Contemporary Mathematics

What we still don't know about loop spaces of spheres

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Abstract. We describe a program for computing the Morava K-theory of certain iterated loop spaces of spheres, based on a hypothetical duality with the computation of that of the Eilenberg-Mac Lane spaces by Wilson and the author [**RW80**]. Under this duality the duals of the bar spectral sequence and facts about $K(n)_*(B\mathbf{Z}/p)$ used in [**RW80**] are the Eilenberg-Moore spectral sequence and facts about $K(n)^*(\Omega^2 S^{2m+1})$ respectively. The program depends on the existence of a new geometric structure in those loop spaces dual to the cup product. We get a precise answer under these hypotheses.

Iterated loop spaces of spheres have played a central role in homotopy theory for many years. They have been thoroughly studied but there are still some open questions concerning them.

We will denote $\Omega^{m+1}S^{s+1}$ by $L_{m,s}$. Its homology has long been known for many years and is given in [**CLM76**]. Its BP homology is known only for $m \leq 1$; see [**Rav93**]. Its Morava K-theory was computed for $m = 1$ by Yamaguchi [**Yam88**] and for $m = 2$ by Tamaki (unpublished). In §5 we will describe a speculative program to compute the Morava K-theory more generally in a way that is analogous to the Ravenel-Wilson computation [**RW80**] of the Morava K-theory of Eilenberg-Mac Lane spaces, which is reviewed in §4. It depends on a hypothetical new geometric structure on the stable Snaith summands of iterated loop spaces.

1. Notation

- p is an odd prime. (See §5.4 for the case $p = 2$.)
- h denotes either ordinary mod p homology H or $K(n)$.
- $h^{*,*}(X) = \mathrm{Tor}^{h^*(X)}(h^*, h^*)$, the E_2-term of the Eilenberg-Moore spectral sequence converging (in favorable circumstances) to $h^*(\Omega X)$.
- For a loop space X, $h_{*,*}(X) = \mathrm{Tor}^{h_*(X)}(h_*, h_*)$, the E_2-term of the bar spectral sequence converging to $h_*(BX)$.
- $K_m = K(\mathbf{Z}/(p), m)$.
- $L_{m,s} = \Omega^{m+1}S^{s+1}$ for $0 \leq m < s$.
- $E(x)$ is the exterior algebra on x.
- $P(x)$ is the polynomial algebra on x.
- $T(x)$ is the truncated polynomial algebra of height p on x.

2. Two old splitting theorems

2.1. The James splitting. We begin by recalling the splitting theorem of James [**Jam55**]

$$\Sigma L_{0,s} = \Sigma\Omega S^{s+1} \simeq \bigvee_{i>0} S^{si+1}. \tag{2.1}$$

This means projection onto the ith summand gives a map

$$\Sigma\Omega S^{s+1} \longrightarrow S^{si+1},$$

which is adjoint to

$$\Omega S^{s+1} \xrightarrow{\quad H_i \quad} \Omega S^{si+1}, \tag{2.2}$$

the ith James-Hopf map. James also showed that ΩS^{s+1} is homotopy equivalent to a CW complex with a single cell in each dimension divisible by s. He identified the ks-skeleton as a certain topological quotient of the k-fold Cartesian product of S^s. It is called the reduced product or the kth James construction and is denoted by $J_k S^s$, i. e.,

$$J_k S^s = S^s \cup e^{2s} \cup \ldots \cup e^{ks} \simeq (\Omega S^{s+1})^{ks}.$$

Earlier Hopf ([**Hop30**] and [**Hop35**]) studied the map of (2.2) for $i = 2$. When s is odd, its fiber is S^s, i.e., we have a fibration

$$S^{2m-1} \longrightarrow \Omega S^{2m} \longrightarrow \Omega S^{4m-1}. \tag{2.3}$$

Serre [**Ser53**] showed that this splits after localization at any odd prime p, i.e., that

$$\Omega S^{2m}_{(p)} \simeq S^{2m-1}_{(p)} \times \Omega S^{4m-1}_{(p)},$$

so

$$\pi_{2m+k}(S^{2m}_{(p)}) \cong \pi_{2m-1+k}(S^{2m-1}_{(p)}) \oplus \pi_{2m+k}(S^{4m-1}_{(p)}).$$

In other words, from the point of view of computing homotopy groups, *even dimensional spheres are uninteresting at odd primes*; one need only study odd dimensional spheres.

James also identifed the p-local fiber of (2.2) when s is even and i is a power of p. We get a fiber sequence

$$J_{p^j-1} S^{2s}_{(p)} \longrightarrow \Omega S^{2s+1}_{(p)} \xrightarrow{\quad H_{p^j} \quad} \Omega S^{2sp^j+1}_{(p)}$$

This is of particular interest for $j = 1$. We regard $J_{p-1} S^{2s}_{(p)}$ as a p-local substitute for S^{2s}, and denote it by $\widehat{S}^{2s}$. (Note that for $p = 2$, $\widehat{S}^{2s}$ is simply $S^{2s}_{(2)}$.) Thus we have a p-local fiber sequence

$$\widehat{S}^{2s} \longrightarrow \Omega S^{2s+1} \xrightarrow{\quad H_p \quad} \Omega S^{2ps+1}. \tag{2.4}$$

Toda [**Tod62**] showed that there is also a p-local fiber sequence

$$S^{2s-1} \longrightarrow \Omega \widehat{S}^{2s} \longrightarrow \Omega S^{2ps-1}; \tag{2.5}$$

for $p=2$ this coincides with (2.3). The long exact sequences of homotopy groups associated with these two fiber sequences are called the *EHP sequence*. They give us a wonderful inductive procedure for computing the homotopy groups of spheres. Some additional references for this are [**Whi53**], [**Mah67**], [**BCK$^+$66**], [**Mah82**], [**Gra84**], [**Gra85**], [**MR87**], [**Rav86**, Chapter 1] and [**MR93**].

2.2. The Snaith splitting. In [**Sna74**] Snaith generalized (2.1) and gave a *stable* splitting of $L_{m,s}$ of the form

$$\Sigma^\infty \Omega^{m+1} S^{s+1} \simeq \bigvee_{i>0} \Sigma^{i(s-m)} D^i_{m,s}, \tag{2.6}$$

where each stable summand $D^i_{m,s}$ is a certain finite spectrum with bottom cell in dimension 0, defined in terms of configuration spaces. When an element $x \in H_*(\Omega^{m+1}S^{s+1})$ comes from the i the Snaith summand, we will say that its *Snaith degree*, denoted by $||x||$, is i.

By the James, splitting, $D^i_{0,s} = S^0$ for all i and s. One also knows that $D^1_{m,s} = S^0$ for all m and s.

One also has pairings

$$D^i_{m,s} \wedge D^j_{m,s} \longrightarrow D^{i+j}_{m,s} \tag{2.7}$$

having degree one on the bottom cell.

These summands also have the following properties after localization at a prime p.

- The stable summands of $\Omega^2 S^{2s+1}$ are

$$D^{pi+e}_{1,2s} = \begin{cases} \text{pt.} & \text{if } e \neq 0, 1 \bmod (p) \\ \Sigma^{(p-2)i} B_i & \text{if } e = 0 \text{ or } 1, \end{cases} \tag{2.8}$$

  where B_i is the ith Brown-Gitler spectrum. For $p=2$ this fact is due to Mahowald [**Mah77**] and Brown-Peterson [**BP78**]. For odd primes part of it was proved by Ralph Cohen [**Coh81**] and the rest by David Hunter and Nick Kuhn [**HK**].
- For every odd m, $D^i_{m,2s}$ is contractible unless i is congruent to 0 or 1 mod p. The bottom p-local cell of $D^{pi}_{m,2s}$ has dimension $(p-2)i$. (For even m it has dimension 0.)
- The top cell of $D^i_{m,2s}$ is in dimension $m(i-\alpha_p(i))$ where $\alpha_p(i)$ denotes the sum of the digits in the p-adic expansion of i.
- For fixed i and m, the p-local homotopy type of $D^i_{m,2s}$ depends only on the congruence of s modulo $p^{f(m)}$ for a certain arithmetic function f. In particular $f(1)=0$ and $f(2)=1$; see [**CCKN83**] for more information. Using this fact we can define $D^i_{m,2s}$ for all integers s. In particular we can define a ring spectrum

$$L_m = \bigvee_{i\geq 0} D^i_{m,0}. \tag{2.9}$$

3. Some new spectra

3.1. A colimit of Snaith summands. The pairings (2.7) with $j=1$ give us maps $D^i_{m,2s} \to D^{i+1}_{m,2s}$. For even m we define

$$D^\infty_{m,2s} = \operatorname{hocolim}_{\overrightarrow{i}} D^i_{m,2s}. \tag{3.1}$$

For odd m this would be contractible at odd primes, so we need to modify the definition to get something interesting. Consider the composite

$$S^{p-2} \wedge D^{pi}_{m,2s} \longrightarrow D^p_{m,2s} \wedge D^{pi}_{m,2s} \longrightarrow D^{p(i+1)}_{m,2s}.$$

This has degree 1 on the bottom cell, which lies in dimension $(p-2)(i+1)$, so we define

$$D^\infty_{m,2s} = \operatorname{hocolim}_{\overrightarrow{i}} \Sigma^{i(2-p)} D^{pi}_{m,2s}. \tag{3.2}$$

Question 1. *What is the spectrum $D^\infty_{m,2s}$ defined by (3.1) and (3.2)?*

The James splitting (2.1) gives

$$D^\infty_{0,2s} = S^0$$

and (2.8) gives

$$D^\infty_{1,2s} = \operatorname{hocolim}_{\overrightarrow{i}} B_i = H/p, \tag{3.3}$$

the mod p Eilenberg-Mac Lane spectrum. This latter fact plays a central role in the proofs of the nilpotence theorem of Devinatz-Hopkin-Smith [**DHS88**] (see also [**Rav92**, Chapter 9]) and of Nishida's theorem [**Nis73**].

The spectrum $D^\infty_{1,2s}$ can be identified with the Thom spectrum of certain p-local spherical fibration over $\Omega^2 S^3$. For $p=2$ it is the vector bundle given by the double loop map $\Omega^2 S^3 \to BO$ extending the nontrivial map from S^1. The identification of the corresponding Thom spectrum with $H/2$ (which does *not* require knowing that $D^{2i}_{1,2s}$ is a Brown-Gitler specturm) is originally due to Mahowald [**Mah79**].

To see (3.3) computationally, note that

$$H_*(\Omega^2 S^{2s+1}) = \begin{cases} P(e_0, e_1, \dots) & \text{for } p=2 \\ E(e_0, e_1, \dots) \otimes P(f_1, f_2 \dots) & \text{for } p \text{ odd} \end{cases}$$

where $|e_i| = 2sp^i - 1$, $|f_i| = 2sp^i - 2$ and both generators have Snaith degree p^i. The action of the Steenrod algebra is given by the following formulas, with all other operations on the generators being zero.

$$\begin{aligned} \mathrm{Sq}^1(e_{i+1}) &= e_i^2 && \text{for } p=2 \\ \beta(e_i) &= f_i && \text{for } p \text{ odd} \\ \mathcal{P}^1(f_{i+1}) &= f_i^p. \end{aligned}$$

In the mapping the Snaith summands to $D^\infty_{1,2s}$, we have

$$\begin{aligned} e_0 &\mapsto 1, \\ f_1 &\mapsto 1, \\ f_{i+1} &\mapsto \xi_i, \\ \text{and} \qquad e_{i+1} &\mapsto \tau_i \end{aligned}$$

for odd primes (where ξ_i and τ_i denote the standard generators of the dual Steenrod algebra) and

$$\begin{array}{rrcl} & e_0 & \mapsto & 1 \\ \text{and} & e_i & \mapsto & \xi_i \end{array}$$

for $p = 2$.

We can use (3.3) to get information about $D^\infty_{2k+1,2s}$ in the following way. We have the $2k$-fold suspension map

$$\Omega^2 S^{2s-2k+1} \longrightarrow \Omega^{2k+2} S^{2s+1},$$

which induces maps of Snaith summands

$$D^i_{1,2s-2k} \longrightarrow D^i_{2k+1,2s},$$

making the spectrum $D^\infty_{2k+1,2s}$ a module spectrum over $D^\infty_{1,2s-2k}$, i.e. a generalized mod p Eilenberg-Mac Lane spectrum.

However, *the homotopy type of* $D^\infty_{m,2s}$ *for even* m *is far less obvious.* $D^\infty_{2,0}$ is the Thom spectrum for the complex vector bundle induced by the composite

$$\Omega^3_0 S^3 = \Omega^3_0 SU(2) \longrightarrow \Omega^3_0 SU = BU.$$

Thus MU is a module spectrum over $D^\infty_{2,0}$, which is *not* an Eilenberg-Mac Lane spectrum. In §6 we will see that certain telescopes are module spectra over it.

For odd primes we have

$$H_*(\Omega^3 S^{2ps+1}) = P(u_k \colon k \geq 0) \otimes E(x_{i,j} \colon i > 0, j \geq 0) \otimes P(y_{i,j} \colon i > 0, j \geq 0),$$

with

$$\begin{array}{rcl} |u_k| & = & 2sp^{k+1} - 2 \\ ||u_k|| & = & p^k \\ |x_{i,j}| & = & 2sp^{i+j+1} - 2p^j - 1 \\ ||x_{i,j}|| & = & p^{i+j} \\ |y_{i,j}| & = & 2sp^{i+j+2} - 2p^{j+1} - 2 \\ ||y_{i,j}|| & = & p^{i+j+1}, \end{array}$$

where $||x||$ denotes the Snaith degree of x. For $p = 2$ we have a similar description with $y_{i,j} = x^2_{i,j}$.

For odd primes the action of the Steenrod algebra A is given by

$$\begin{array}{rcl} \beta(u_i) & = & x_{i,0} \quad \text{for } i > 0 \\ \mathcal{P}^{p^k}(u_i) & = & \begin{cases} u^p_{i-1} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \\ \beta(x_{i,j}) & = & y_{i,j-1} \quad \text{for } j > 0 \\ \mathcal{P}^{p^k}(x_{i,j}) & = & \begin{cases} x_{i-1,j+1} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \\ \beta(y_{i,j}) & = & 0 \\ \mathcal{P}^{p^k}(y_{i,j}) & = & \begin{cases} y_{i-1,j+1} & \text{if } k = j+1 \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

For $p = 2$ we have

$$\begin{aligned}
\mathrm{Sq}^{2^k}(u_i) &= \begin{cases} x_{i,0} & \text{if } k = 0 \\ u_{i-1}^2 & \text{if } k = 1 \\ x_{i-k+1,k-2}^2 & \text{otherwise} \end{cases} \\
\mathrm{Sq}^{2^k}(x_{i,j}) &= \begin{cases} x_{i,j-1}^2 & \text{if } k = 0 \text{ and } j > 0 \\ x_{i-1,j+1} & \text{if } k = j+1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

More generally, $D_{2m,2s}^{\infty}$ is the Thom spectrum for the stable complex vector bundle induced by the composite

$$\Omega_0^{2m+1} S^{2m+1} \longrightarrow Q_0 S^0 = B\Sigma_\infty^+ \xrightarrow{(s-m)\rho} BU$$

where ρ denotes the standard complex representation of the infinite symmetric group Σ_∞. (For the equivalence of $Q_0 S^0 = B\Sigma_\infty^+$, see [**Ada78**].) It follows that MU is a module spectrum over each $D_{2m,2s}^{\infty}$.

3.2. A colimit of dual Snaith summands. It is convenient here to restate the p-local stable Snaith splitting [**Sna74**]

$$L_{m,2s} \simeq \bigvee_{i>0} \Sigma^{2si} C_{m,2s}^i \tag{3.4}$$

where $C_{m,2s}^i$ is a certain suspension of the summand $D_{m,2s}^i$ of (2.6). It has top cell in dimension $-m\alpha_p(i)$, where $\alpha_p(i)$ is the sum of the digits in the p-adic expansion of i. As before the p-local homotopy of $C_{m,2s}^i$ for fixed m and i depends only on the congruence of s modulo $p^{f(m)}$. When s is divisible by $p^{f(m)}$, we abbreviate $C_{m,2s}^i$ by C_m^i. L_m will denote the spectrum

$$\bigvee_{i>0} C_m^i.$$

Knowing its cohomology is equivalent to knowing that of the space $L_{m,2s}$ for s highly divisible in light of the Snaith splitting. We can use this isomorphism to define cup products in $h^*(L_m)$.

The Hopf map

$$L_{m,2s} \xrightarrow{H_p} L_{m,2ps},$$

induces maps

$$C_{m,2s}^{pi} \longrightarrow C_{m,2ps}^i$$

having degree 1 on the top cell. For $m = 1$ and $p = 2$ this is due to [**CMM78**], and the general case is due to [**Kuh**]. For s highly divisible by p this gives us

$$C_m^{pi} \longrightarrow C_m^i.$$

This leads to an inverse system of spectra, but we want to look instead at the homotopy direct limit of the Spanier-Whitehead duals. Let

$$\tilde{K}_m = \operatorname*{hocolim}_{\overrightarrow{DH}} DC_m^{p^j}.$$

It has bottom cell in dimension m.

CONJECTURE 1. *For $p = 2$ the spectrum $\tilde{K}_m$ has the suspension spectrum of the Eilenberg-Mac Lane space K_m as a retract. For an odd prime p $\tilde{K}_m$ has a nontrivial retract of the suspension spectrum of the Eilenberg-Mac Lane space K_m as a retract.*

For $m = 1$ and $p = 2$ this can be deduced from a theorem of Carlsson [**Car83**], which is needed for Miller's proof of the Sullivan conjecture [**Mil84**]. Kuhn [**Kuh**] has recently shown that $H^*(\tilde{K}_m)$ is an unstable A-module having $H^*(K_m)$ as a summand for all m, and has identified the other summands as well.

4. The Ravenel-Wilson computation of $K(n)_*(K_m)$

Conjecture 1 implies that $K(n)_*(\tilde{K}_m)$ and $K(n)_*(K_m)$ have a common nontrivial summand. In this section we will describe the computation of the latter by Wilson and the author [**RW80**]. The approach used there is also good for computing the ordinary homology of the Eilenberg-Mac Lane spaces, and with it one can get complete information *with no prior knowledge of Steenrod operations.* For more details, see Wilson's primer [**Wil82**]. In the next section we will suggest a way in which this approach might carry over to a computation of $K(n)_*(L_{m,2s})$.

We have the bar spectral sequence with

$$E_2^{*,*} = h_{*,*}(K_m) \implies h_*(K_{m+1}).$$

It always collapses for $h_* = H_*$ but not for $h_* = K(n)_*$. Since $K_1 = B\mathbf{Z}/p$ is an S^1-bundle over $\mathbf{C}P^\infty$, $h_*(K_1)$ can be also calculated with the Gysin sequence. This result implies that in the bar spectral sequence for $m = 0$ there are both nontrivial differentials and multiplicative extensions at E_∞.

A key computational ingredient for this bar spectral sequence is the cup product map

$$K_\ell \wedge K_m \longrightarrow K_{\ell+m},$$

(induced by the cup product of the fundamental classes in $H^*(K_\ell \times K_m)$) which induces a spectral sequence pairing

$$h_*(K_\ell) \otimes h_{*,*}(K_m) \longrightarrow h_{*,*}(K_{\ell+m}) \tag{4.1}$$

known as the *circle product*. The image of $x \otimes y$ under this map is denoted by $x \circ y$.

This enables us to determine the behavior of the bar spectral sequence for each $m > 0$ once we know it for $m = 0$. In particular we find that

$$\overline{K(n)}_*(K_m) = 0 \qquad \text{iff} \quad m > n,$$

and it is concentrated in even dimensions for $m \le n$.

In more detail, K_0 is the discrete group of order p so $h_*(K_0)$ is its group ring over h_*, which is isomorphic to $T(u)$. It follows that for each h,

$$h_{*,*}(K_0) = E(s) \otimes T(a_i : i \ge 0) \tag{4.2}$$

with $|s| = 1$ and $|a_i| = 2p^i$. The bar spectral sequence collapses for $h = H$, and for $h = K(n)$, there is a differential

$$d_r(a_n) = v_n s, \qquad \text{with } r = 2p^n - 1, \tag{4.3}$$

which gives

$$E^0h^*(K_1) = E_\infty = T(a_i : 0 \le i < n). \tag{4.4}$$

There is a multiplicative extension

$$a_{n-1}^p = v_n a_0. \tag{4.5}$$

Now here is a useful mnenomic device not used in [**RW80**]. *We can ignore the multiplicative extension and still get the right answer in the spectral sequence.* We pretend that $E_\infty = h_*(K_1)$, and find that the resulting Tor group (using our transpotent formula $\tau(V(x)) = a_0 \circ x$) is

$$E(s \circ a_i : 0 \le i < n) \otimes T(a_j \circ a_{i+j+1} : 0 \le j, 0 \le i < n).$$

We can regard this as the E_1-term of the bar spectral sequence. Our formulas give differentials

$$d_r(a_{n-1-i} \circ a_n) = -v_n s \circ a_{n-1-i}$$

(this is a d_1 for $i = n-1$) which leads to the correct value of E_∞.

For $h = H$ the bar spectral sequence collapses and we have

$$H_*(K_2) = H_{*,*}(K_1) = T(b_i : i \ge 0) \otimes T(a_j \circ a_{i+j+1} : i, j \ge 0) \tag{4.6}$$

with $|b_i| = 2p^i$. The element b_i is in the image of the map $\mathbf{C}P^\infty \to K_2$ and it is the dual (with respect to basis described below) of $\mathcal{P}^{\Delta_i}x_2$, where $x_m \in H^m(K_m)$ is the fundamental class. The element $a_j \circ a_{i+j+1}$ is dual to $Q_jQ_{i+j+1}x_2$. The basis in question is that of monomials in generators obtained by the action on x_m of suitable Milnor basis elements of the Steenrod algebra.

5. A speculative computation of $K(n)_*(L_m)$

5.1. A possible analog of the cup product.

Conjecture 2. *The cup product pairing*

$$K_\ell \wedge K_m \longrightarrow K_{\ell+m}$$

has an analog

$$\tilde{K}_\ell \wedge \tilde{K}_m \quad \longrightarrow \quad \tilde{K}_{\ell+m}$$

which also has degree one on the bottom cell and similar formal properties.

I had hoped for a similar copairing

$$C_\ell^i \wedge C_m^i \longleftarrow C_{\ell+m}^i$$

but this cannot exist as the following example illustrates. Let $p = 2$ and $i = 2$. Then

$$C_k^i = \Sigma^{-k} RP_{-k}^0$$

and the desired map up to suspension is

$$RP_{-\ell}^0 \wedge RP_{-m}^0 \longleftarrow RP_{-\ell+m}^0,$$

which cannot exist for A-module reasons. Perhaps there is a copairing of the form

$$C_\ell^i \wedge C_m^j \longleftarrow C_{\ell+m}^{i+j}.$$

The 'dual' of the bar spectral sequence is the Eilenberg-Moore spectral sequence, in which we have

$$E_2^{*,*} = h^{*,*}(L_m) \implies h^*(L_{m+1}).$$

Tamaki ([**Tam94**] and [**Tam**]) has shown that it converges for Morava K-theory. Like the bar spectral sequence, it collapses for all m for $h^* = H^*$ but not for $h^* = K(n)^*$.

If Conjecture 2 holds, one could hope for spectral sequence pairings

$$h^*(L_\ell) \otimes h^{*,*}(L_m) \longrightarrow h^{*,*}(L_{\ell+m}),$$

analogous to (4.1), which I will refer to as the *cocircle product.* Again we denote the image of $x \otimes y$ by $x \circ y$ As in the Ravenel-Wilson computation, these should enable us to use the behavior of the spectral sequence for $m = 0$ to determine its behavior for all $m > 0$. This leads to an explicit description of $K(n)^*(L_m)$ in all cases. It agrees with results of Yamaguchi for $m = 1$ and with an unpublished computation of Tamaki for $m = 2$. We will give more details in the next subsections.

5.2. An analog of the Ravenel-Wilson computation. I believe that the computation of $h^*(L_m)$ by induction on m via the Eilenberg-Moore spectral sequence, which converges by Tamaki's theorem, is parallel to that of $h_*(K_m)$ by the bar spectral sequence. To emphasize this analogy, I will use notation for elements in $h^*(L_m)$ similar to that for the corresponding the elements in $h_*(K_m)$, except that there will be an extra index in the former related to the Snaith degree.

Here is a table illustrating this. It includes a description of the Verschiebung V when it is nontrivial. Our indexing convention is the following. *All indices are assumed to be nonnegative, with any additional conditions stated explicitly.* When there are two subscripts, the Snaith degree is determined by their sum, and the Verschiebung lowers the first index.

Eilenberg-Mac Lane spaces	Loop spaces of spheres
$u \in h_0(K_0)$	$u_j \in h^0(L_0)$ $\lvert\lvert u_j\rvert\rvert = p^j$ $V(u_{j+1}) = u_j$ $H(u_j) = u_{j+1}$
$s \in h_1(K_1)$	$s_j \in h^{-1}(L_1)$ $\lvert\lvert s_j\rvert\rvert = p^j$ $H(s_j) = s_{j+1}$
$a_i \in h_{2p^i}(K_1)$ $V(a_{i+1}) = a_i$	$a_{i,j} \in h^{-2p^i}(L_1)$ $\lvert\lvert a_{i,j}\rvert\rvert = p^{i+j+1}$ $V(a_{i+1,j}) = a_{i,j}$ $H(a_{i,j}) = a_{i,j+1}$
$b_i \in h_{2p^i}(K_2)$ $b_0 = s \circ s$ $V(b_{i+1}) = b_i$	$b_{i,j} \in h^{-2p^i}(L_2)$ $b_{0,j} = s_j \circ s_j$ $\lvert\lvert b_{i,j}\rvert\rvert = p^{i+j}$ $V(b_{i+1,j}) = b_{i,j}$ $H(b_{i,j}) = b_{i,j+1}$

By James' theorem, L_0 is a wedge of 0-spheres, and

$$h^*(L_0) = T(u_j).$$

The element u_j is the unit for the cocircle product in Snaith degree p^j.

The following formulas should be compared with (4.2–4.5). We have

$$h^{*,*}(L_0) = E(s_j) \otimes T(a_{i,j}) \tag{5.1}$$

The Eilenberg-Moore spectral sequence collapses for $h = H$, and for $h = K(n)$, there is a differential

$$d_r(a_{n,j}) = v_n s_{n+j+1}, \tag{5.2}$$

which gives

$$E_\infty = E(s_j : j \leq n) \otimes T(a_{i,j} : i < n). \tag{5.3}$$

This is compatible with Yamaguchi's computation [**Yam88**] of $K(n)_*(L_1)$. There is a multiplicative extension (proved in [**Rav93**])

$$a_{n-1,j}^p = v_n a_{0,j+n}. \tag{5.4}$$

As before, (5.4) can be ignored when computing Tor. Using (5.3) as a substitute for $h^*(L_1)$, and assuming that a generator $V(x)$ of Snaith degree p^j has desuspension $s_j \circ V(x)$, and transpotent $a_{0,j} \circ x$, we get

$$\begin{array}{rcl} h^{*,*}(L_1) & = & T(b_{i,j} : j \leq n) \otimes \\ & & E(s_{i+j+1} \circ a_{i,j} : i < n) \otimes \\ & & T(a_{k,i+j+1} \circ a_{i+k+1,j} : i < n). \end{array} \tag{5.5}$$

There are differentials and extensions formally implied by (5.2) and (5.4), and we get

$$\begin{array}{rcl} h^*(L_2) & = & T(b_{i,j} : j \leq n) \otimes \\ & & E(s_{i+j+1} \circ a_{i,j} : i + j < n) \otimes \\ & & T(a_{k,i+j+1} \circ a_{i+k+1,j} : i + k + 1 < n). \end{array} \tag{5.6}$$

Note that $b_{i,j}$ is a permanent cycle because there is no suitable target for a nontrivial differential on it. The resulting value of $K(n)_*(\Omega^3 S^{odd})$ is equivalent to Tamaki's (unpublished).

In ordinary cohomology we have

$$\begin{array}{rcl} H^*(L_1) & = & E(s_j) \otimes T(a_{i,j}) \\ H^*(L_2) & = & \Gamma(b_{0,j}) \otimes E(s_{i+j+1} \circ a_{i,j}) \\ & & \quad \otimes \Gamma(a_{0,i+j+1} \circ a_{i+1,j}) \\ & = & T(b_{k,j}) \otimes E(s_{i+j+1} \circ a_{i,j}) \\ & & \quad \otimes T(a_{k,i+j+1} \circ a_{k+i+1,j}) \end{array}$$

In terms of Dyer-Lashof operations acting on the fundamental class $x_{-m} \in H_{-m}(L_m)$,

$$\begin{array}{rcl} s_j & \text{is dual to} & Q_1^j x_{-1}, \\ a_{i,j} & \text{is dual to} & Q_0^i \beta Q_1^j x_{-1}, \\ b_{k,j} & \text{is dual to} & Q_0^k \beta Q_1^j x_{-2}, \\ s_{i+j+1} \circ a_{i,j} & \text{is dual to} & Q_1^i \beta Q_1^j x_{-2} \quad \text{and} \\ a_{k,i+j+1} \circ a_{k+i+1,j} & \text{is dual to} & Q_0^k \beta Q_1^i \beta Q_1^j x_{-2}. \end{array}$$

5.3. A computational conjecture. Now I will state a conjecture about the structure of the 'Hopf Ring' $K(n)^*(L_*)$.

Here is some more notation. For a sequence of n zeroes and ones

$$I = (i_0, i_1, \dots i_{n-1}),$$

let

$$\begin{array}{rrcl} & \lambda(I) & = & \max(s : i_s > 0), \\ & |I| & = & i_0 + i_1 + \dots + i_{n-1}, \\ \text{and} & ||I|| & = & i_0 + p i_1 + \dots + p^{n-1} i_{n-1}. \end{array}$$

For $t \geq \lambda(I)$, let

$$a_{I,t} = a_{0,t}^{\circ i_0} \circ a_{1,t-1}^{\circ i_1} \circ \cdots a_{n-1,t+1-n}^{\circ i_{n-1}} \in K(n)^{-2||I||}(L_{|I|})$$

For a sequence of $n+1$ nonnegative integers

$$J = (j_0, j_1, \dots j_n),$$

let

$$\begin{array}{rrcl} & \lambda(J) & = & \max(s : j_s > 0), \\ & |J| & = & j_0 + j_1 + \dots + j_n, \\ \text{and} & ||J|| & = & j_0 + p j_1 + \dots + p^n j_n. \end{array}$$

For $t \geq \lambda(J)$, let

$$b_{t,J} = b_{t,0}^{\circ j_0} \circ b_{t-1,1}^{\circ j_1} \circ \cdots \circ b_{t-n,n}^{\circ j_n}. \in K(n)^{-2||J||}(L_{2|J|}).$$

CONJECTURE 3.

$$\bigotimes_{m \geq 0} K(n)^*(L_m) = E(s_t \circ b_{t,J} \circ a_{I,t-1} : t \leq n) \otimes T(b_{t,J} \circ a_{I,t-1})$$

with all possible values of I, J and t, subject to the multiplicative extension of (4.5). If I and J consist entirely of zeroes, then $t \geq 0$, and $s_t \circ b_{t,J} \circ a_{I,t-1}$ and $b_{t,J} \circ a_{I,t-1}$ are understood to be s_t and u_t respectively. For $n = \infty$ (ordinary homology) this should lead to the usual description of $H^*(L_m)$.

Now consider the Hopf map $H : L_{m,2s} \to L_{m,2ps}$. For $m \equiv 0$ this induces maps $C_m^{pi} \to C_m^i$. In cohomology we have

$$\begin{array}{rrcl} & s_i & \mapsto & s_{i+1}, \\ & a_{i,j} & \mapsto & a_{i,j+1}, \\ \text{and} & b_{i,j} & \mapsto & b_{i,j+1}. \end{array}$$

Since s_j and $b_{i,j}$ are trivial for $j > n$, only the generators in the limit

$$\varinjlim_{i} h^*(C_m^{p^i})$$

are m-fold cocircle products of $a_{i,j}$s, so the limit contains a summand isomorphic to the vector space spanned by the generators of $K(n)_*(K_m)$, as one would expect in light of Conjecture 1.

Next we note that Conjecture 3 leads to a description of $K(n)^*(L_{m,2s})$ for general s as follows. In the Snaith splitting 3.4, each summand $C^i_{m,2s}$ is up to suspension the Thom spectrum of a certain complex vector bundle over a certain configuration space. The stable fiber homotopy type of this bundle depends only on the congruence class of s module $p^{f(m)}$ for a known function f. Our divisibility condition on s meant that we were considering the cases where the bundle is fiber homotopically trivial. Since $K(n)$ is complex orientable, it follows that $K(n)^*(C^i_{m,2s})$ is Thom isomorphic to $K(n)^*(C^i_m)$, which is described in principle by Conjecture 3.

It may be possible to prove Conjecture 3 without proving the other two. The former describes the answer in the language (the cocircle product) provided by Conjecture 2. Without this language it could be translated into a description of the differentials and multiplicative extensions occuring in the Eilenberg-Moore spectral sequence. Without the cocircle product we still have a spectral sequence of Hopf algebras which must respect the Snaith splitting and the Hopf map. The conjecture says that there all the differentials are formally implied (via the Hopf algebra structure or the Hopf map) by those occuring in Snaith degree p^{n+1}. The differentials in Snaith degree p^{n+1} may be detected by the Milnor operation Q_n. The absence of any other differentials could be forced on us by all the structure at hand if we do the required bookkeeping carefully enough.

5.4. Computations for $p = 2$. In this section we will describe the computation for $p = 2$. *Again it is understood that all indices range over the nonnegative integers unless otherwise specified.* The notation here is simpler. The table of §5.2 gets replaced by

Eilenberg-Mac Lane spaces	Loop spaces of spheres
$u \in h_0(K_0)$	$u_j \in h^0(L_0)$ $\lvert\lvert u_j\rvert\rvert = 2^j$ $V(u_{j+1}) = u_j$ $H(u_j) = u_{j+1}$
$a_i \in h_{2^i}(K_1)$ $V(a_{i+1}) = a_i$	$a_{i,j} \in h^{-2^i}(L_1)$ $\lvert\lvert a_{i,j}\rvert\rvert = 2^{i+j}$ $V(a_{i+1,j}) = a_{i,j}$ $H(a_{i,j}) = a_{i,j+1}$
$b_i = a_i \circ a_i$	$b_{i,j} = a_{i,j} \circ a_{i,j}$

In the odd primary case we had $a_i \circ a_i = 0$ and $a_{i,j} \circ a_{i,j} = 0$ due to sign considerations. For $p = 2$ these products need not vanish, and it is convenient to use the indicated notation for them.

In ordinary homology we have

$$\begin{aligned} H_*(K_0) &= E(u) \\ H_*(K_1) = H_{*,*}(K_0) &= \Gamma(a_0) \\ &= E(a_i \colon i \geq 0) \end{aligned}$$

where $a_i \in H_{2^i}(K_1)$ is dual to $\mathrm{Sq}^{\Delta_i} x_1 \in H^{2^i}(K_1)$, with $x_m \in H^m(K_m)$ being the fundamental class. More generally we have

$$\begin{aligned} H_*(K_m) &= \Gamma(a_0 \circ a_{i_1} \circ a_{i_1+i_2} \circ \ldots \circ a_{i_1+\cdots+i_{m-1}}) \\ &= E(a_{i_0} \circ a_{i_0+i_1} \circ a_{i_0+i_1+i_2} \circ \ldots \circ a_{i_0+\cdots+i_{m-1}}) \end{aligned}$$

where

$$a_{i_0} \circ a_{i_0+i_1} \circ a_{i_0+i_1+i_2} \circ \ldots \circ a_{i_0+\cdots+i_{m-1}} \in H_{2^{i_0}+2^{i_0+i_1}+\ldots+2^{i_0+\cdots+i_{m-1}}}(K_m)$$

is dual to

$$\mathrm{Sq}^{\Delta_{i_0}+\Delta_{i_0+i_1}+\ldots+\Delta_{i_0+\ldots i_{m-1}}} x_m.$$

Similarly for loop spaces we have

$$H^*(L_m) = E(a_{i_0, i_1+\ldots+i_m} \circ a_{i_0+i_1, i_2+\ldots+i_m} \circ \ldots \circ a_{i_0+\ldots+i_{m-1}, i_m})$$

where $a_{i,j} \in H^{-2^i}(L_1)$ has Snaith degree 2^{i+j}. In terms of Dyer-Lashof operations on the fundamental class $x_{-m} \in H_{-m}(L_m)$, the element

$$a_{i_0, i_1+\ldots+i_m} \circ a_{i_0+i_1, i_2+\ldots+i_m} \circ \ldots \circ a_{i_0+\ldots+i_{m-1}, i_m}$$

is dual to

$$Q_0^{i_0} Q_1^{i_1} \ldots Q_m^{i_m} x_{-m}.$$

For $h = K(n)$ we have differentials

$$\begin{aligned} d_r(a_{n+1}) &= v_n a_0 \\ \text{and} \qquad d_r(a_{n+1,j}) &= v_n a_{0,i+n+1}, \end{aligned}$$

and multiplicative extensions

$$\begin{aligned} a_n^2 &= v_n a_1 \\ \text{and} \qquad a_{n,j}^2 &= v_n a_{1,n+j}. \end{aligned}$$

Thus we have (using the same mnemonic device as before)

$$\begin{aligned} h_*(K_0) &= E(u) \\ h_{*,*}(K_0) &= E(a_i \colon i \geq 0) \\ E^0 h_*(K_1) &= E(a_{i+1} \colon 0 \leq i < n) \\ h_{*,*}(K_1) &= E(a_{i_0} \circ a_{1+i_0+i_1} \colon i_1 < n) \\ E^0 h_*(K_2) &= E(a_{1+i_0} \circ a_{2+i_0+i_1} \colon i_0 + i_1 < n-1) \\ &\vdots \\ E^0 h_*(K_m) &= E(a_{1+i_0} \circ a_{2+i_0+i_1} \circ \ldots \circ a_{m+i_0+\ldots+i_{m-1}} \colon \\ &\qquad\qquad i_0 + \ldots + i_{m-1} < n+1-m). \end{aligned}$$

Note that the conditions on the subscripts cannot be satisfied for $m > n$, so in that case K_m is $K(n)$-acyclic.

We can show that $a_i \circ a_i = 0$ by induction on i as follows. Since a_1 is primitive, $a_1 \circ a_1$ is primitive and there are no primitives of its dimension in $K(n)_*(K_2)$. Inductively $a_i \circ a_i$ is primitive and hence zero.

For loop spaces we have

$$
\begin{array}{rcl}
h^*(L_0) & = & E(u_j) \\
h^{*,*}(L_0) & = & E(a_{i,j}) \\
E^0h^*(L_1) & = & E(a_{0,j} \colon j \leq n) \otimes E(a_{1+i,j} \colon i < n) \\
h^{*,*}(L_1) & = & \Gamma(a_{0,j} \circ a_{0,j} \colon j \leq n) \otimes \Gamma(a_{0,1+i_1+j} \circ a_{1+i_1,j} \colon i_1 < n) \\
 & = & E(b_{i_0,j} \colon j \leq n) \otimes E(a_{i_0,1+i_1+j} \circ a_{1+i_0+i_1,j} \colon i < n) \\
E^0h^*(L_2) & = & E(b_{i_0,j} \colon j \leq n) \otimes E(a_{0,1+i_1+j} \circ a_{1+i_1,j} \colon i_1 + j < n) \\
 & & \quad \otimes E(a_{1+i_0,1+i_1+j} \circ a_{2+i_0+i_1,j} \colon i_0 + i_1 < n-1) \\
 & \vdots &
\end{array}
$$

Notice that the last factor of $E^0h^*(L_1)$ and of $E^0h^*(L_1)$ resemble $E^0h_*(K_1)$ and $E^0h_*(K_2)$ respectively, but the other factors in the former have no analogs in the latter.

6. Triple loop spaces and telescopes

In this section we will outline the computation of $h^*(L_2)$ without relying on the conjectures stated above. We want to do this for some additional homology theories h which we now describe. The first is $k(n)$, connective Morava K-theory.

The next is $y(n)$, which for $p = 2$ is the Thom spectrum associated with the composite map

$$\Omega J_{p^n-1}(S^2) \to \Omega^2 S^3 \to BO$$

Here $J_k(S^2)$ denotes the kth James reduced product on S^2 (which is the same as the $2k$-skeleton of ΩS^3), and the last map is the double loop map induced by the nontrivial element of $\pi_1(BO)$. For odd primes $y(n)$ is the Thom spectrum associated with a certain p-local spherical fibration induced from one over $\Omega^2 S^3$ which Thomifies to H/p.

Then we have

$$H_*(y(n)) = \begin{cases} P(\xi_1, \ldots, \xi_n) & \text{for } p = 2 \\ E(\tau_0, \ldots \tau_{n-1}) \otimes P(\xi_1, \ldots, \xi_n) & \text{for } p \text{ odd} \end{cases}$$

as comodule algebras over the dual Steenrod algebra. There is a map $y(n) \to k(n)$ inducing a surjection in ordinary mod p homology. It is an equivalence through dimension $2p^{n+1} - 4$.

The spectrum $Y(n)$ is the telescope obtained from $y(n)$ by iterating a map

$$v_n : \Sigma^{2p^n-2} y(n) \to y(n),$$

which is obtained as follows. We have a fiber sequence of spaces

$$\Omega^3 S^{2p^n+1} \to \Omega J_{p^n-1}(S^2) \to \Omega^2 S^3$$

which Thomifies to a stable map

$$\Omega^3 S^{2p^n+1} \to y(n).$$

The bottom cell of the source gives us an element of $v_n \in \pi_{2p^n-2}(y(n))$ which (since $y(n)$ is a ring spectrum) gives us the desired self map. *This map makes $Y(n)$ a module spectrum over $D_{2,0}^{\infty}$.*

Note that $K(n)$ and $k(n)$ are module spectra over $Y(n)$ and $y(n)$ respectively, while $Y(n)_*$ and $y(n)_*$ are (unnaturally) modules over $K(n)_*$ and $k(n)_*$.

Now consider the Atiyah-Hirzebruch spectral sequence for $h^*(L_1)$, where we have

$$E_2 = H^*(L_1, h^*).$$

For each theory other than H^* the first differential is induced by the Milnor operation Q_n, and we have (modulo v_n-torsion for $h = y(n)$ and $k(n)$)

$$E_{2p^n} = E(s_j : j \le n) \otimes T(a_{i,j} : i < n).$$

We want to show that there are no other differentials. Note that this spectral sequence has the following structure.

- All differentials must respect the Snaith splitting, i.e., they must preserve Snaith degree.
- It is a spectral sequence of Hopf algebras. If s_j or $a_{i,j}$ supports a nontrivial differential, its target must be a linear combination (over h^*) of other such generators of the same Snaith degree.
- Differentials must commute with the Hopf map H, which raises the index j by one.

Concerning the Hopf map, we remark that Tamaki's spectral sequence for the homology of a space of the form $\Omega^3\Sigma^3 X$ is functorial on X while the Hopf map is not. However Tamaki has shown [**Tam**] that his spectral sequence coincides from E_2 onward with the Eilenberg-Moore spectral sequence, which is natural with respect to all loop maps including the Hopf map H.

Now the primitives of Snaith degree p^k surviving to E_{2p^n} are

$$\begin{cases} \{s_k, a_{0,k-1}, a_{1,k-2}, \ldots a_{k-1,0}\} & \text{for } k \le n \\ \{a_{0,k-1}, a_{1,k-2}, \ldots a_{n-1,k-n}\} & \text{for } k > n. \end{cases}$$

Thus within each Snaith degree the dimensions of these elements are within $2p^{n-1}$ of each other, so there is no room for any more differentials. Hence we have (subject to the multiplicative extensions of (5.4))

$$\begin{array}{rrcl} & Y(n)^*(L_1) & = & Y(n)^* \otimes_{K(n)^*} K(n)^*(L_1) \\ & & = & E(s_j : j \le n) \otimes T(a_{i,j} : i < n) \\ \text{and} & y(n)^*(L_1) & = & y(n)^* \otimes_{k(n)^*} k(n)^*(L_1) \\ & & & \quad \text{modulo } v_n\text{-torsion.} \end{array}$$

Now we can proceed to the computation of $h^*(L_2)$. We will study Tamaki's formulation of the Eilenberg-Moore spectral sequence. In the case $h^* = Y(n)^*$, $h^*(L_1)$ is a free module over the coefficient ring, so the E_2-term is Tor as before. In the case $h^* = y(n)^*$ we can make similar computations modulo v_n-torsion. With this understanding we have

$$E_2 = \Gamma(\sigma s_j : j \le n) \otimes E(\sigma a_{i,j} : i < n) \otimes \Gamma(\tau a_{i,j} : i < n).$$

We expect differentials

$$d_{2p^{n-i-1}-1}(\gamma_{p^{n-i-1}}(\tau a_{i,j})) = v_n \sigma a_{n-i-1,i+j+1} \qquad \text{for } i < n. \tag{6.1}$$

If we can prove this for $j = 0$, the Hopf map will give it to us for $j > 0$. The element $\gamma_{p^{n-i-1}}(\tau a_{i,0})$ has Snaith degree p^{n+1}, and there are no differentials (in either the Atiyah-Hirzebruch spectral sequence or the Eilenberg-Moore spectral sequence) in lower Snaith degrees. In ordinary cohomology one can show that

$$Q_n \gamma_{p^{n-i-1}}(\tau a_{i,0}) = \sigma a_{n-i-1,i+j+1},$$

so a similar differential occurs in the Atiyah-Hirzebruch spectral sequence. The failure of such a differential to occur in the Eilenberg-Moore spectral sequence would lead to the wrong description of $h^*(C_2^{p^{n+1}})$, so the differentials of (6.1) must occur.

This means that $E_{2p^{n-1}}$ is a subquotient of

$$T(\gamma_{p^k}(\sigma s_j) : j \leq n) \otimes E(\sigma a_{i,j} : i + j < n) \otimes T(\gamma_{p^k}(\tau a_{i,j}) : k + i < n - 1). \tag{6.2}$$

In Snaith degree p^ℓ for $\ell \geq n+1$, the generators listed are

$$\begin{aligned} \{\gamma_{p^{\ell-i-j-1}}(\tau a_{i,j}) \in h^{-2p^{\ell-j}-2p^{\ell-i-j}-2p^{\ell-i-j-1}} : i < n-1, \ell - j < n\} \\ \bigcup \{\gamma_{p^{\ell-j}}(\sigma s_j) \in h^{-2p^{\ell-j}} : j \leq n\}; \end{aligned} \tag{6.3}$$

The dimensions shown are the ones the elements would have if they survived. Differentials in this spectral sequence raise dimensions. Elements in the first family are in dimensions so high that they cannot support any nontrivial differentials.

Elements of the second family of (6.3) (where there is no restriction on $\ell - j$) occur in lower dimensions. Each is annihilated by some iterate of the Hopf map, so it cannot support a differential hitting an element of the first family. The only remaining possibility is a differential of the form

$$d_r(\gamma_{p^{\ell-j_1}}(\sigma s_{j_1})) = x\gamma_{p^{\ell-j_2}}(\sigma s_{j_2})$$

for some $x \in h^*$ and $j_1 < j_2$. Since we have a spectral sequence of Hopf algebras, the target of the first such differential must be primitive. (This is dual to the statement that a pth power can support a differential only if its pth root supports an earlier differential.) However since $j_2 \leq n$ and $\ell \geq n+1$ the target above is never primitive.

It follows that no other differentials can occur so E_∞ (modulo v_n-torsion) is the Hopf algebra of (6.2).

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