a = d$ this means that we leave out the sets from groups d and 1). In dimension a, we will encode a choice of vertices in the sets given by $T^{(a)}$.

For any candidate solution S, we let $t_S^{(a)}$ denote the encoding of S in the a-th dimension according to $T^{(a)}$, i.e.,

$$
\begin{aligned}
t_S^{(a)} = (&\underbrace{v_1^{(a)}, \dots, v_g^{(a)}}_{\text{main part}},\\
&\underbrace{v_g^{(1)}, \dots, v_g^{(a-1)}, v_g^{(a+2)}, \dots, v_g^{(d)}}_{g\text{-th elements of redundant groups}},\\
&\dots,\\
&\underbrace{v_{g-g/d+1}^{(1)}, \dots, v_{g-g/d+1}^{(a-1)}, v_{g-g/d+1}^{(a+2)}, \dots, v_{g-g/d+1}^{(d)}}_{(g-g/d+1)\text{-st elements of redundant groups}}).
\end{aligned}
$$

As before, we will use a lexicographic encoding to map any $t_S^{(a)}$ to an interval in $[0, U)$: Given any tuple $t = (t_1, \dots, t_L) \in \{0, \dots, n-1\}^L$, we use the corresponding index

$$\mathrm{ind}(t) = t_1 \cdot n^{L-1} + t_2 \cdot n^{L-2} + \dots + t_L,$$

with the understanding that $[\mathrm{ind}(t), \mathrm{ind}(t)+1)$ represents the tuple t. We define $Q(S) = \prod_{a=1}^{d} [\mathrm{ind}(t_S^{(a)}), \mathrm{ind}(t_S^{(a)})+1)$.

The prefix of the first ℓ positions of $t = (t_1, \dots, t_L)$ will be denoted as $t[..\ell] = (t_1, \dots, t_\ell)$. Similar to before, tuples with prefix $p = (p_1, \dots, p_\ell)$ are uniquely represented by a corresponding interval

$$
\begin{aligned}
I(p) = [&\mathrm{ind}(p_1, \dots, p_\ell, 0, \dots, 0),\\
&\mathrm{ind}(p_1, \dots, p_\ell, 0, \dots, 0) + n^{L-\ell}).
\end{aligned}
$$

Note that this interval is equal to the union of all $[\mathrm{ind}(t), \mathrm{ind}(t)+1)$ such that $p_1, \dots, p_\ell$ is a prefix of t.

We are ready to define the set of boxes $\mathcal{B}$ for the given dg-partite 3-uniform hyperclique instances G.

Edge-checking box: For all $a, b, c \in \binom{[d]}{3}, 0 \le i, j, k \le L$ with $i + j + k \le (3 - 3/d)g + 1$ and all $v^{(a)} \in \{0, \dots, n-1\}^i, v^{(b)} \in \{0, \dots, n-1\}^j, v^{(c)} \in \{0, \dots, n-1\}^k$, we include the **edge-checking box**

$$C_{v^{(a)}, v^{(b)}, v^{(c)}} = B(a : I(v^{(a)}), b : I(v^{(b)}), c : I(v^{(c)}))$$

if and only if $v^{(a)}, v^{(b)}, v^{(c)}$ give a valid encoding[12] (according to $T^{(a)}, T^{(b)}, T^{(c)}$) of a partial solution $S_{\mathrm{partial}} \subseteq V$ that contains a non-edge $\{w_1, w_2, w_3\} \notin E$, i.e., $w_1, w_2, w_3 \in S_{\mathrm{partial}}$.

Consistency-checking box: For all $a \in [d], a' \in [d] \setminus \{a, a+1\}$, as well as $v \in \{0, \dots, n-1\}^g, v' \in \{0, \dots, n-1\}^g$ and $v_{\mathrm{other}} \in \{0, \dots, n-1\}^{(1-3/d)g}$, we consider all $0 \le b < g/d$ and $\bar{v} \in \{0, \dots, n-1\}$ such that $\bar{v} \ne v'_{g-b}$. We let v_R denote the encoding of the redundant part $R^{(a)}$ given by using $(v'_g, \dots, v'_{g-b+1}, \bar{v}, v'_{g-b-1}, v'_{g-g/d+1})$ for the vertices corresponding to the group a', and using v_{other} for the vertices corresponding to the remainings groups $[d] \setminus \{a, a+1, a'\}$. We define the **consistency-checking box**

$$
\begin{aligned}
D_{a, a', b, v, v', v_{\mathrm{other}}, \bar{v}} = B(&a : I(v \circ v_R[..(d-2)b]),\\
&a' : I(v'[..g-b])).
\end{aligned}
$$

[12]More precisely, an encoding is valid, if the set obtained by choosing the vertices for the first i sets of $T^{(a)}$ according to $v^{(a)}$, the vertices for the first j sets of $T^{(b)}$ according to $v^{(b)}$ and the vertices for the first k sets of $T^{(c)}$ according to $v^{(c)}$ yields a set S_{partial} of vertices, where for each set $V_{b'}^{(a')}$ we chose at most one vertex.

Recall that we call a unit hypercube $H = \prod_{i=1}^{d} [h_i, h_i + 1)$ with $h_i \in \{0, \dots, U-1\}$ *covered* by $\mathcal{B}$, if it is contained in some $B \in \mathcal{B}$. Otherwise, observe that it does not intersect any $B \in \mathcal{B}$, and we call it *uncovered*.

Claim VI.2. Let $H = [h_1, h_1 + 1) \times \dots \times [h_d, h_d + 1)$ be a unit hypercube in $[0, U)^d$ with $h_i \in \{0, \dots, n^L - 1\}$. Then H is uncovered by $\mathcal{B}$ if and only if $H = Q(S)$ for some set $S = (v_1^{(1)}, \dots, v_g^{(1)}, \dots, v_1^{(d)}, \dots, v_g^{(d)}) \in V_1^{(1)} \times \dots \times V_g^{(1)} \times \dots \times V_1^{(d)} \times \dots \times V_g^{(d)}$ that forms a clique in G.

Proof. Consider a clique $S = (v_1^{(1)}, \dots, v_g^{(1)}, \dots, v_1^{(d)}, \dots, v_g^{(d)}) \in V_1^{(1)} \times \dots \times V_g^{(1)} \times \dots \times V_1^{(d)} \times \dots \times V_g^{(d)}$ and let $H = Q(S)$, i.e., $H = [h_1, h_1+1) \times \dots \times [h_d, h_d+1)$ with $h_a = \mathrm{ind}(t_S^{(a)})$. We claim that it cannot intersect any box in $\mathcal{B}$:

- Consider the case that some $D_{a, a', b, v, v', v_{\mathrm{other}}, \bar{v}}$ intersects H: Then $I(v'[..g-b])$ intersecting $[h_{a'}, h_{a'}+1)$ requires that $(v_1^{(a')}, \dots, v_{g-b}^{(a')}) = (v'_1, \dots, v'_{g-b})$, in particular $v_{g-b}^{(a')} = v'_{g-b}$. Similarly, $I(v \circ v_R[..(d-2)b])$ intersecting $[h_a, h_a + 1)$ requires, by definition of v_R, particularly that $v_{g-b}^{(a')} = \bar{v}$. We obtain the contradiction $v_{g-b}^{(a')} = \bar{v} \ne v'_{g-b} = v_{g-b}^{(a')}$.
- It remains to consider the case that some $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ with $\tilde{v}^{(a)} \in \{0, \dots, n-1\}^i$, $\tilde{v}^{(b)} \in \{0, \dots, n-1\}^j$ and $\tilde{v}^{(c)} \in \{0, \dots, n-1\}^k$ intersects H: Then let S_{partial} denote the choice of vertices given by using $\tilde{v}^{(a)}$ to choose vertices for the first i sets of $T^{(a)}$, as well as $\tilde{v}^{(b)}$ for the first j sets of $T^{(b)}$ and $\tilde{v}^{(c)}$ for the first k sets of $T^{(c)}$. Since $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ intersects H, we must have $S_{\mathrm{partial}} \subseteq S$. Since S_{partial} contains vertices w_1, w_2, w_3 forming a non-edge in G, we obtain a contradiction to S being a clique.

It remains to show that if $H = [h_1, h_1 + 1) \times \dots \times [h_d, h_d + 1)$ with $h_a \in \{0, \dots, n^L - 1\}$ is not equal to $Q(S)$ for some clique S, then H is contained in $\cup_{B \in \mathcal{B}} B$. For all $a \in [d]$, let $v_1^{(a)}, \dots, v_g^{(a)}, \bar{v}_1^{(a)}, \dots, \bar{v}_{(1-2/d)g}^{(a)}$ such that $h_i = \mathrm{ind}(v_1^{(a)}, \dots, v_g^{(a)}, \bar{v}_1^{(a)}, \dots, \bar{v}_{(1-2/d)g}^{(a)})$. Let $S = (v_1^{(1)}, \dots, v_g^{(1)}, \dots, v_1^{(d)}, \dots, v_g^{(d)})$. If $Q(S) \ne H$, then there are $a, a' \in [d]$ and $0 \le b \le g/d$ such that $v_{g-b}^{(a')}$ is different from the entry $\bar{v}$ in $t_S^{(a)}$ corresponding to $V_{g-b}^{(a')}$ according to $T^{(a)}$. We consider the smallest such b and show that a corresponding consistency-checking box contains H: Define $v = (v_1^{(a)}, \dots, v_g^{(a)})$, $v' = (v_1^{(a')}, \dots, v_g^{(a')})$, and obtain v_{other} as the vertices chosen by $\bar{v}_1^{(a)}, \dots, \bar{v}_{(1-2/d)g}^{(a)}$ for the redundant vertices of the groups except $a, a+1$ and a' (according to $T^{(a)}$). It is straightforward to see that H is contained in the corresponding consistency-checking box $D_{a, a', b, v, v', v_{\mathrm{other}}, \bar{v}}$.

Thus, it remains to show that if $H(S)$ is uncovered by $\mathcal{B}$, then S must indeed form a clique in G. Thus, assume for contradiction that there are $w_1, w_2, w_3 \in S$ such that $\{w_1, w_2, w_3\} \notin E$. Let $a, b, c \in [d]$ and $i, j, k \in [g]$ be such that $w_1 = v_i^{(a)}, w_2 = v_j^{(b)}$ and $w_3 = v_k^{(c)}$. First, assume that $\{a, b, c\}$ are not all distinct, say that $b = c = a'$

for some $a' \in [d]$. Then define $\tilde{v}^{(a)} = (v_1^{(a)}, \dots, v_g^{(a)})$, $\tilde{v}^{(a')} = (v_1^{(a')}, \dots, v_g^{(a')})$ and $\tilde{v}^{(a'')}$ as empty prefix for an arbitrary $a'' \notin [d] \setminus \{a, a'\}$, and observe that the edge-checking box $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ exists (in particular, it chooses at most $2g \le (3 - 3/d)g + 1$ vertices) and contains $H(S)$. Thus, the most interesting case in which $\{a, b, c\}$ are all distinct remains.

We consider two main cases: If i, j, k are all at most $g - g/d$, then none of $v_i^{(a)}, v_j^{(b)}, v_k^{(c)}$ are redundantly encoded anywhere. However, we observe that the edge-checking box $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ with $\tilde{v}^{(a)} = (v_1^{(a)}, \dots, v_i^{(a)})$, $\tilde{v}^{(b)} = (v_1^{(b)}, \dots, v_j^{(b)})$ and $\tilde{v}^{(c)} = (v_1^{(c)}, \dots, v_k^{(c)})$ exists, since $\{v_i^{(a)}, v_j^{(b)}, v_k^{(c)}\} \notin E$ and $i + j + k \le 3(g - g/d) \le (3 - 3/d)g + 1$. Observe that $H(S)$ would be covered by this edge-checking box, which would give a contradiction. It remains to consider the second main case, which is that at least one of i, j, k is at least $g - g/d + 1$. We consider the case that i is largest among i, j, k, the other two case are symmetric. We observe that at least one of $T^{(b)}$ and $T^{(c)}$ contains $V_i^{(a)}$ in its redundant part, since $T^{(b)}$ does not encode the a-th group redundantly only if $a = b + 1$ (since $a \neq b$), but then $V_i^{(a)}$ must be encoded in $T^{(c)}$, which encodes all groups except $c \neq a$ and $c + 1 \neq b + 1$. For notational convenience, re-order b, c if necessary such that $V_i^{(a)}$ is encoded in $T^{(b)}$, i.e., $a \notin \{b, b+1\}$. We define $\tilde{v}^{(a)}$ as empty prefix,

$$\begin{aligned}
\tilde{v}^{(b)} =& t_S^{(b)}[..g + (d-2)(g - i + 1)] \\
=& (\underbrace{v_1^{(b)}, \dots, v_g^{(b)}}_{\text{main part}}, \\
& \underbrace{v_g^{(1)}, \dots, v_g^{(b-1)}, v_g^{(b+2)}, ..., v_g^{(d)}}_{g\text{-th elements of redundant groups}}, \\
& \dots, \\
& \underbrace{v_i^{(1)}, \dots, v_i^{(b-1)}, v_i^{(b+2)}, ..., v_i^{(b)}}_{i\text{-th elements of redundant groups}})
\end{aligned}$$

and $\tilde{v}^{(c)} = (v_1^{(c)}, \dots, v_k^{(c)})$. We observe that the corresponding edge-checking box $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ exists, since $\{v_i^{(a)}, v_j^{(b)}, v_k^{(c)}\} \notin E$ and

$$\begin{aligned}
i + j + k &= g + (d-2)(g - i + 1) + k \\
&= (d-1)g - (d-2)i + k + (d-2) \\
&\le (d-1)g - (d-3)i + (d-2) \\
&\le (d-1)g - (d-3)(g - g/d + 1) + (d-2) \\
&= \left(3 - \frac{3}{d}\right) g + 1,
\end{aligned}$$

where we used $k \le i$ (since i is largest among i, j, k by assumption) in the third line and $i \ge g - g/d + 1$ in the fourth line. Thus, $C_{\tilde{v}^{(a)}, \tilde{v}^{(b)}, \tilde{v}^{(c)}}$ would contain $H(S)$, which is a contradiction. This concludes the proof of the claim. $\square$

We can finally prove the lower bound for general $d \ge 3$.

Proof of Theorem VI.1.. Assume that we can solve Klee's measure problem in $\mathbb{R}^d$ in time $O(n^{\frac{d}{3-3/d} - \varepsilon})$. We show that there would be some g' and $\epsilon' > 0$ such that we can count g'-hypercliques in 3-uniform graphs in time $O(n^{g' - \varepsilon'})$, contradicting the 3-uniform hyperclique counting hypothesis. To this end, fix an arbitrary integer $g \ge d/\epsilon$ such that d divides g. Take any dg-partite 3-uniform hypergraph $G = (V, E)$ with $V = V_1^{(1)} \cup \dots \cup V_g^{(1)} \cup \dots \cup V_1^{(d)} \cup \dots \cup V_g^{(d)}$ and construct the set of boxes $\mathcal{B}$ as described above in time $O(|\mathcal{B}|)$. Observe that the number of edge-checking boxes is at most $\binom{d}{3} L^3 n^{(3-3/d)g+1} = O(n^{(3-3/d)g+1})$ and the number of consistency-checking boxes is at most $d(d-2) \cdot \frac{g}{d} \cdot n^{(3-3/d)g+1} = O(n^{(3-3/d)g+1})$, thus $|\mathcal{B}| = O(n^{(3-3/d)g+1})$.

By the above claim, we have that $n^{Ld} - \mathrm{vol}(\cup_{B \in \mathcal{B}} B)$ is the number of gd-cliques in G. By running the supposed algorithm for Klee's measure problem, we can compute $\mathrm{vol}(\cup_{B \in \mathcal{B}} B)$ in time

$$O(|B|^{\frac{d}{3-3/d} - \epsilon}) = O(n^{((3 - \frac{3}{d})g + 1)(\frac{d}{3-3/d} - \epsilon)}) = O(n^{gd - d}),$$

where we use that $d \ge 3$ and $g \ge d/\epsilon$ imply $((3 - 3/d)g + 1)(\frac{d}{3-3/d} - \epsilon) \le gd + \frac{d}{3-3/d} - \epsilon(3 - 3/d)g \le gd + \frac{d}{2} - 2\epsilon g \le gd - d$. This would contradict the 3-uniform hyperclique counting conjecture, and the theorem statement follows. $\square$

VII. CONCLUSION

Our conditional lower bound sheds light on the reasons why Klee's measure problem resisted any algorithmic improvement below $n^{3/2 \pm o(1)}$ in $\mathbb{R}^3$. We show that any polynomial improvement for the problem must necessarily lead to $O(n^{k-\epsilon})$-time algorithms for k-hyperclique counting, which in turn would give novel algorithms for various other problems, including: exact $O((2-\epsilon)^n)$-algorithms for Max-3SAT [24], $O(m^{k-\epsilon})$ model-checking for certain first-order properties [26], and $O(n^{k-\epsilon})$-time detection of k-sized solutions of 3-CNFs and other Boolean CSPs [29].

We hope that our work inspires further work on Klee's measure problem in high dimensions: Can we show higher non-combinatorial lower bounds than $n^{\frac{d}{3-3/d} - o(1)}$ (in particular, any $n^{\gamma d - o(d)}$ non-combinatorial lower bound for $\gamma > 1/3$) or can we give an improved algorithm that crucially exploits large dimension d? Note that by Chan's combinatorial lower bound, it is expected that any algorithmic improvement must rely on fast matrix multiplication techniques.

Finally, can we apply and extend our ideas to other problems with a similar fine-grained complexity status (i.e., problems with tight conditional lower bounds for combinatorial algorithms, where no improved non-combinatorial algorithms are known), or use them to clarify the complexity of special cases of Klee's measure problem, such as the hypervolume indicator, orthants, or arbitrary fat boxes?

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