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TOGS (GENERALIZATIONS OF TWO-GRAPHS)

Thomas Zaslavsky,
Department of Mathematical Sciences,
State University of New York,
University Center at Binghamton,
Binghamton, New York 13901, U.S.A.

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Abstract

We generalize two-graphs ("unitogs") to "togs", structures consisting of sets of polygons or cycles in a graph that satisfy specified relations. Togs are equivalent to switching classes of signings of a particular base graph. We develop a mechanism for generating and proving examples and apply it to unitogs (based on complete graphs), bipartite, tripartite, and multipartite togs (based on complete multipartite graphs), circular togs (based on complete circular multipartite graphs), Hamming togs (based on Hamming graphs), and Johnson togs (based on Johnson graphs, whose nodes are the r-subsets of a set, with adjacency corresponding to $(r-1)$-element overlap).

We also explore the possibility of generalizing unitogs to set families which are determined by the sets on any one point. In some examples such a "residually determined" family must be a tog or one of a few exceptional families.

1. Introduction

A *two-graph* (in our terminology a *unitog*) is a class of (unordered) triples of elements of a given point set, such that each quadruple of points contains an even number of triples in the class. Unitogs have found application to strongly regular graphs, equiangular line sets, and permutation groups (see the survey [**6**]). In this article we explore the combinatorics of generalizing unitogs, concentrating on a generalization based on signed graphs which we call a *tog* (a rough acronym for "generalized Two-Graph" which we like because it is short and distinctive[1]) and which is due more or less independently to Cameron (see [**10**, Section 2.1] and [**2**]) and myself [**13**]. We give a general definition and a technique for verifying examples (Section 3), a collection of examples illustrating the technique (Section 4), and a discussion of the connection between togs and another possible generalization related to extension of designs (Section 5). Our hope is that some of the interesting properties and applications of unitogs will have analogs for other examples, but we do not treat such questions here.

Unitogs were first defined explicitly by G. Higman (see [**6**, **8**]). But they had earlier shown up implicitly in the work of Seidel and others ([**4**], [**5**], and see [**6**], [**7**]) on sets of equiangular lines and switching equivalence classes of adjacency matrices of signed complete graphs. Since the latter standpoint was the inspiration for our work, we describe it here.

A *signed graph* is a graph with signed edges. A cycle is called *positive* or *negative* depending on the product of its edge signs. Let $\mathcal{T}(\Sigma)$ be the class of negative triangles in a signed complete graph Σ. Then $\mathcal{T}(\Sigma)$ is a unitog. (Strictly speaking, the unitog consists of the node triples which support negative triangles.) Furthermore, every unitog arises in this way, and uniquely so in the following sense. Let Σ_1 and Σ_2 be *switching equivalent*, $\Sigma_1 \sim \Sigma_2$, if one is obtained from the other by (*signed*) *switching*: reversing the signs of the edges between a node set and its complement. Then $\mathcal{T}(\Sigma_1) = \mathcal{T}(\Sigma_2)$ if and only if $\Sigma_1 \sim \Sigma_2$. Thus unitogs correspond to switching equivalence classes of signed complete graphs. The adjacency matrix of Σ is $A = (a_{ij})$ where $a_{ii} = 0$ and $a_{ij} = \pm 1$ for $i \neq j$, a_{ij} taking the sign of the edge joining the i-th and j-th nodes. Switching Σ corresponds to conjugating A by a diagonal matrix whose diagonal elements are ± 1. We call A a *matrix of* $\mathcal{T}(\Sigma)$. (In work on unitogs [**4-8**] it is customary to suppress the positive edges, leaving a graph G. Then A is called the $(0, -1, +1)$-*adjacency matrix* of G.)

In a tog we replace the complete graph by an arbitrary graph Γ and the triangles by a class $\mathcal{D}$ of distinguished cycles that spans the binary vector space of all cycles. A tog is then a subclass of distinguished cycles which is the class of negative distinguished cycles in some signing of Γ. The heart and the hard part of the generalization is to find analogs of the quadruple constraints: that is, to find combinatorial properties of a subclass of $\mathcal{D}$ which are valid if and only if the subclass is a tog. This is what makes a true generalization of unitogs. The tog itself is an epiphenomenon upon the "foundation" consisting of Γ, $\mathcal{D}$, and the combinatorial constraints. Our theory is intended to help find and establish the validity of such constraints, in other words to treat foundations. The togs themselves are not directly our concern.

One can take quite a different approach, as Cameron did in [**1**]. Suppose we want a class $\mathcal{T}$ of (unordered) triples in an n-element point set such that the triples on any one point v determine the rest. The motivation comes from the fact that, if $\mathcal{T}$ were a design, it would be determined by the residual design $\mathcal{T}/v = (\{x, y\} : \{v, x, y\} \in \mathcal{T}\}$. The simplest rule is that the number of triples of $\mathcal{T}$ in any set of four points should belong to a fixed set S of permitted numbers. Evidently, if $S \subseteq \{0, 2, 4\}$ then $\mathcal{T}$ is a unitog. One can show that, for arbitrary S, there are only two kinds of possible classes $\mathcal{T}$ that are not unitogs. Thus we are led to the same evenness condition as was suggested by the signed-graph approach. In Section 5 we show that this near agreement of definitions holds true for several other examples of togs, although not for all.

Having now outlined the entire article, we present some of the examples to be treated in detail in Section 4.

Unitogs have already been defined. A set N of n lines in $\mathbb{R}^d$ is *equiangular* if any two lines make the same angle. Such a set gives rise to a signed complete graph Σ on N.

[1]The term "2-graph" has several unrelated senses in the literature. Besides a unitog it can mean a 2-regular graph or a 2-uniform hypergraph, for instance.

Choosing a unit vector u_i parallel to each line l_i, give the edge e_{ij} the sign of $u_i \cdot u_j$ for $i \neq j$. Reversing some of the unit vectors has the effect of switching Σ. Hence, an equiangular line set determines a unitog; the converse is also true (see [**6**, Theorem 5.4]).

The most important unitogs are *regular*: each pair of points lies in the same number λ of triples of the unitog $\mathcal{T}$. That is, $\mathcal{T}$ is a $2-(n,3,\lambda)$ design. (See [**6**] and [**8**] for regular unitogs.) There are close connections with other combinatorial objects, for instance strongly regular graphs. If Σ is a signed complete graph for which $\mathcal{T}(\Sigma)$ is a regular unitog, and if in the negative-edge subgraph G a node w is isolated, then $G \backslash w$ is strongly regular. Seidel showed that regular unitogs are characterized by having a matrix A with exactly two eigenvalues, whose product is $1-n$, where the point set has n elements. One aim of the theory of togs is to find definitions of regularity in other examples that have characterizations and connections to other areas as interesting and fruitful as those of unitogs.

Bipartite togs, briefly *bitogs*, are based on complete bipartite graphs. A bitog can be defined as a class $\mathcal{T}$ of (unordered) quadruples of points, half of each quadruple drawn from each of two disjoint sets X and Y with cardinalities l and m, such that any quintuple contains an even number of quadruples in $\mathcal{T}$. The matrix of a bitog corresponding to the matrix A of a unitog is an l by m matrix B consisting of -1's and $+1$'s, such that

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

is the adjacency matrix of a signing corresponding to $\mathcal{T}$ of the underlying graph.

One can define various kinds of regularity. (For regular bitogs see [**13**, Section 4].) Let (i,j)*-regularity* mean that any set of i points in X and j points in Y is contained in the same number of bitog quadruples. Bitogs that are both $(2,0)$- and $(0,2)$-regular arise from $2-(v,k,\lambda)$ designs. Hadamard designs yield bitogs that are also $(1,1)$-regular. Some matrix characterizations of regularity are: $\mathcal{T}$ is $(2,0)$-regular if and only if $BB^T - lI = pA$ where $p \geq 0$ and A is a matrix of a unitog. (The unitog is uniquely determined by $\mathcal{T}$ if $p \neq 0$.) $\mathcal{T}$ is $(1,1)$-regular if and only if BB^T has at most one non-zero eigenvalue.

In the next three examples we let $N = X_1 \cup X_2 \cup \cdots \cup X_r$ be a partition of the point set N, that is, the X_i are nonempty and pairwise disjoint. A set of points is *transverse* if it meets no X_i in more than one point; it is a *complete transversal* if it meets every X_i in exactly one point.

A *tripartite tog* has $r = 3$; it consists of transverse triples and has an even number of members contained in each set $W \subseteq N$ consisting of exactly two points from each X_i.

A *multipartite tog* has $r \geq 4$; it consists of transverse triples and has an even number of members in each transverse quadruple.

A *circular tog* has $r \geq 4$; it consists of complete transversals subject to the condition that there is an even number of tog elements in any set of the form $T \cup \{x_i, x_j\}$ where T is a complete transversal, $x_i \in X_i \backslash T$, $x_j \in X_j \backslash T$, and $i - j \not\equiv 0,\ \pm 1 \pmod{r}$.

A *Hamming tog* has point set $N = \{0, 1, \cdots, q-1\}^d$, where $q, d \geq 2$. A *Johnson tog* has for points all the r-element subsets of an m-element set, where $0 < r < m$. These togs are hard to describe briefly, except for Johnson togs with $r = 2$. The latter can be regarded as having for points the edges of the complete graph K_m. A tog consists of triples of edges which either form a triangle or are concurrent at a node of K_m, subject to the two conditions that in any concurrent foursome of edges, an even number of triples belong to the tog, and any K_4 subgraph of K_m contains an even number of tog triples.

The idea of generalizing unitogs by means of signed graphs occurred to Cameron (see [**10**, Section 2.1]) and me and possibly several others. The theory of togs presented in this paper is a generalization of the technique of [**13**, Sections 1 and 2] (which was perhaps too unsystematic to be called a "theory"), based on the cohomological theory of "exact" triples developed by Cameron and Wells in [**10**] and [**2**]. Our theory is substantially equivalent to theirs but we have chosen to employ the graph-theoretic language of [**13**] rather than the language of exact sequences. There is some cost in the amount of preparatory work required (in Section 2), but perhaps a gain in accessibility of the statements and proofs of examples. Another difference should be mentioned: our Theorem 10, upon which we rely heavily in attacking the examples. It is based on the more special technique discussed at the end of [**13**, Section 2]. Despite the number of lemmas we need to prove it, it is really quite elementary; but it is useful.

Some of the examples were independently discovered both by Cameron-Wells and me: bitogs and multipartite togs. Tripartite and circular togs were introduced in [**13**], the latter with an incomplete proof. Hamming and Johnson togs are new, but the former are generalized from the cases where q or $d = 2$, which are due to Cameron and Wells. These authors have several other interesting examples, such as folded-cube togs [**2**, (7.9)], togs of polygons ([**2**, (6.2)], based on [**11**, Theorem 6]), and togs of induced polygons ([**2**, (6.4)], based on [**9**]). For these and more we refer the reader to [**2**].

The work in Section 5 on residual determinacy is, as far as I know, new except for uniform strong numerical determinacy in the unitog case, which is discussed in [**1**].

2. Graphs, sets, and signed graphs

A graph $\Gamma = (N, E)$ may have multiple edges and loops. We write $n = |N|$ for the order of the node set N. A *cycle* is an edge set which has even degree at every vertex. A *polygon* is the edge set of a simple, closed path; the class of polygons [of length k] in Γ is denoted by $\mathcal{C}(\Gamma)$ [resp., $\mathcal{C}^{(k)}(\Gamma)$].

The class $P(S)$ of subsets of a set S forms a binary vector space under the operation of symmetric difference, which we call "sum" and write $+$. The span of a subset $W \subseteq P(S)$ is denoted by $\langle W\rangle$. In particular the polygons of Γ span the subspace of $P(E)$ consisting of all cycles (including the empty set); we call $\langle\mathcal{C}(\Gamma)\rangle$ the *cycle space* of Γ. A basis of the cycle space is called a *cycle basis*. Any subset $\mathcal{D}$ of the cycle space spans a subspace $\langle\mathcal{D}\rangle$ of $P(E)$; we speak of independent, spanning, and basic subsets of $\mathcal{D}$ meaning, e.g., basic in $\langle\mathcal{D}\rangle$.

A *relator* in $\mathcal{D}$ is a subset $\mathcal{R} \subseteq \mathcal{D}$ whose sum is 0 (that is, the empty set). The class $\mathbf{R}(\mathcal{D})$ of all relators in $\mathcal{D}$ forms a subspace of $P(\mathcal{D})$ and hence of $P(P(E))$. The significance of relators is that they correspond to the linear relations in $\mathcal{D}$; one may think of $\mathbf{R}(\mathcal{D})$ as being the class of all linear relations in $\mathcal{D}$. Let $\mathcal{B} \subseteq \mathcal{D}$ and $\mathbf{R} \subseteq \mathbf{R}(\mathcal{D})$; we say $\mathcal{B}$ *respects* $\mathbf{R}$ if $|\mathcal{R}\backslash\mathcal{B}|$ is even for every $\mathcal{R} \in \mathbf{R}$. (In [**11**] we used the term *additive* for a subset $\mathcal{B}$ that respects $\mathbf{R}(\mathcal{D})$.) The meaning of "respect" may be suggested by Lemma 1. For $\mathcal{S} \subseteq \mathcal{D}$ let $1_{\mathcal{S}}$ be the *characteristic function* of $\mathcal{S}$, that is $1_{\mathcal{S}} : \mathcal{D} \to \mathbb{Z}_2$ defined by $1_{\mathcal{S}}(C) = 1$ if $C \in \mathcal{S}$, $= 0$ if $C \notin \mathcal{S}$. For $\mathbf{S} \subseteq P(\mathcal{D})$, let $\mathbf{S}^* = \{1_{\mathcal{S}} : \mathcal{S} \in \mathbf{S}\}$. Then $P(\mathcal{D})^*$ is the dual vector space of $P(\mathcal{D})$; and every element of $P(\mathcal{D})^*$ has the form $1_{\mathcal{S}}$ for some $\mathcal{S} \subseteq \mathcal{D}$. In $P(\mathcal{D})^*$ we have the inner product

$$1_{\mathcal{S}} \cdot 1_{\mathcal{T}} = \sum_{C\in\mathcal{D}} 1_{\mathcal{S}}(C) 1_{\mathcal{T}}(C) \equiv |\mathcal{S} \cap \mathcal{T}| \pmod{2}.$$

We write $\mathbf{S}^{*\perp}$ for the orthogonal dual space of $\mathbf{S}^*$ in $P(\mathcal{D})^*$.

Lemma 1. *Let $\mathcal{B} \subseteq \mathcal{D}$ and $\mathbf{R} \subseteq \mathbf{R}(\mathcal{D})$. The following are equivalent:*

(i) $\mathcal{B}$ *respects* $\mathbf{R}$.

(ii) $\sum_{D \in \mathcal{R}} 1_{\mathcal{D}\backslash\mathcal{B}}(D) = 0$ *for every* $\mathcal{R} \in \mathbf{R}$.

(iii) $1_{\mathcal{D}\backslash\mathcal{B}} \in \mathbf{R}^{*\perp}$.

Proof. Let $\mathcal{R} \in \mathbf{R}$. Then $\sum 1_{\mathcal{D}\backslash\mathcal{B}}(D) = 1_{\mathcal{D}\backslash\mathcal{B}} \cdot 1_{\mathcal{R}} = |\mathcal{R}\backslash\mathcal{B}| \pmod 2$. The lemma follows. ■

Lemma 2. *Let $\mathbf{R}_1 \subseteq \mathbf{R}_2 \subseteq \mathbf{R}(\mathcal{D})$. Then (i) $\mathbf{R}_1$ spans $\langle \mathbf{R}_2 \rangle$ if and only if (ii) every $\mathcal{B} \subseteq \mathcal{D}$ that respects $\mathbf{R}_1$ also respects $\mathbf{R}_2$.*

Proof. By Lemma 1, (ii) can be stated: $\mathbf{R}_1^{*\perp} \subseteq \mathbf{R}_2^{*\perp}$. Since $\mathbf{R}^{*\perp} = \langle \mathbf{R}^* \rangle^{\perp}$ and $\langle \mathbf{R}^* \rangle = \langle \mathbf{R} \rangle^*$, this is equivalent to (ii$'$) $\langle \mathbf{R}_1 \rangle^{*\perp} \subseteq \langle \mathbf{R}_2 \rangle^{*\perp}$. Since $\mathbf{R}_1 \subseteq \mathbf{R}_2$, we have $\langle \mathbf{R}_1 \rangle^{*\perp} \supseteq \langle \mathbf{R}_2 \rangle^{*\perp}$ in any case. So (ii$'$) is equivalent to $\langle \mathbf{R}_1 \rangle^{*\perp} = \langle \mathbf{R}_2 \rangle^{*\perp}$, which is equivalent to $\langle \mathbf{R}_1 \rangle = \langle \mathbf{R}_2 \rangle$. ■

The cases $\mathbf{R}_2 = \langle \mathbf{R}_1 \rangle$ and $\mathbf{R}_2 = \mathbf{R}(\mathcal{D})$ will be most useful for us.

The next result is not required for our work; we include it because it throws more light on the significance of "respect".

Proposition 3. *If $\mathcal{B} \subseteq \mathcal{D}$ respects $\mathbf{R}(\mathcal{D})$, then $\langle \mathcal{B} \rangle \cap \mathcal{D} = \mathcal{B}$.*

Proof. Let $B \in \langle \mathcal{B} \rangle \cap \mathcal{D}$, so $B = B_1 + \cdots + B_k$ where $B_1, \cdots, B_k \in \mathcal{B}$. Then $\mathcal{R} = \{B, B_1, \cdots, B_k\} \in \mathbf{R}(\mathcal{D})$. Since all $B_i \in \mathcal{B}$ and evenly many elements of $\mathcal{R}$ are not in $\mathcal{B}$, B must lie in $\mathcal{B}$. ■

Let $\mathbf{R}$ be any set of relators in $\mathcal{D}$, and let $\mathcal{D}' \subseteq \mathcal{D}$. The $\mathbf{R}$-*closure* of $\mathcal{D}'$ is

$$\mathcal{D}' \cup \{D \in \mathcal{D}\backslash\mathcal{D}' : \exists\, \mathcal{R} \in \mathbf{R} \text{ such that } D \in \mathcal{R} \subseteq \mathcal{D}' \cup \{D\}\}.$$

We call $\mathcal{R}$ a *generating relator for D over $\mathcal{D}'$* because it corresponds to the linear relation $D = \sum \{D' \in \mathcal{R} : D' \neq D\}$ which expresses D as a linear combination of elements of $\mathcal{D}'$.

Lemma 4. *Let $\mathcal{B}' \subseteq \mathcal{D}' \subseteq \mathcal{D} \subseteq \langle \mathcal{C}(\Gamma) \rangle$ where $\mathcal{D}'$ spans $\mathcal{D}$ and $\mathcal{B}'$ respects $\mathbf{R}(\mathcal{D}')$. Let $\mathbf{R} \subseteq \mathbf{R}(\mathcal{D})$ be such that $\mathcal{D}$ lies in the $\mathbf{R}$-closure of $\mathcal{D}'$. Choose a fixed generating relator (in $\mathbf{R}$) over $\mathcal{D}'$ for each $D \in \mathcal{D}\backslash\mathcal{D}'$. Define $\mathcal{B} \subseteq \mathcal{D}$ by $\mathcal{B} \cap \mathcal{D}' = \mathcal{B}'$ and $D \in \mathcal{D}\backslash\mathcal{D}'$ is in $\mathcal{B}$ if and only if an even number of members of its chosen generating relator are contained in $\mathcal{D}'\backslash\mathcal{B}'$. Then $\mathcal{B}$ respects $\mathbf{R}(\mathcal{D})$ and is independent of the choices of generating relators.*

Proof. The lemma is an extension of [**11**, Lemma 3], which concerns the case where $\mathcal{D} \subseteq \mathcal{C}(\Gamma)$ and $\mathbf{R} = \mathbf{R}(\mathcal{D})$, and whose proof is adapted here (with a missing step supplied).

First we show that $\mathcal{B}$ is independent of the choice of generating relators. Let $C \in \mathcal{D}\backslash\mathcal{D}'$ have two generating relators (in $\mathbf{R}$) over $\mathcal{D}'$, $\{C, C_1, \cdots, C_q\}$ and $\{C, D_1, \cdots, D_r\}$. Then $C_1 + \cdots + C_q = C = D_1 + \cdots + D_r$, whence $\{C_1, \cdots, C_q, D_1, \cdots, D_r\} \in \mathbf{R}(\mathcal{D}')$. By the hypothesis that $\mathcal{B}'$ respects $\mathbf{R}(\mathcal{D}')$, the numbers of C_i and D_j not in $\mathcal{B}'$ have the same parity. Consequently, the two generating relators for C agree on whether or not it lies in $\mathcal{B}$.

Now we prove $\mathcal{B}$ respects $\mathbf{R}(\mathcal{D})$. Suppose $\mathcal{R} = \{D_1, \cdots, D_p\} \in \mathbf{R}(\mathcal{D})$. Let the chosen generating relator for D_i be $\{D_i, D_{i1}, \cdots, D_{ir_i}\}$, where all $D_{ij} \in \mathcal{D}'$. (If $D_i \in \mathcal{D}'$ we let $r_i = 1$ and $D_{i1} = D_i$.) By hypothesis,

$$1_{\mathcal{D}\backslash\mathcal{B}}(D_i) = \sum_j 1_{\mathcal{D}'\backslash\mathcal{B}'}(D_{ij}).$$

Thus we have

$$\sum_{i,j} D_{ij} = \sum_i D_i = 0$$

and

$$\sum_i 1_{\mathcal{D}\setminus\mathcal{B}}(D_i) = \sum_{i,j} 1_{\mathcal{D}'\setminus\mathcal{B}'}(D_{ij}).$$

Let $\mathcal{R}' = \{D_{ij} : D_{ij}$ appears oddly often in the p generating relations$\}$. Then $\mathcal{R}' \in \mathbf{R}(\mathcal{D}')$; so $\mathcal{B}'$ respects it. Therefore, $\sum 1_{\mathcal{D}'\setminus\mathcal{B}'}(D_{ij}) = 0$. It follows that $\mathcal{B}$ respects $\mathcal{R}$. ■

A *signed graph* $\Sigma = (\Gamma, \sigma)$ consists of an underlying graph Γ and a *signing* $\sigma : E \to \{+, -\}$. For $S \subseteq E$ we define $\sigma(S) =$ product of the edge signs of S. A polygon C is *balanced* or *unbalanced* according as $\sigma(C) = +$ or $-$. An edge set (or subgraph) is *balanced* when every polygon in it is; otherwise it is *unbalanced*. (Signed graphs and balance were introduced by Harary [**3**].) The class of balanced polygons of Σ is denoted by $\mathcal{B}(\Sigma)$. We define $\hat{\sigma}$ to be σ regarded as a function from the cycle space to $\mathbb{Z}_2$. Then $\hat{\sigma}$ is obviously linear, $\mathcal{B}(\Sigma) = \mathcal{C}(\Gamma) \cap$ Ker $\hat{\sigma}$ by definition, and by standard linear algebra Ker $\hat{\sigma}$ is either the whole space or a hyperplane (subspace of codimension 1).

Lemma 5. *Let $\delta : \langle \mathcal{C}(\Gamma) \rangle \to \mathbb{Z}_2$ be a linear functional. Then there is a signed graph Σ such that $\hat{\sigma} = \delta$.*

Proof. Let F be a maximal forest in Γ and, for an edge $e \notin F$, let $C(e)$ be the unique polygon in $F \cup \{e\}$. Let $\sigma(e) = +$ if $e \in F$ and $\sigma(e) = (-)^{\delta(C(e))}$ if $e \notin F$. This determines Σ. Because $\hat{\sigma}$ is linear, the polygons $C(e)$ are a cycle basis, and $\hat{\sigma}(C(e)) = \delta(C(e))$, we have $\hat{\sigma} = \delta$. ■

Switching Σ by a function $\mu : N \to \{+, -\}$ means forming $\Sigma^\mu = (\Gamma, \sigma^\mu)$ where $\sigma^\mu(e) = \mu(v)\sigma(e)\mu(w)$ if e is an edge whose endpoints are v and w. We call Σ and Σ^μ *switching equivalent*; their equivalence class is the *switching class* $[\Sigma]$ of Σ. Switching leaves $\hat{\sigma}$ and $\mathcal{B}(\Sigma)$ unaltered. Fundamental to our generalization of two-graphs is the characterization of switching equivalence classes in Lemma 6.

Lemma 6. *Let Σ_1 and Σ_2 be signed graphs on the same underlying graph. The following are equivalent:*

(i) Σ_1 *and* Σ_2 *are switching equivalent.*
(ii) $\hat{\sigma}_1 = \hat{\sigma}_2$.
(iii) $\mathcal{B}(\Sigma_1) = \mathcal{B}(\Sigma_2)$.

Proof. This is a well-known lemma; see for instance [**12**, Proposition 3.2]. We give a proof for the sake of completeness. The equivalence of (ii) and (iii) follows from the fact that $\mathcal{B}(\Sigma) = \mathcal{C}(\Gamma) \cap \text{Ker } \hat{\sigma}$. We know (i) implies (iii). Suppose (iii) holds true. Choose a root node w_j and a spanning tree F_j in each component Γ_j of Γ. For $i = 1, 2$ and $v \in N(\Gamma_j)$, let P be the path in F_j from w_j to v and $\mu_1(v) = \sigma_i(P)$. Then $\Sigma_1^{\mu_1} = \Sigma_2^{\mu_2}$, so the signed graphs are switching equivalent. ■

Lemma 7. *Let $\mathcal{B}' \subseteq \mathcal{D}' \subseteq \langle \mathcal{C}(\Gamma) \rangle$. $\mathcal{B}'$ respects $\mathbf{R}(\mathcal{D}')$ if and only if $1_{\mathcal{D}'\setminus\mathcal{B}'} : \mathcal{D}' \to \mathbb{Z}_2$ extends to a linear functional $\langle \mathcal{C}(\Gamma) \rangle \to \mathbb{Z}_2$.*

Proof. The "if" part is obvious. The "only if" is obvious if $\mathcal{D}'$ is a subspace of the cycle space. Suppose $\mathcal{D}'$ is arbitrary and $\mathcal{B}'$ respects $\mathbf{R}(\mathcal{D}')$. In Lemma 4 let $\mathcal{D} = \langle \mathcal{D}' \rangle$ and $\mathbf{R} = \mathbf{R}(\mathcal{D})$. By that lemma there exists $\mathcal{B} \subseteq \mathcal{D}$ which respects $\mathbf{R}(\mathcal{D})$. Equivalently, $1_{\mathcal{D}\setminus\mathcal{B}}$ is a linear functional on $\mathcal{D}$. Therefore it extends to a functional on $\langle \mathcal{C}(\Gamma) \rangle$. ■

Lemma 8. *Let $\mathcal{D}$ be a subset of the cycle space $\langle \mathcal{C}(\Gamma) \rangle$. If $\mathcal{D}$ spans $\langle \mathcal{C}(\Gamma) \rangle$, then the mapping $\Sigma \to \mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}$ induces a one-to-one correspondence between switching classes of signed graphs on Γ and subsets of $\mathcal{D}$ that respect $\mathbf{R}(\mathcal{D})$.*

But if $\mathcal{D}$ does not span $\langle \mathcal{C}(\Gamma) \rangle$, then the mapping is not injective.

Proof. This lemma is an extension of Theorem 2 and Corollary 8 of [**11**], which concern the case in which $\mathcal{D} \subseteq \mathcal{C}(\Gamma)$. The proof here is analogous to that in [**11**] but is more complete.

First, let $\mathcal{D}$ be any subset of the cycle space.

Let Σ be a signed graph. Then $\mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}$ respects $\mathbf{R}(\mathcal{D})$ because $\hat{\sigma}$ is linear. Switching Σ leaves $\hat{\sigma}$ unaltered. Thus the mapping $\Sigma \to f(\Sigma) = \mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}$ is a well-defined map from switching classes to subsets of $\mathcal{D}$ that respect $\mathbf{R}(\mathcal{D})$.

Suppose $\mathcal{B} \subseteq \mathcal{D}$ respects $\mathbf{R}(\mathcal{D})$. By Lemma 7, $1_{\mathcal{D} \setminus \mathcal{B}}$ extends to a functional δ on $\langle \mathcal{C}(\Gamma) \rangle$. By Lemma 5, there is a signed graph Σ such that $\hat{\sigma} = \delta$; thus $\mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma} = \mathcal{B}$. This shows that f is surjective.

Now suppose $\mathcal{D}$ spans the cycle space and $f(\Sigma_1) = f(\Sigma_2)$; that is, $\mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}_1 = \mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}_2$. Then $\hat{\sigma}_1 = \hat{\sigma}_2$. By Lemma 6, $[\Sigma_1] = [\Sigma_2]$. Thus f is injective.

On the other hand, suppose $\langle \mathcal{D} \rangle$ is a proper subspace of the cycle space. Let δ_1 and δ_2 be different linear functionals on the cycle space that agree on $\langle \mathcal{D} \rangle$. By Lemma 5, there exist signed graphs Σ_1 and Σ_2 for which $\hat{\sigma}_1 = \delta_1$ and $\hat{\sigma}_2 = \delta_2$. By Lemma 6, $[\Sigma_1] \neq [\Sigma_2]$. So f is not injective. ■

3. Theory of togs

Since we take the fundamental object of the general theory to be a switching class of signed graphs, the problems are then to describe the switching classes efficiently and to find simple criteria for recognizing them. A switching class can be described by its class of positive cycles and the latter by its restriction to a spanning class of polygons–the triangles, in the case of unitogs—or more generally to a spanning class $\mathcal{D}$ of cycles. Recognition depends (by Lemma 8) on verifying that a possible tog respects $\mathbf{R}(\mathcal{D})$; efficient recognition depends on finding a small but adequate set of test relations—for unitogs, the four triangles in each quadruple. We now present our general definition.

A tog consists of a foundation and the tog proper. The *foundation* F is composed of a graph Γ, a spanning subset $\mathcal{D}$ of the cycle space, and a set $\mathbf{R}$ of relators in $\mathcal{D}$ that spans the relator space $\mathbf{R}(\mathcal{D})$. A *tog on* F is then a subset $\mathcal{T} \subseteq \mathcal{D}$ such that $\mathcal{D} \setminus \mathcal{T}$ respects $\mathbf{R}$: each relator in $\mathbf{R}$ contains an even number of members of $\mathcal{T}$.

Theorem 9. *Given Γ, $\mathcal{D} \subseteq \langle \mathcal{C}(\Gamma) \rangle$, and $\mathbf{R} \subseteq \mathbf{R}(\mathcal{D})$. If $F = (\Gamma, \mathcal{D}, \mathbf{R})$ is a tog foundation, then togs $\mathcal{T}$ on F are in one-to-one correspondence with the switching classes of signed graphs based on Γ, by the rule $\mathcal{T} = \mathcal{D} \setminus \mathrm{Ker}\, \hat{\sigma}$. This correspondence holds only if $\mathcal{D}$ spans $\langle \mathcal{C}(\Gamma) \rangle$ and $\mathbf{R}$ spans $\mathbf{R}(\mathcal{D})$.*

Proof. First, suppose F is a tog foundation. By Lemma 8, switching classes correspond bijectively to subsets of $\mathcal{D}$ that respect $\mathbf{R}(\mathcal{D})$. Thus, what we have to prove is that a subset $\mathcal{B}$ of $\mathcal{D}$ respects $\mathbf{R}$ if and only if it respects $\mathbf{R}(\mathcal{D})$. Since $\mathbf{R}$ spans $\mathbf{R}(\mathcal{D})$, this follows from Lemma 2. That proves the first part of the theorem.

Suppose $\mathbf{R}$ does not span $\mathbf{R}(\mathcal{D})$. Then, by Lemma 2, there exists $\mathcal{B}$ which respects $\mathbf{R}$ but not $\mathbf{R}(\mathcal{D})$. Lemma 8 shows that $\mathcal{B}$ is certainly not of the form $\mathcal{D} \cap \mathrm{Ker}\, \hat{\sigma}$, regardless of whether or not $\mathcal{D}$ spans $\langle \mathcal{C}(\Gamma) \rangle$.

Suppose $\mathbf{R}$ spans $\mathbf{R}(\mathcal{D})$ but $\mathcal{D}$ does not span $\langle\mathcal{C}(\Gamma)\rangle$. Then $\mathcal{B}$ respects $\mathbf{R}$ precisely when it respects $\mathbf{R}(\mathcal{D})$, but as Lemma 8 indicates, $\mathcal{B}$ corresponds to more than one switching class. That concludes the proof. ■

As it stands, Theorem 9 is not that easy to apply. What kinds of relator sets $\mathbf{R}$ should one look for? Given $\mathcal{D}$ and $\mathbf{R}$, how does one prove they form a tog foundation? We present a method that is effective in several examples, as the next section will show.

Theorem 10. *Let Γ be a graph, $\mathcal{D} \subseteq \langle\mathcal{C}(\Gamma)\rangle$, and $\mathbf{R} \subseteq \mathbf{R}(\mathcal{D})$. Suppose $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}_k = \mathcal{D}$ where each $\mathcal{D}_i$ lies in the $\mathbf{R}$-closure of $\mathcal{D}_{i-1}$ and $\mathcal{D}_0$ is a cycle basis of Γ. Then $F = (\Gamma, \mathcal{D}, \mathbf{R})$ is a tog foundation, and a tog on F is precisely a set $\mathcal{T} \subseteq \mathcal{D}$ such that $\mathcal{D}\backslash\mathcal{T}$ respects $\mathbf{R}$.*

Remark. The hypothesis that $\mathcal{D}_0$ is a cycle basis may be replaced by the two assumptions that $\mathcal{D}$ spans the cycle space and that $\mathcal{D}_0$ is independent or has the same size as a cycle basis. The reason is that the $\mathbf{R}$-closure of $\mathcal{D}_{i-1}$ is contained in $\langle\mathcal{D}_{i-1}\rangle$; hence $\langle\mathcal{D}_0\rangle = \langle\mathcal{D}_k\rangle$.

Proof. $\mathcal{D}$ spans the cycle space because $\mathcal{D}_0$ does so.

By Lemma 4, a subclass $\mathcal{B}$ of $\mathcal{D}$ respects $\mathbf{R}(\mathcal{D})$ precisely when it respects $\mathbf{R}(\mathcal{D}_0)$ and $\mathbf{R}$. Since $\mathcal{D}_0$ is a basis, $\mathbf{R}(\mathcal{D}_0)$ is empty. Therefore, any subclass respects it. Hence, $\mathcal{B}$ respects $\mathbf{R}(\mathcal{D})$ if and only if it respects $\mathbf{R}$. By Lemma 2, $\mathbf{R}$ spans $\mathbf{R}(\mathcal{D})$. ■

This theorem provides not just a mechanism for proving the validity of a tog foundation but also a means of discovery. Suppose Γ and a spanning cycle set $\mathcal{D}$ have been chosen; the task is to find $\mathbf{R}$. One way to do so is by picking a basis $\mathcal{D}_0$ and looking for a set $\mathbf{R}_1$ of relators such that the $\mathbf{R}_1$-closure of $\mathcal{D}_0$ contains a larger set $\mathcal{D}_1$, then a relator set $\mathbf{R}_2$ (perhaps the same as $\mathbf{R}_1$) so the $\mathbf{R}_2$-closure of $\mathcal{D}_1$ contains a larger set $\mathcal{D}_2$, and so on until $\mathcal{D}_k = \mathcal{D}$ is obtained. Then $\mathbf{R} = \mathbf{R}_1 \cup \cdots \cup \mathbf{R}_k$ is a relator set which, with Γ and $\mathcal{D}$, forms a tog foundation. Another way to proceed is to guess $\mathbf{R}$ first, then find a basis $\mathcal{D}_0$ that generates $\mathcal{D}$ through one or more intermediate stages $\mathcal{D}_i$ as in Theorem 10. We used both approaches to find the examples of the next section.

4. Examples

Here we treat in detail several examples of tog foundations. Most of them were already described in the introduction in the form of constraints on sets of points. Here we describe the foundations in terms of cycles of the underlying graph; it should be apparent how the two descriptions are equivalent.

Consider the triple $(K_n, \mathcal{C}^{(3)}(K_n), \mathbf{R}^{(4)})$, where

$$\mathbf{R}^{(4)} = \{\mathcal{C}^{(3)}(G) : G \text{ is an induced subgraph of } K_n \text{ of order } 4\}.$$

We prove that this is a tog foundation. Plainly a tog on it is the same as a unitog (a two-graph), defined as a set of triangles having an even number in each set of four vertices.

Theorem 11. $(K_n, \mathcal{C}^{(3)}(K_n), \mathbf{R}^{(4)})$ *is a tog foundation. Its togs are precisely the unitogs.*

Proof. We have $\mathcal{D} = \mathcal{C}^{(3)}(K_n)$. In Theorem 10 let $\mathcal{D}_0$ consist of all triangles at a fixed vertex v, let $\mathcal{D}_1 = \mathcal{D}$, and let $\mathbf{R} = \mathbf{R}^{(4)}$. ■

Thus (by Theorem 9) unitogs correspond bijectively to switching classes of signed complete graphs. This, of course, is well known.

In our next example the base graph is the complete bipartite graph K_{lm} with left node set X, of size l, and right node set Y, of size m. Let

$$\mathbf{R}^{(5)} = \{\mathcal{C}^{(4)}(G) : G \text{ is a subgraph of } K_{lm} \text{ isomorphic to } K_{2,3} \text{ or } K_{3,2}\}.$$

Theorem 12. *$(K_{lm}, \mathcal{C}^{(4)}(K_{lm}), \mathbf{R}^{(5)})$ is a tog foundation. Its togs are the sets $\mathcal{T} \subseteq \mathcal{C}^{(4)}(K_{lm})$ having an even number of members in each $K_{2,3}$ and each $K_{3,2}$ subgraph of K_{lm}.*

Proof. Fix $a \in X$ and $b \in Y$. In Theorem 10 let $\mathcal{D} = \mathcal{C}^{(4)}(K_{lm})$ and $\mathbf{R} = \mathbf{R}^{(5)}$. Let $\mathcal{D}_0$ consist of all quadrilaterals meeting a and b; let $\mathcal{D}_1$ consist of all meeting a or b; and let $\mathcal{D}_2 = \mathcal{D}$. ■

By Theorem 9 these *bipartite togs* (for short, *bitogs*) correspond one-for-one to switching classes of signed complete bipartite graphs.

Other complete multipartite graphs lead to togs consisting of triangles, thus resembling unitogs. Let $K_{n_1 n_2 \cdots n_r}$ be the complete r-partite graph on node sets $N_1, N_2, \cdots, N_r$ of orders $n_1, n_2, \cdots, n_r$. For $r \geq 4$, let

$$\mathbf{R}^{(4)} = \{\mathcal{C}^{(3)}(G) : G \text{ is an induced subgraph isomorphic to } K_4\}.$$

For $r = 3$, let

$$\mathbf{R}^{(6)} = \{\mathcal{C}^{(3)}(G) : G \text{ is an induced subgraph isomorphic to } K_{2,2,2}\}.$$

Theorem 13. *$(K_{n_1 n_2 n_3}, \mathcal{C}^{(3)}(K_{n_1 n_2 n_3}), \mathbf{R}^{(6)})$ is a tog foundation.*

Theorem 14. *For $r \geq 4$, $(K_{n_1 \cdots n_r}, \mathcal{C}^{(3)}(K_{n_1 \cdots n_r}), \mathbf{R}^{(4)})$ is a tog foundation.*

Proofs. Let $a_i \in N_i$ be fixed elements for $i = 1, 2, 3$. Let $\mathcal{D}_0$ consist of all triangles which contain a_1, or contain a_2 and meet N_1, or contain a_3 and meet N_1 and N_2. It is easy to see that $\mathcal{D}_0$ is independent and has the cardinality of a basis for $\mathcal{C}(K_{n_1 \cdots n_r})$; hence it is a basis.

Suppose $r = 3$. Then each triangle not meeting a_1, a_2, or a_3 lies in the $\mathbf{R}^{(6)}$-closure of $\mathcal{D}_0$. So Theorem 10 applies with $k = 1$.

Suppose $r \geq 4$. Each triangle not meeting N_1 is in the $\mathbf{R}^{(4)}$-closure of $\mathcal{D}_0$. Let $\mathcal{D}_1$ consist of $\mathcal{D}_0$ with all such triangles adjoined. Each triangle meeting N_1 but not N_2 is in the $\mathbf{R}^{(4)}$-closure of $\mathcal{D}_1$. Let $\mathcal{D}_2$ be $\mathcal{D}_1$ with those triangles adjoined. Let $\mathcal{D}_3$ be $\mathcal{D}_2$ together with all triangles meeting N_1 and N_2 but not N_3. Finally, let $\mathcal{D}_4$ be $\mathcal{D}_3$ with all remaining triangles meeting N_1, N_2, and N_3 added on. Since every $\mathcal{D}_i$ lies in the $\mathbf{R}^{(4)}$-closure of $\mathcal{D}_{i-1}$, Theorem 10 applies with $k = 4$. ■

The togs based on the foundations in Theorems 13 and 14 we call *multipartite togs*. Those with $r \geq 4$ are quite similar to unitogs; in fact they are a direct generalization since setting all $n_i = 1$ gives unitogs. Tripartite togs, on the other hand, are unexpectedly different and complicated.

A tog can be based on a sparser graph than a complete multipartite one. The *complete circular multipartite graph* $\Gamma = C_r[N_1, N_2, \cdots, N_r]$ has node set $N = N_1 \cup \cdots \cup N_r$, where the N_i are disjoint and nonempty, and edge set $E = \{x_{i-1}x_i : i = 1, 2, \cdots, r\}$, where by convention subscripts indicate part membership (e.g., $x_i \in N_i$) and are taken modulo r. We call a polygon of the form $x_1 x_2 \cdots x_r$ *whole*. Let $\mathcal{C}^*(\Gamma)$ be the class of whole polygons and let $\mathbf{R}^*$ consist of all $\mathcal{C}^*(G)$ where G is a subgraph of Γ induced by a node set $\{x_1, \cdots, x_r, y_i, y_j\}$ with $y_i \neq x_i$, $y_j \neq x_j$, and $j \not\equiv i, i \pm 1 \pmod r$.

Theorem 15. *Let* $\Gamma = C_r[N_1, \cdots, N_r]$*, where* $r \geq 4$*. Then* $(\Gamma, \mathcal{C}^*(\Gamma), \mathbf{R}^*)$ *is a tog foundation.*

We call the togs on this foundation *circular togs.*

Proof. We identify whole polygons with elements of $N_1 \times \cdots \times N_r$. Let $a_1, \cdots, a_r$ be fixed elements and $A = a_1 a_2 \cdots a_r$. For $X \in N_1 \times \cdots \times N_r$ we define the *deviation* $d(X)$ to be $|X \backslash A|$ if the A-elements in X appear in one consecutive string, $|X \backslash A| + 1$ otherwise. The largest possible deviation is r; the only X with this deviation are those disjoint from A. Let

$$\mathcal{D}_k = \{X \in N_1 \times \cdots \times N_r : d(X) \leq k + 2\}.$$

We need to check that $\mathcal{D}_0$ is a cycle basis. Let $n_i = |N_i|$. $\mathcal{D}_0$ has the size of a cycle basis, namely $|E| - (n-1) = \sum n_i n_{i+1} - n + 1$, because it has 1 element of the form $a_1 \cdots a_r$, $\sum (n_i - 1)$ of the form $a_1 \cdots x_i \cdots a_r$ (one $x_i \notin A$), and $\sum (n_i - 1)(n_{i+1} - 1)$ of the form $a_1 \cdots x_i x_{i+1} \cdots a_r$ (two adjacent nodes not in A). To see $\mathcal{D}_0$ is independent, suppose it had a linear relation. No $a_1 \cdots x_i x_{i+1} \cdots a_r$ can participate in the relation, for the edge $x_i x_{i+1}$ belongs to no other element of $\mathcal{D}_0$. Then no $a_1 \cdots x_i \cdots a_r$ can participate, for $x_i a_{i+1}$ belongs to no other polygon in the relation. That leaves only A to participate, but $A \neq 0$.

We show that $\mathcal{D}_k$ lies in the $\mathbf{R}^*$-closure of $\mathcal{D}_{k-1}$ when $1 \leq k \leq r - 3$. Suppose $d(X) = k + 2$. If X contains a substring $a_{i-1} y_i a_{i+1}$ (where $y_i \notin A$), then it contains also a substring of the form $x_{j-1} y_j a_{j+1}$ (where $y_j \notin A$) such that $j \neq i$, because $|X \backslash A| \geq k + 1 \geq 2$. On the other hand if X contains a substring $y_{i-1} y_i a_{i+1}$ (where $y_{i-1}, y_i \notin A$), then it contains a substring $a_{j-1} y_j x_{j+1}$ (where $y_j \notin A$) such that $j \neq i - 1$; for otherwise $d(X)$ would be 2. In either case let

$$X' = X \backslash \{y_i\} \cup \{a_i\},$$
$$X'' = X \backslash \{y_j\} \cup \{a_j\},$$
$$X''' = X \backslash \{y_i, y_j\} \cup \{a_i, a_j\}.$$

Now X', X'', $X''' \in \mathcal{D}_{k-1}$ and $\{X, X', X'', X'''\} \in \mathbf{R}^*$. And any X with $d(X) = k + 2$ falls into one or both cases, for $k < r - 2$. Consequently indeed the $\mathbf{R}^*$-closure of $\mathcal{D}_{i-1}$ contains $\mathcal{D}_i$; moreover, in the generating relators we can take $j \not\equiv i$, $i \pm 1 \pmod{r}$.

We can obtain $\mathcal{D}_{r-2}$ similarly from $\mathcal{D}_{r-3}$ if we take $i = 2$ and $j = r$. Since $r \geq 4$, we have $j \not\equiv i$, $i - 1$.

Now by Theorem 10 we have the result. ∎

The *Hamming graph* (or *lattice graph*) $H_d(q)$ of dimension d on q symbols has for its nodes the ordered d-tuples $x = (x_1, x_2, \cdots, x_d)$ with $x_i \in \{0, 1, \cdots, q-1\}$, two nodes being adjacent when they differ in precisely one coordinate. $H_d(2)$, for instance, is the graph of the d-dimensional hypercube. A *δ-flat* $f_S = f_S(y_1, \cdots, y_{d-\delta})$ in $H_d(q)$, where $S \subseteq \{1, 2, \cdots, d\}$ and $|S| = \delta$, is the induced subgraph that has for its nodes those x whose coordinates x_l for $l \notin S$, arranged in order of increasing subscript, are respectively $y_1, \cdots, y_{d-\delta}$. In particular a 1-flat or *line* $l_i = f_{\{i\}}(y_1, \cdots, y_{d-1})$ is isomorphic to K_q and its nodes are naturally labelled $0, 1, \cdots, q-1$; a *plane* π_{ij} is a 2-flat $f_{\{i,j\}}$; a *space* s_{ijk} is a 3-flat $f_{\{i,j,k\}}$. Any triangle or K_4 subgraph of $H_d(q)$ is necessarily collinear. A *square* is an induced subgraph on the four coplanar points, say in a plane π_{ij}, given by $x_i = a_i$ or b_i and $x_j = a_j$ or b_j. A *cube* is an induced subgraph on eight cospatial points, say in a space s_{ijk}, given by $x_i = a_i$ or b_i, $x_j = a_j$ or b_j, and $x_k = a_k$ or b_k; it is an $H_3(2)$.

A *prism* is an induced subgraph on six coplanar points, say in π_{ij}, given by $x_i = a_1$ or b_1 and $x_j = a_2$ or b_2 or c_2, or vice versa (so there are two possible orientations for a prism, "horizontal" or "vertical").

Let $\mathcal{S}(H_d(q))$ be the class of squares of $H_d(q)$ and let

$$\begin{aligned}
\mathbf{R}^{(4)} &= \{\mathcal{C}^{(3)}(\Gamma) : \Gamma \text{ is a } K_4 \text{ subgraph of } H_d(q)\},\\
\mathbf{R}^{(8)} &= \{\mathcal{S}(\Gamma) : \Gamma \text{ is a cube in } H_d(q)\},\\
\mathbf{R}^{(6)} &= \{\mathcal{C}^{(3)}(\Gamma) \cup \mathcal{C}^{(4)}(\Gamma) : \Gamma \text{ is a prism in } H_d(q)\}.
\end{aligned}$$

Theorem 16. *Let $d \geq 1$, $q \geq 2$, and let $\mathcal{D} = \mathcal{C}^{(3)}(H_d(q)) \cup \mathcal{S}(H_d(q))$ and $\mathbf{R} = \mathbf{R}^{(4)} \cup \mathbf{R}^{(6)} \cup \mathbf{R}^{(8)}$. Then $(H_d(q), \mathcal{D}, \mathbf{R})$ is a tog foundation.*

We call the togs on this foundation *Hamming togs*. They generalize the togs on $H_d(2)$ and $H_2(q)$ found by Cameron and Wells (see [**2**, (7.7) and (7.8)]).

Proof. We apply Theorem 10 with $k = 1 + d$. If S is a square, its *height* is 2 less the number of a_i, b_i, a_j, b_j which equal zero. A plane $\pi_{ij}(y_1, \cdots, y_{d-2})$ is *l-initially zero* if $y_h = 0$ for $h < \min(i, l)$, where we assume $i < j$. Let $\mathcal{S}_{h,0}$ be the set of squares S of height h lying in d-initially zero planes. Let $\mathcal{C}_0^{(3)} = \{C \in \mathcal{C}^{(3)}(H_d(q)) : C$ contains the 0-labelled node in its line$\}$. In Theorem 10 let

$$\begin{aligned}
\mathcal{D}_0 &= \mathcal{C}_0^{(3)} \cup \mathcal{S}_{0,0},\\
\mathcal{D}_{h+1} &= \mathcal{C}^{(3)}(H_d(q)) \cup \mathcal{S}_{h,0} \quad \text{for} \quad h = 0, 1, 2,\\
\mathcal{D}_{3+m} &= \mathcal{C}^{(3)}(H_d(q)) \cup \{S \in \mathcal{S}(H_d(q)) : S \text{ lies in a } (d-2-m)\text{-initially zero plane}\}
\end{aligned}$$

for $m = 0, 1, \cdots, d-2$.

A cycle basis has cardinality $\frac{1}{2}d(q-1)q^d - q^d + 1$, since $H_d(q)$ is regular of degree $d(q-1)$ and has q^d nodes. Because there are dq^{d-1} lines, $|\mathcal{C}_0^{(3)}| = dq^{d-1}\binom{q-1}{2}$. The number of squares of height 0 in a particular plane is $(q-1)^2$. The number in all d-initially zero planes π_{ij} with a fixed value of i (where we take $i < j$) is $(d-i)q^{d-i-1}$. Therefore,

$$|\mathcal{S}_{0,0}| = (q-1)^2 \sum_{i=1}^{d-1} (d-i) q^{d-i-1} = (q-1)^2 f'(q),$$

where $f(q) = \sum_{0}^{d-1} q^h = (q^d - 1)/(q-1)$. Consequently,

$$|\mathcal{S}_{0,0}| = (q-1)dq^{d-1} - (q^d - 1),$$

so

$$\begin{aligned}
|\mathcal{D}_0| &= |\mathcal{C}_0^{(3)}| + |\mathcal{S}_{0,0}|\\
&= \frac{1}{2}dq^{d-1}(q-1)(q-2) + dq^{d-1}(q-1) - q^d + 1\\
&= \frac{1}{2}dq^{d-1}(q-1)q - q^d + 1,
\end{aligned}$$

which is the cardinality of a cycle basis.

Because of the relators $\mathcal{C}^{(3)}(\Gamma)$ where Γ is a K_4 subgraph of $H_d(q)$, the $\mathbf{R}$-closure of $\mathcal{D}_0$ includes every triangle. Hence it contains $\mathcal{D}_1$.

For $h = 1, 2$, the $\mathbf{R}$-closure of $\mathcal{D}_{h-1}$ contains $\mathcal{D}_h$, by prism relators in each d-initially zero plane.

For $m = 1, 2, \cdots, d-2$, the $\mathbf{R}$-closure of $\mathcal{D}_{2+m}$ contains $\mathcal{D}_{3+m}$ because of the cube relators

$$\{S_h(y_h), S_h(0), S_i(a_i), S_i(b_i), S_j(a_j), S_j(b_j)\},$$

where $d-2-m = h < i < j$, the plane $\pi_{ij}(y_1, \ldots, y_{d-2})$ is h-initially zero, and $S_l(\alpha)$ for $l \in \{h, i, j\}$ denotes the square given by $x_l = \alpha$ in the cube $f_{\{h,i,j\}}(y_1, \cdots, y_{h-1}, y_{h+1}, \cdots, y_{d-2})$.

Consequently, each $\mathcal{D}_i$ lies in the $\mathbf{R}$-closure of $\mathcal{D}_{i-1}$. It remains to show that $\mathcal{D}_{1+d}$ spans the cycle space.

Lemma 17. *$\mathcal{D}$ spans the cycle space of $H_d(q)$.*

Proof of Lemma. If not, let C be a cycle of minimum length not in $\langle \mathcal{D} \rangle$. Evidently C is a polygon. It cannot have three consecutive collinear nodes, because then summing it with a triangle would reduce its length. Suppose C had three consecutive nodes wxy, and let x' be the fourth node of the square they determine. If x' were a node of C, then summing the square $wxyx'$ into C would either shorten C (if $x'wxy$ or $wxyx'$ were consecutive in C) or form a nonpolygonal cycle, which would contradict the minimality of C.

Let $m(C)$ be the maximum number of nodes of C in a hyperplane $x_i = \alpha$ that does not contain C, and choose C (of minimum length not in $\langle \mathcal{D} \rangle$) to maximize $m(C)$. Letting $x_i = \alpha$ be a hyperplane containing $m(C)$ nodes of C, suppose xyz are consecutive nodes with $x_i = \alpha$ and $y_i \neq \alpha$. If $z_i = \alpha$, then $x = z$, an impossible case. Thus, $z_i \neq \alpha$. Let $xyzy'$ be the square containing x, y, z. (The square exists because C has no three consecutive collinear nodes.) Adding this square into C replaces y by y' and consequently increases $m(C)$ by one. But C already had the maximum value of $m(C)$. This is a contradiction. It follows that $\mathcal{D}$ spans the cycle space. ∎

Consequently, Theorem 10 applies to $(H_d(q), \mathcal{D}, \mathbf{R})$, proving Theorem 16. ∎

The Hamming graph is the cross product of d copies of K_q. There is a notion of product of tog foundations under which the Hamming foundation is the product of unitog foundations. Since the definition and proof are best stated in terms of tensor products of associated exact sequences, we omit them here.

The *Johnson graph* $J_m(r)$ has for node set the class $P^{(r)}(V)$ of r-element subsets of an m-element set V. Two nodes X, Y are adjacent when their overlap has order $r-1$, in other words when $|X \backslash Y| = 1$. $J_m(1)$ is K_m; $J_m(2)$ is $L(K_m)$, the line graph of K_m (also known as the *triangular graph*). We assume $m > r > 0$ for a Johnson graph.

A *minor* of $J_m(r)$ is the induced subgraph on a node set of the form $\{X \in N(J_m(r)) : U \subseteq X \subseteq W\}$ where U and W are fixed sets with $U \subseteq W \subseteq V$ and $|U| < r < |W|$. We say the minor has *type* $J_{|W \backslash U|}(r - |U|)$.

For notational convenience we abbreviate $A \cup \{b_1, b_2, \cdots, b_k\}$, where $A \subseteq V$ and all $b_i \in V \backslash A$, by the notation $Ab_1b_2 \cdots b_k$.

Theorem 18. *Let $m > r > 0$. Then $(J_m(r), \mathcal{C}^{(3)}(J_m(r)), \mathbf{R})$ is a tog foundation, where $\mathbf{R} = \{\mathcal{C}^{(3)}(\Gamma) : \Gamma$ is a minor of $J_m(r)$ of type $J_4(1)$, $J_4(2)$, or $J_4(3)\}$.*

We call the togs on this foundation *Johnson togs*. For $r = 1$ they are the unitogs, since $J_m(1)$ has minors of type $J_4(1) = K_4$ only. Consequently, we need to prove the theorem only for $r \geq 2$. We do so by induction on r. But first, we establish that the triangles span the cycle space.

Lemma 19. *$\mathcal{C}^{(3)}(J_m(r))$ spans the cycle space of $J_m(r)$.*

Proof. Suppose it did not. Then there would be a polygon C of minimum length not spanned by the triangles. C has length $l = 2k$ or $2k+1$ (where $k \geq 2$), according as it is even or odd. Let its nodes, in order, be $X_0 X_1 X_2 \cdots X_l$. (We take subscripts modulo l, so $X_l = X_0$.) Minimality of C implies that

$$(*) \qquad d(X_i, X_{i+j}) = j \quad \text{for} \quad 1 \leq j \leq k,$$

where $d(X,Y) = |X \backslash Y|$ is the distance between X and Y in the graph. In particular, $k \leq r$. Property $(*)$ implies that

$$X_i = W x_{i+1} x_{i+2} \cdots x_{i+k} \quad \text{for} \quad i = 0, 1, 2, \cdots, k,$$

$$X_{k+1} = \begin{cases} W x_1 x_{k+2} x_{k+3} \cdots x_{2k} & \text{if } l = 2k, \\ W x_{k+2} x_{k+3} \cdots x_{2k} x_{2k+1} & \text{if } l = 2k+1. \end{cases}$$

Let $X_1' = W x_2 \cdots x_k x_{k+2}$ and let C' be C with X_1 replaced by X_1'. Then C' is a polygon with length l, but $d(X_1', X_{k+1}) = k-1$. Therefore C' is triangle-generated. Moreover, the quadrilateral $Q = X_0 X_1 X_2 X_1' X_0$ has $X_1 X_1'$ as a chord so it is the sum of triangles. It follows that C is a sum of triangles, since $C = Q + C'$ (as an edge set). This is a contradiction. ■

Proof of Theorem. First, some notation. We let H, I, J denote point sets of sizes $r-3$, $r-2$, $r-1$ respectively. Any edge in the Johnson graph has the form $\{Ji, Jj\}$; we call this edge e_J^{ij}. There are two kinds of triangles in $J_m(r)$, namely

$$\Delta_I^{ijk} = \{e_{Ii}^{jk}, e_{Ij}^{ik}, e_{Ik}^{ij}\}$$

and

$$\Delta_J^{ijk} = \{e_J^{ij}, e_J^{ik}, e_J^{jk}\}.$$

The latter are analogs of the vertex triangles in $L(K_m)$; we call them *verticial* triangles. The former are analogous to the triangles in $L(K_m)$ arising from triangles in K_m; we call them *essential.* When we name a triangle of either type, we imply that $i < j < k$ and that i, j, k are not in I or J (as appropriate). The three kinds of relators in $\mathbf{R}$ are those of type $J_4(1)$, namely

$$(1) \qquad \{\Delta_J^{ijk}, \Delta_J^{ijl}, \Delta_J^{ikl}, \Delta_J^{jkl}\},$$

those of type $J_4(2)$, namely

$$(2) \qquad \{\Delta_I^{ijk}, \Delta_I^{ijl}, \Delta_I^{ikl}, \Delta_I^{jkl}, \Delta_{Ii}^{jkl}, \Delta_{Ij}^{ikl}, \Delta_{Ik}^{ijl}, \Delta_{Il}^{ijk}\},$$

and those of type $J_4(3)$, namely

$$(3) \qquad \{\Delta_{Hi}^{jkl}, \Delta_{Hj}^{ikl}, \Delta_{Hk}^{ijl}, \Delta_{Hl}^{ijk}\}.$$

We leave it to the reader to verify by taking the appropriate 4-point minor that each of these does indeed belong to $\mathbf{R}(\mathcal{C}^{(3)}(J_m(r)))$. The third type appears only when $r \geq 3$.

The size of a cycle basis in $J_m(r)$ is

$$b_m(r) = \frac{r(m-r)}{2}\binom{m}{r} - \binom{m}{r} + 1$$

since there are $\binom{m}{r}$ nodes and each one has degree $r(m-r)$.

Proof of the case $r=2$. Let $\mathcal{D}_0(2)$ consist of all verticial triangles Δ_{i+1}^{ijk} (where $j,k \neq i+1$, of course) and all essential triangles $\Delta_\emptyset^{1jk}$. Then

$$|\mathcal{D}_0(2)| = m\binom{m-2}{2} + \binom{m-1}{2} = b_m(2).$$

Let $\mathcal{D}_1(2)$ consist of $\mathcal{D}_0(2)$ and all verticial triangles. Then $\mathcal{D}_1(2)$ lies in the $\mathbf{R}$-closure of $\mathcal{D}_0(2)$ because of (1). By (2) with $i=1$, the $\mathbf{R}$-closure of $\mathcal{D}_1(2)$ contains every essential triangle $\Delta_\emptyset^{jkl}$, in other words it contains $\mathcal{D}_2(2) = \mathcal{C}^{(3)}(J_m(2))$. Now we apply Theorem 10 with $k=2$.

Proof of the case $r>2$. Let $J_m(r)/1$ be the minor induced by the nodes X that contain 1; it is of type $J_{m-1}(r-1)$. Let $\mathcal{D}_0(r)$ consist of the cycle basis $\mathcal{D}_0(r-1)$ for $J_m(r)/1$ together with the essential triangles Δ_I^{jkl} where $1 \notin I$ and $j<k<l>\max I$ and the verticial triangles Δ_J^{1kl} where $1 \notin J$ and $1<k<l>\max J$. The size of $\mathcal{D}_0(r)$ is therefore

$$b_{m-1}(r-1) + \sum_{l=r+1}^{m}\binom{l-2}{r-2}\binom{l-r+1}{2} + \sum_{l=r+2}^{m}\binom{l-2}{r-1}\binom{l-r-1}{1} = b_m(r).$$

By the previous step of the induction we know that $\mathcal{D}_p(r-1) = \mathcal{C}^{(3)}(J_m(r)/1)$ for $p = 3r-7$. Let $\mathcal{D}_q(r) = \mathcal{D}_0(r) \cup \mathcal{D}_q(r-1)$ for $q \leq p$. Our proof starts with $\mathcal{D}_p(r)$.

Let Δ_I^{ijk} satisfy $l = \max I > k$. Then, setting $H = I\backslash\{l\}$, we see $\Delta_I^{ijk} \in \mathbf{R}$-closure of $\mathcal{D}_p(r)$ because of (3). Let $\mathcal{D}_{p+1}(r) = \mathcal{D}_p(r) \cup \{\text{essential triangles}\}$.

Let Δ_J^{1jk} satisfy $l = \max J > k$. Then, setting $I = J\backslash\{l\}$, we see that $\Delta_J^{1jk} \in \mathbf{R}$-closure of $\mathcal{D}_{p+1}(r)$ because of (2) with $i=1$. Let $\mathcal{D}_{p+2}(r) = \mathcal{D}_{p+1}(r) \cup \{\text{verticial triangles } \Delta_J^{1jk}\}$.

Finally, $\mathcal{D}_{p+3}(r) = \mathcal{C}^{(3)}(J_m(r))$ lies in the $\mathbf{R}$-closure of $\mathcal{D}_{p+2}(r)$, for $\Delta_J^{jkl} \in \mathbf{R}$-closure of $\mathcal{D}_{p+2}(r)$ by (1) with $i=1$. Therefore Theorem 10 applies with $k = p+3$ and $\mathcal{D}_q = \mathcal{D}_q(r)$ for $q \leq k$. ■

One can generalize Theorem 18 for $r=2$ to a construction of a tog foundation on the line graph $L(\Gamma)$ of a graph Γ supporting a given tog foundation. It would be desirable to extend this result to all r; probably that would depend on finding a suitable definition of togs on hypergraphs.

5. Residual determinacy

Cameron in [**1**] presents unitogs as classes $\mathcal{T}$ of triples in a set N such that the *residuum* $\mathcal{T}/p = \{T \in \mathcal{T} : p \in T\}$ on any point p determines the whole class in a natural way. (If we regard $\mathcal{T}/p$ as a graph on vertex set $N\backslash\{p\}$, then a unitog is an extension, in the sense of design theory, of a graph by a point such that the graph determines the whole extension.) Specifically, there is a set S of integers such that whether a triple T not on p is in $\mathcal{T}$ is determined by the requirement that $|\mathcal{T}(T \cup \{p\})| \in S$. (By $\mathcal{T}(U)$ we mean $\{T \in \mathcal{T} : T \subseteq U\}$.) Evidently S must satisfy $|i-j| \neq 1$ for $i, j \in S$. If $S \subseteq \{0, 2, 4\}$, $\mathcal{T}$ is a unitog. One can show, as Cameron remarks, that there are only two other examples: $\mathcal{T} =$ all triples on a certain point, and the complementary class $\mathcal{T}^C$.

If one could generalize this observation to other situations, showing that a class obeying a rule of residual determinacy (such as the numerical rule above) is a tog or is one of a small list of exceptions, then choosing to base a generalization of unitogs on signed graphs would seem less arbitrary. We explore this idea here in two directions: we show that the weakest possible rule of global residual determination gives no new examples in the case of classes of triples, and we study other kinds of togs from the same point of view. Several kinds do have nice characterizations: bitogs, for example, are nearly characterized by residual determinacy. On the other hand, a rule of residual determination like Cameron's rule for triples fails completely to characterize quadripartite togs. Thus one cannot expect a completely general characterization theorem. Unfortunately, we have not even a fairly general theorem. Here we present the evidence of examples and a few remarks on what one might hope to discover.

Let $\mathcal{D}$ be a fixed class of subsets of a set N. We wish to find subsets $\mathcal{T} \subseteq \mathcal{D}$ such that, for each $p \in N$, the residuum $\mathcal{T}/p$ determines $\mathcal{T}$ in a natural way. The most natural way seems to be to let the residual structure in T determine whether or not T belongs to $\mathcal{T}$. If this is the case we call $\mathcal{T}$ *weakly residually determined*. Formally the definition is that, for each p, there is a class $\hat{\mathcal{S}}(p)$ such that a set $T \in \mathcal{D}(N\backslash\{p\})$ belongs to $\mathcal{T}$ if and only if $(\mathcal{T}/p)(T)$ is isomorphic to an element of $\hat{\mathcal{S}}(p)$. Note that $\hat{\mathcal{S}}(p)$ may vary with p; if it does not, we call the determinacy *uniform*.

A more restrictive notion is that, for each p, there is a class $\mathcal{S}(p)$ such that a set $T \in \mathcal{D}(N\backslash\{p\})$ is in $\mathcal{T}$ if and only if $\mathcal{T}(T \cup \{p\})$ is isomorphic to an element of $\mathcal{S}(p)$. Such a class $\mathcal{T}$ we call *strongly residually determined*. One result one might hope for is that weak determinacy implies strong determinacy. We shall see that this is true for systems $\mathcal{D}$ related to unitogs and bitogs.

These are the broadest versions of residual determinacy. In most of our examples we adopt a simpler rule. Call $\mathcal{T}$ *weakly* [or, *strongly*] *numerically determined* if for each p there is a set $\hat{S}(p)$ [or, $S(p)$] of integers such that a set $T \in \mathcal{D}(N\backslash\{p\})$ is in $\mathcal{T}$ if and only if $|(\mathcal{T}/p)(T)| \in \hat{S}(p)$ [or respectively, $|\mathcal{T}(T \cup \{p\})| \in S(p)$]. For strong numerical determinacy to be well defined $S(p)$ must not contain two consecutive integers. We call numerical determinacy *uniform* if $\hat{S}(p)$ [or, $S(p)$] is independent of p. Any tog is uniformly determined with $S(p) \subseteq \{0, 2, 4, \cdots\}$. (This assumes the tog can be described as a class of sets. The definitions can be generalized to classes of cycles in a graph.)

In the first example we take $\mathcal{D} = P^{(3)}(N)$, the class of 3-element subsets of N. Then residual and numerical determinacy are the same. We strengthen the observation in [**1**] that a class with uniform, strong residual determinacy is a unitog or one of two exceptions.

Theorem 20. *Let $\mathcal{T}$ be a class of (unordered) triples from a set N. If $\mathcal{T}$ is weakly residually determined, then it is a unitog or it is the class of triples on a fixed point of N or the complement of such a class.*

Proof. We may assume, complementing $\mathcal{T}$ if necessary, that there is a quadruple $W = pq_1q_2q_3$ in which $\mathcal{T}(W) = \{q_1q_2q_3\}$. (Otherwise $\mathcal{T}$ is a unitog and we are done.)

There is only one way to extend to a fifth point r, namely with $\mathcal{T}(W \cup \{r\}) = P^{(3)}(q_1q_2q_3r)$. To show this we examine the number k of triples on r in $\mathcal{T}(q_1q_2q_3r)$. From $\mathcal{T}(W)$ we know $0 \in \hat{S}(p)$ and $1 \notin \hat{S}(q_i)$.

If $k = 0$, we may take prq_1 and prq_2 either both in $\mathcal{T}$ or both in $\mathcal{T}^c$. The former contradicts $1 \notin \hat{S}(q_1)$, the latter contradicts $0 \in \hat{S}(p)$.

If $k = 1$, say $rq_2q_3 \in \mathcal{T}$. This contradicts $1 \notin \hat{S}(q_1)$.

If $k = 2$, say q_1q_2r and $q_1q_3r \in \mathcal{T}$. Then from $\mathcal{T}(q_1q_2q_3r)$ we have $2 \in \hat{S}(q_2)$. From $\mathcal{T}(pq_2q_3r)$ we have pq_2r or $pq_3r \in \mathcal{T}$, say the former. Then $2 \in \hat{S}(q_2)$ applied to $\mathcal{T}(pq_1q_2r)$ shows $pq_1r \in \mathcal{T}$. Since $2 \in \hat{S}(r)$, $\mathcal{T}(pq_2q_3r)$ gives $pq_3r \notin \mathcal{T}$. Then $\mathcal{T}(pq_1q_3r)$ contradicts $1 \notin \hat{S}(q_3)$.

If $k = 3$, then $P^{(3)}(q_1q_2q_3r) \subseteq \mathcal{T}$. We have to decide which triples pq_ir are in $\mathcal{T}$. If (say) $pq_2r \in \mathcal{T}$, then $3 \in \hat{S}(r)$ applied to $\mathcal{T}(pq_1q_2r)$ implies $pq_1r \notin \mathcal{T}$. If $pq_1r \notin \mathcal{T}$, then $1 \notin \hat{S}(q_1)$ implies no $pq_ir \in \mathcal{T}$. Thus $\mathcal{T} = P^{(3)}(q_1q_2q_3r)$.

The proof can be completed by induction on $|N|$. ■

In the second example we take disjoint sets X and Y and $\mathcal{D} = \mathcal{D}(X,Y) = \{T \subseteq X \cup Y : |T \cap X| = |T \cap Y| = 2\}$. Residual and numerical determinacy coincide.

Theorem 21. *Let X and Y be disjoint, nonempty sets and let $\mathcal{T} \subseteq \mathcal{D}(X,Y)$. If $\mathcal{T}$ is weakly residually determined, then it or its complement is a bitog or is constructed by partitioning $X = X_1 \cup X_2$ (or, $Y = Y_1 \cup Y_2$) and choosing all $T \in \mathcal{D}(X,Y)$ which meet both X_1 and X_2 (or, Y_1 and Y_2).*

The first step of the proof is conveniently stated as a lemma.

Lemma 22. *If $\mathcal{T}$ is weakly residually determined, it is strongly numerically determined with $S(x) =$ constant $S(X)$ for $x \in X$ and $S(y) =$ constant $S(Y)$ for $y \in Y$.*

Proof. We write x for elements of X and y for elements of Y. We may assume $|X| \geq 3$, $|Y| \geq 2$. Let $S^1(x) = \{m :$ for some $x_1x_2y_1y_2 \in \mathcal{T}$, exactly m of $xx_1y_1y_2$ and $xx_2y_1y_2$ are in $\mathcal{T}\}$. Let $S^0(x) =$ same but for $x_1x_2y_1y_2 \notin \mathcal{T}$. These sets are disjoint because of weak residual determinacy. Let $S^*(x) = S^0(x) \cup \{m+1 : m \in S^1(x)\}$. Then $S(x) \supseteq S^*(x)$. Define $S^1(y)$, etc., similarly.

We wish to prove that a set $S(X)$ exists for which all $S^*(x) \subseteq S(X)$; then we may let $S(x) = S(X)$. Also, that $S(Y)$ exists for which all $S^*(y) \subseteq S(Y)$.

Case 1. Let $|Y| = 2$. Let $T = \{xx' : xx' \cup Y \in \mathcal{T}\}$. This defines a graph on X. Fix $p \in X$ and let $Q = \{q \in X : pq \in T\}$, $R = \{r \in X : r \neq p,\ pr \notin T\}$. By weak numerical determinacy, Q is a clique or a coclique, so is R, and either all edges qr are in T or none is. Let us assume $|Q| \geq |R|$, so $|Q| \geq 2$.

Suppose Q is a clique and R is not empty. Then $qr \notin T$ because $2 \in S^1(q)$. Also, R is a clique, because $0 \in S^1(r)$ but if $r_1r_2 \notin T$ then $0 \in S^0(r_1)$, a contradiction.

Suppose Q is a coclique and R is not empty. Then $1 \in S^1(q)$ so $qr \in T$. The triple pqr shows that $2 \in S^0(q)$. This implies $r_1r_2 \notin T$. So R is a coclique.

These arguments leave us with six possibilities for T.

(a1) $T = K(X)$, the complete graph on X. In this case all $S^0(x) = \emptyset$, $S^1(x) = \{2\}$.

(a2) $T = K(X \backslash \{r\}) \cup K(r)$ for some $r \in X$. Here $S^0(x) = \{1\}$ if $x \neq r$ and $S^1(x) = \{0\}$ if $|X| = 3$, $\{0, 2\}$ if $|X| \geq 4$.

(a3) $T = K(P) \cup K(R)$ for some partition $X = P \cup R$ with $|P| \geq |R| \geq 2$. Then $S^0(x) = \{1\}$, $S^1(x) = \{0, 2\}$.

(b1) $T = K(X)^c$, the edgeless graph. Here all $S^0(X) = \{0\}$ and $S^1(x) = \emptyset$.

(b2) $T = K(X, r)$, the star centered at r, for some $r \in X$. In this case $S^0(x) = \{2\}$ if $|X| = 3$, $\{0, 2\}$ if $|X| \geq 4$, and $S^1(x) = \{1\}$ if $x \neq r$.

(b3) $T = K(Q, P)$, the complete bipartite graph, where Q, P partition X and $|P| \geq |Q| \geq 2$. Here $S^0(x) = \{0, 2\}$ and $S^1(x) = \{1\}$.

Evidently, in every case we can take $S(x) = S(X) = \{0, 2\}$ or $\{1, 3\}$. We can take $S(y) = S(Y)$ to be any set.

Case 2. Let $|Y| \geq 3$. For each pair $y_i, y_j \in Y$ we get a graph T_{ij} on X and sets $S^\epsilon_{ij}(x)$, $\epsilon = 0, 1$. We have $S_{ij}(x) \subseteq \{0, 2\}$ for all i, j, x if every T_{ij} is of type (a), all $S_{ij}(x) \subseteq \{1, 3\}$ if every T_{ij} is of type (b). The only case in which we cannot take all $S(x) = \{0, 2\}$ or all equal to $\{1, 3\}$ is that in which, say, T_{12} has type (a) and T_{ij} has type (b) for some $ij \neq 12$.

Suppose that is the case. Since $S^\epsilon(x)$ is the union of $S^\epsilon_{ij}(x)$ for all ij, we must have $S^1_{12}(x) \cap S^0_{ij}(x) = \emptyset$. Inspecting the list of possible T graphs shows that $T_{12} = K(X)^c$ or $T_{ij} = K(X)$. We must also have $S^0_{12}(x) \cap S^1_{ij}(x) = \emptyset$. Thus if $T_{12} = K(X)^c$, T_{ij} must be $K(X)$, while if $T_{ij} = K(X)$, then T_{12} must be $K(X)^c$.

We see that if all T_{ij} are not of the same type, (a) or (b), then each is $K(X)$ or its complement and we can take all $S(x) = \{0, 3\}$. That concludes Case 2.

We have shown that, if $|X| \geq 3$, we can take all $S(x) = S(X) = \{0, 2\}$, $\{1, 3\}$, or $\{0, 3\}$. By the same argument, when $|Y| \geq 3$ we can take all $S(y) = S(Y)$.

That completes the proof of the lemma. ■

Proof of Theorem. We may assume $|X|$, $|Y| \geq 3$. If $S(X) = S(Y) = \{0, 2\}$ or $\{1, 3\}$, then $\mathcal{T}$ is a bitog or bitog complement.

Suppose $S(X) = \{0, 3\}$. Then for each y and y', $\mathcal{T}_{yy'} = \{T \in \mathcal{T} : y, y' \in T\}$ is either $\emptyset$ or $\mathcal{D}_{yy'}$. If $S(Y) = \{0, 3\}$, then $\mathcal{T}_{xx'} = \emptyset$ or $\mathcal{D}_{xx'}$; it follows that $\mathcal{T} = \emptyset$ or $\mathcal{D}$. If $S(Y) = \{1, 3\}$, then Y partitions into parts Y_1 and Y_2 so that $\mathcal{T}_{y_1 y_2} = \emptyset$ and $\mathcal{T}_{yy'} = \mathcal{D}_{yy'}$ if y and y' are in the same part. The case $S(Y) = \{0, 2\}$ is complementary.

The last case, where $S(X) = \{1, 3\}$ and $S(Y) = \{0, 2\}$, cannot occur. Let $W = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and let $t_i = |\mathcal{T}(W \backslash \{x_i\})|$, $u_j = |\mathcal{T}(W \backslash \{y_j\})|$. Then $t_1 + t_2 + t_3 = |\mathcal{T}(W)| = u_1 + u_2 + u_3$. But the former sum is even and the latter is odd.

This concludes the proof. ■

A third variety of tog that is nearly determined by a residual rule is the multipartite togs on five or more parts. If $X_1, X_2, \cdots, X_r$ are disjoint sets, a subset of their union is *transverse* when it has no two elements in any one X_i.

Theorem 23. *Let $N = X_1 \cup X_2 \cup \cdots \cup X_r$ be a partition with $r \geq 5$. Let $\mathcal{T}$ be a class of transverse (unordered) triples which is weakly residually determined. Then $\mathcal{T}$ is a multipartite tog or else it or its complement consists of all transverse triples meeting some fixed set $W_1 \subseteq X_1$.*

Proof. This is a corollary of Theorem 20. ■

The next two examples are a new kind insofar as the residuum is taken over a set P which is a pair or triple of points. The rules we consider are uniform, strong, and numerical. Perhaps weaker rules could have been used. First we take up tripartite togs.

Theorem 24. *Let X_1, X_2, X_3 be disjoint sets with all $|X_i| \geq 3$. Let $\mathcal{T}$ be a class of transverse triples such that, for some set S of integers, no two consecutive, $|\mathcal{T}(T \cup P)| \in S$ for every two disjoint transverse triples T and P. Then $\mathcal{T}$ is a tripartite tog.*

Outline of Proof. We restrict so $X_1 = x_{11}x_{12}x_{13}$, $X_2 = x_{21}x_{22}x_{23}$, and $X_3 = x_{31}x_{32}$. In $\mathcal{T}(x_{11}x_{12}x_{13}x_{21}x_{22}x_{31}x_{32})$ let $r_i =$ the number of transverse triples on x_{1i}. Thus $r_1 + r_2$, $r_1 + r_3$, $r_2 + r_3 \in S$. If all r_i have the same parity, we may take $S = \{\text{evens}\}$; then $\mathcal{T}$ is a tripartite tog. If they do not, then $R = \{r_1, r_2, r_3\} = \{0, 3\}$ or $\{1, 4\}$. (The latter case is complementary to the former.) If, say, $r_1 = 0$ and $r_2 = 3$, then $|\mathcal{T}(x_{11}x_{12}y_2x_{23}x_{31}x_{32})|$, for $y_2 = x_{21}$ and x_{22}, differ by one. This is absurd. The conclusion follows. ■

Now we look at circular togs. A *complete transversal* of $X_1, \cdots, X_r$ is a set consisting of one element from each X_i. A *nonconsecutive pair* is a pair $x_i x_j$ where $j - i \not\equiv 0, \pm 1 \pmod r$. (We take x_i, y_i to be elements of X_i.)

Theorem 25. *Let $X_1, \cdots, X_r$ (where $r \geq 4$) partition N, with $|X_i|$ and $|X_j| \geq 3$ for some two nonconsecutive integers i, j (mod r). Let $\mathcal{T}$ be a set of complete transversals such that, for some set S of integers, no two of which are consecutive, we have $|\mathcal{T}(T \cup x_k x_l)| \in S$ for every disjoint complete transversal T and nonconsecutive pair $x_k x_l$. Then $\mathcal{T}$ is a circular tog.*

Outline of Proof. We can show that $S \not\subseteq \{1, 3\}$ by counting elements of $\mathcal{T}(W)$ where $W = T \cup x_i y_i x_j y_j$ with T a complete transversal and $x_i, y_i, x_j, y_j \notin T$.

The case $S = \{1, 4\}$ is converted to $S = \{0, 3\}$ by complementing $\mathcal{T}$, so we may assume $S = \{0, 3\}$. Let Γ be the bipartite graph on $X_i \cup X_j$ defined by choosing a complete transversal A of all X_k except X_i and X_j, and letting $x_i x_j \in \Gamma$ if $A x_i x_j \in \mathcal{T}$. Then Γ satisfies: in each quadruple $x_i y_i x_j y_j$, Γ has 0 or 3 edges. Fixing x_i, let Q be its neighborhood and $R = X_j \backslash Q$. Each $x'_i \neq x_i$ is adjacent to all but one $q \in Q$; it follows that $|Q| \leq 2$. From Γ restricted to a subset $x_i x'_i r r'$ we see that no x'_i is adjacent to any r if $|R| \geq 2$. Thus if $|R| \geq 2$ and $|Q| \geq 1$ we have a contradiction. It follows that either $Q = \emptyset$, or $|R| = 1$ and $|Q| = 2$. In the latter case every $x'_i \neq x_i$ is adjacent to all x_j, which leads to a contradiction. In the former case $\mathcal{T}(A \cup X_i \cup X_j) = \emptyset$ so we may take $S = \{0, 2, 4\}$. Thus we are done. ■

We have postponed considering quadripartite togs for a good reason: they do not behave nicely in regard to residual determination.

Example 26. Let X_1, X_2, X_3, X_4 be disjoint, nonempty sets. Let $\mathcal{T}$ be a set of transverse triples such that for some set S of integers, no two consecutive, $|\mathcal{T}(U)| \in S$ for every transverse quadruple U. If $S \subseteq \{0, 2, 4\}$, $\mathcal{T}$ is a quadripartite tog. But there are many examples where $S \not\subseteq \{0, 2, 4\}$ and they can be quite complicated.

To see this we reinterpret everything. Let $X = X_1 \times X_2 \times X_3 \times X_4$; we identify transverse quadruples with points $x = (x_1, \cdots, x_4) \in X$ and transverse triples with *lines*, that is sets of the form $\{x : \text{three } x_i \text{ are constant}\}$. $\mathcal{T}$ becomes a set of lines; the *degree* of each point x, which we define as the number of lines of $\mathcal{T}$ on which it lies, must belong to S. There are many ways to choose $\mathcal{T}$ so the set of degrees is $\{1\}$, $\{1, 3\}$, or $\{1, 4\}$, or by complementing $\mathcal{T}$, $\{3\}$ or $\{0, 3\}$. Here are some examples:

(a) Any set of disjoint lines that cover X. The degrees are all 1.
(b) Let $\emptyset \subset Y \subset X_4$. In each 3-flat $\{x : x_4 = \text{constant} \in Y\}$, take a covering set of disjoint lines. In each other 3-flat take all lines. This example has degrees 1 and 3.
(c) If all $|X_i| = q$, choose a set of q points, no two in a line, and take all lines through these points. The degrees are 1 and 4.
(d) Various combinations of (a)-(c).

Thus in several cases, but not all, a form of residual determination characterizes togs with at most a few exceptions that can be classified easily.

Problem. *Find a general theorem of this kind.*

This problem is admittedly vague. One reason is that I do not have a fully general definition either of a class $\mathcal{D}$ of potential elements of $\mathcal{T}$ or of numerical (or residual) determinacy. Another reason is that it is unclear, in view of Example 26, how broad a theorem could be. But any such result not only would help to justify the generalization of two-graphs to togs but also might suggest other generalizations. For instance, if we take $\mathcal{T}$ to consist of r-element subsets of N where $r \geq 4$, does numerical determinacy imply that $\mathcal{T}$ obeys a simple rule like that defining unitogs? If so, we might be able to interpret this rule in terms of signed hypergraphs and thereby deduce a definition of the latter which could be what we need to understand and generalize Johnson togs.

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