Voltage-Graphic Geometry and the Forest Lattice[1]

by

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1. We begin with a theorem that provides a focal point for the general theory. Let $\Gamma = (N, E)$ be a graph, $n = |N|$, $f_k =$ the number of k-tree spanning forests in Γ, and $t(\Delta) =$ the number of tree components of the graph Δ. Let $\mathfrak{F} =$ the set of forests of Γ, including the null graph, ordered in the following way: $F \leq F'$ if F' consists of some (or no) trees of F plus (optionally) additional edges linking some of these trees.

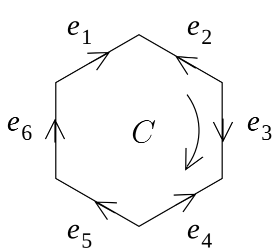
Forest Theorem. *$\mathfrak{F}$ is a geometric lattice of rank n. Its rank function is $\operatorname{rk} F = n - t(F)$. Its characteristic polynomial (when Γ is finite) is*

$$p_{\mathfrak{F}}(\lambda) = (-1)^n \sum_{k=0}^{n} (1-\lambda)^k f_k.$$

Some other facts about $\mathfrak{F}$: its 0 element is $(N, \emptyset)$, its 1 is $(\emptyset, \emptyset)$, its atoms are (N, e) for each link e and $(N \setminus \{v\}, \emptyset)$ for each vertex v.

The Forest Theorem can be proved directly, e.g. by deletion-contraction, but it is more interesting to derive it from the theory of voltage-graphic matroids.

2. A *voltage graph* [now called a *gain graph*] is a pair (Γ, φ) consisting of a graph $\Gamma = (N, E)$ and a *voltage* [now *gain function*], a mapping $\varphi : E \to \mathfrak{G}$ where $\mathfrak{G}$ is a group called the *voltage group* [now *gain group*]. The voltage [gain] on an edge depends on the sense in which the edge is traversed: if for e in one direction the voltage is $\varphi(e)$, then in the opposite direction it is $\varphi(e)^{-1}$. The voltage [gain] on a circle is the product of the edge voltages taken in order with consistent direction; if the product equals 1 the circle is called *balanced*. (While in general the starting point and orientation of C influence its voltage, they have no effect on whether it is balanced.) A subgraph is balanced if every circle in it is balanced. For $S \subseteq E$, let $b(S) =$ the number of balanced components of (N, S).



$$\varphi(C) = \varphi(e_1)\varphi(e_2)^{-1}\varphi(e_3)\varphi(e_4)^{-1}\varphi(e_5)\varphi(e_6)$$

Matroid Theorem. *The function $\operatorname{rk} S = n - b(S)$ is the rank function of a matroid $G(\Gamma, \varphi)$ on the set E. A set $A \subseteq E$ is closed iff every edge $e \notin A$ has an endpoint in a balanced*

[1]Abstract for the Denison–OSU Conference, May, 1980. In: *Report on the XVth Denison–OSU Math Conference* (Granville, Ohio, 1980), pp. 85–89. Dept. of Math., Ohio State University, Columbus, Ohio, 1980.

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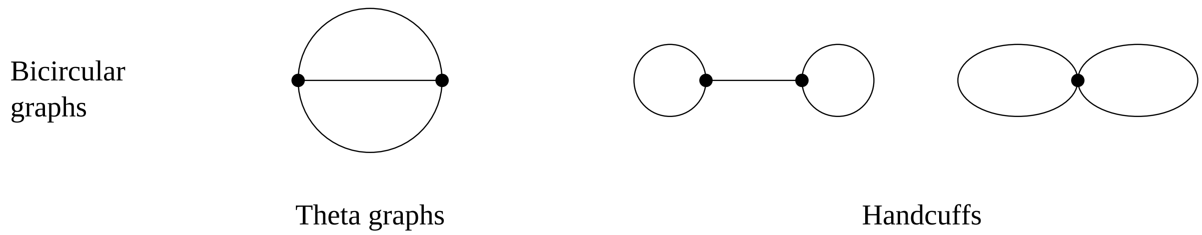
component of (N, A) *but does not combine with edges in* A *to form a balanced circle. A set is a circuit iff it is a balanced circle or a bicircular graph containing no balanced circle.*



We call $G(\Gamma, \varphi)$ a *voltage-graphic matroid.* [Now, *frame matroid* [6], or *gain-graphic bias matroid* [5].] In case it is a simple matroid it is a subgeometry of the Dowling lattice $Q_n(\mathfrak{G})$ (see [1]).

Example 1. $\varphi \equiv 1$. Then $G(\Gamma, \varphi) = G(\Gamma)$, the usual graphic matroid.

Example 2. $\mathfrak{G} = \{\pm 1\}$. Then (Γ, φ) is a signed graph.

Example 3. Same, with $\varphi \equiv -1$. Then $G(\Gamma, \varphi)$ is the even-circle matroid of Γ (see [2] for references).

Example 4. No balanced circles. Then $G(\Gamma, \varphi) = B(\Gamma)$, the bicircular matroid of Γ (see [4] for references). The balanced sets are the spanning forests. The closed sets correspond to the forests $F = (X, E(F))$ such that the subgraph of Γ induced on X^c has no trees. The circuits are the bicircular graphs (whence the name). The rank function is $\operatorname{rk} S = n - t(S)$.

The first parts of the Forest Theorem follows from these observations, the Matroid Theorem, and:

Lemma. $\mathfrak{F} \cong$ *the lattice of flats of* $B(\Gamma^\circ)$*, where* Γ° *denotes* Γ *with a loop at every node.*

3. Now let Γ be finite and let $\mathfrak{G}$ have finite order g. A *proper* μ*-coloring* of (Γ, φ) is a mapping

$$\kappa : N \to \{0\} \cup (\{1, \dots, \mu\} \times \mathfrak{G})$$

such that, for any edge e from v to w (including loops), we have $\kappa(v) \neq 0$ or $\kappa(w) \neq 0$ and also

$$\kappa_1(v) \neq \kappa_1(w) \quad \text{or} \quad \kappa_2(w) \neq \kappa_2(v)\varphi(e) \qquad \text{if } \kappa(v),\ \kappa(w) \neq 0,$$

where κ_1 and κ_2 are the numerical and group parts of κ. Let $\chi(\mu g + 1) =$ the number of proper μ-colorings of (Γ, φ) and let $\chi^{\mathrm{b}}(\mu g) =$ the number which do not take the value 0.

Chromatic Polynomial Theorem. $\chi(\mu g + 1)$ *is a polynomial in* μ*. Indeed* $\chi(\lambda) = \lambda^{b(E)} p(\lambda)$*, where* $p(\lambda)$ *is the characteristic polynomial of* $G(\Gamma, \varphi)$*.*

Balanced Chromatic Polynomial Theorem. *$\chi^{\mathrm{b}}(\mu g)$ is a polynomial in μ. Indeed $\chi^{\mathrm{b}}(\lambda) = \Sigma_S \lambda^{b(S)}(-1)^{|S|}$, summed over balanced $S \subseteq E$.*

Fundamental Theorem. *Let $\chi^{\mathrm{b}}_X(\lambda)$ denote the balanced chromatic polynomial of the induced voltage graph on $X \subseteq N$. Then*

$$\chi(\lambda) = \sum_{X \text{ stable}} \chi^{\mathrm{b}}_X(\lambda - 1).$$

In particular for the forest lattice we look at $B(\Gamma^\circ)$. The necessary finite voltage group may be, for instance, the power set $\mathcal{P}(E)$ with symmetric difference, with voltage $\varphi(e) = \{e\}$. Then the latter two theorems quickly yield the characteristic polynomial of $\mathfrak{F}$.

References

[1] T. A. Dowling, "A class of geometric lattices based on finite groups," *J. Combinatorial Theory Ser. B*, **14** (1973), 61–86. MR 46 #7066. Erratum, *ibid.*, **15** (1973), 211. MR 47 #8369.

[2] T. Zaslavsky, "Signed graphs," *Discrete Appl. Math.*, **4** (1982), 47–74. Erratum, *ibid.*, 5 (1983), 248. Proof of the Matroid Theorem [for signed graphs].

[3] T. Zaslavsky, "Signed graph coloring," *Discrete Math.*, **39** (1982), 214–228; and "Chromatic invariants of signed graphs," *Discrete Math.*, **42** (1982), 287–312. Proofs of the coloring results [for signed graphs].

[4] T. Zaslavsky, "Bicircular geometry and the lattice of forests of a graph," *Quart. J. Math. Oxford* (2), **33** (1982), 493–511. Has more detail, references, and applications to geometry.

[5] [T. Zaslavsky, "Biased graphs. II. The three matroids," *J. Combinatorial Theory Ser. B* **51** (1991), 46–72. Proof of the Matroid Theorem.]

[6] [T. Zaslavsky, "Frame matroids and biased graphs," *European J. Combin.* **15** (1994), 303–307.]

[7] [T. Zaslavsky, Biased graphs. III. Chromatic and dichromatic invariants. *J. Combin. Theory Ser. B* **64** (1995), 17–88. Proofs of the coloring results.]