Transitive Closure and Transitive Reduction in Bidirected Graphs

Ouahiba Bessouf [a] Abdelkader Khelladi [a]

[a] *Faculté de Mathématiques, USTHB BP 32 El Alia, Bab-Ezzouar 16111, Alger, Algérie*

Thomas Zaslavsky [b]

[b] *Department of Mathematical Sciences, Binghamton University, Binghamton, NY 13902-6000, U.S.A.*

Abstract

In a bidirected graph an edge has a direction at each end, so bidirected graphs generalize directed graphs. We generalize the definitions of transitive closure and transitive reduction from directed graphs to bidirected graphs by introducing new notions of b-path and b-circuit that generalize directed paths and cycles. We show how transitive reduction is related to transitive closure and to the matroids of the signed graph corresponding to the bidirected graph.

Key words: bidirected graph, signed graph, matroid, transitive closure, transitive reduction
Mathematics Subject Classification 2010: Primary 05C22, Secondary 05C20, 05C38.

---

1 Introduction.

Bidirected graphs are a generalization of undirected and directed graphs. Harary defined in 1954 the notion of signed graphs. For any bidirected graph, we can associate a signed graph. Reciprocally, any signed graph can be associated to a bidirected graph. Transitive reduction in directed graphs was introduced by A. V. Aho. The aim of this paper is to extend the concept

Email addresses: obessouf@yahoo.fr (Ouahiba Bessouf), kader_khelladi@yahoo.fr (Abdelkader Khelladi), zaslav@math.binghamton.edu (Thomas Zaslavsky).

of transitive closure,mmq which is denoted $\mathrm{Ft}(G_\tau)$, and transitive reduction, which is denoted $\mathrm{Rt}(G_\tau)$, to bidirected graphs. We seek to find definitions of transitive closure and transitive reduction for bidirected graphs through which the classical concepts would be a special case. We establish for bidirected graphs some properties of these concepts and a duality relationship between transitive closure and transitive reduction.

2 Bidirected Graphs

We allow graphs to have loops and multiple edges. Given an undirected graph $G = (V, E)$, the set of *half-edges* of G is the set $\Phi(G)$ defined as follows:

$$\Phi(G) = \{(e, x) \in E \times V : e \text{ is incident with } x\}.$$

Thus, each edge e with ends x and y is represented by its two half-edges (e, x) and (e, y). For a loop the notation does not distinguish between its two half-edges. There is no very good notation for the two half-edges of a loop, but we believe the reader will be able to interpret our formulas for loops.

A *chain* is a sequence of vertices and edges, $x_0, e_1, x_1, \ldots, e_k.x_k$ such that $k \geq 0$ and x_{i-1} and x_i are the ends of e_i, $\forall\ i = 1, \ldots, k$. It is *elementary* if it does not repeat any vertices or edges. It is *closed* if $x_0 = x_k$. A *partial graph* of a graph is also known as a *subgraph*. The terminology is due to Berge [2].

2.1 Basic Properties of Bidirected Graphs

Definition 2.1 A *biorientation* of G is a signature of its half-edges:

$$\tau : \Phi(G) \to \{-1, +1\}.$$

It is agreed that $\tau(e, x) = 0$ if (e, x) is not a half-edge of G; that makes it possible to extend τ to all of $E \times V$, which we will do henceforth.

A *bidirected graph* is a graph provided with a biorientation; it is written $G_\tau = (V, E; \tau)$.

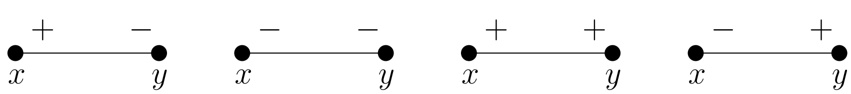


Fig. 1. The four possible biorientations of an edge $\{x, y\}$ of G_τ.

Each edge (including a loop) has four possible biorientations (figure 1); therefore, the number of biorientations of G is $4^{|E|}$.

Definition 2.2 We define two subsets of V:

$$V_{+1} = \{x \in G_\tau : \tau(e,x) = +1, \forall (e,x) \in E_x\} \text{ is the set of } \textit{source} \text{ vertices,}$$
$$V_{-1} = \{x \in G_\tau : \tau(e,x) = -1, \forall (e,x) \in E_x\} \text{ is the set of } \textit{sink} \text{ vertices,}$$

where E_x is the set of all half-edges incident with x.

We observe that $V_{+1} \cap V_{-1}$ is the set of vertices that are not an end of any edge (*isolated* vertices).

Definition 2.3 [3] Let $G_\tau = (V, E; \tau)$be a bidirected graph. Then W (resp., $\overline{W}$) is a function defined on V (resp., E) as follows:

$$W : V \to \mathbb{Z}, \qquad x \mapsto W(x) = \textstyle\sum_{e \in E} \tau(e,x),$$

$$\overline{W} : E \to \{-2, 0, 2\}, \qquad e \mapsto \overline{W}(e) = \textstyle\sum_{x \in V} \tau(e,x).$$

Thus, $\overline{W}(x)$ is the number of positive half-edges incident with x less the number of negative half-edges incident with x.

Definition 2.4 [7] A *signed graph* is a triple $(V, E; \sigma)$ where $G = (V, E)$ is an undirected graph and σ is a signature of the edge set E:

$$\sigma : E \to \{-1, +1\}.$$

A signed graph is written $G_\sigma = (V, E; \sigma)$.

Definition 2.5 [3] Let $G_\sigma = (V, E; \sigma)$ be a signed graph and P a chain (not necessarily elementary) connecting x and y in G_σ:

$$P : x, e_1, x_1, e_2, x_2, \dots, y,$$

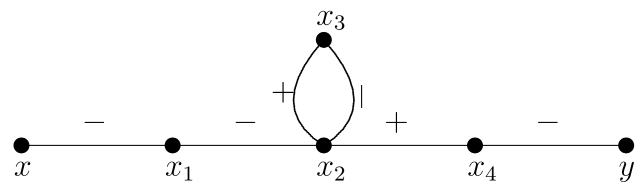
where $x, x_1, \dots, y$ are vertices and $e_1, e_2, \dots$ are edges of G. We put

$$\sigma(P) = \prod_{e_i \in P} \sigma(e_i).$$

We write P^α instead of P, if $\alpha = \sigma(P)$. P^α is called a *signed chain* of sign α connecting x and y; for short, an *s-chain* of sign α. See figure 2.

Especially, a cycle in a signed graph is *positive* if the number of its negative edges is even. In the opposite case, it is *negative*.

Definition 2.6 A signed graph is *balanced* if all its cycles are positive [7].



$P^+ : x, x_1, x_2, x_3, x_2, x_4, y$ is a positive s-chain.
$P^- : x, x_1, x_2, x_4, y$ is a negative s-chain.

Fig. 2. P^+ contains P^- as a subchain, but both P^+ and P^- are minimal s-chains from x to y because their signs differ.

A signed graph is *antibalanced* if, by negating the signs of all edges, it becomes balanced[8].

It follows from the definitions that a cycle is balanced if, and only if, it is positive.

Lemma 2.7 *A signed graph is antibalanced if, and only if, every positive cycle has even length and every negative cycle has odd length.*

Proof. Let G_σ be a signed graph. It is antibalanced if, and only if, $G_{-\sigma}$ has only positive cycles. The sign of a cycle is the same in G_σ and $G_{-\sigma}$ if the cycle has even length and is the opposite if the cycle has odd length. Thus, $G_{-\sigma}$ is balanced if, and only if, every even cycle in G_σ is positive and every odd cycle in G_σ is negative. ■

Definition 2.8 [13] For a biorientation τ of a graph $G_\tau = (V, E; \tau)$, we define a signature σ of E, for any edge e with ends x and y, by:

$$\sigma(e) = -\tau(e, x).\tau(e, y).$$

Definition 2.9 A signed or bidirected graph is *all positive* if all its edges are positive, i.e., for every edge e, $\overline{W}(e) = 0$.

We observe that a bidirected graph that is all positive is a usual directed graph.

Each bidirected graph determines a unique signature. However, the number of biorientations of a signed graph is $2^{|E|}$ because each edge has two possible biorientations.

Definition 2.10 [9,13] Let $G_\tau = (V, E; \tau)$ be a bidirected graph and let X be a set of vertices of G. A biorientation τ_X of G is defined as follows:

$$\tau_x(e, x) = -\tau(e, x), \ \forall \ x \in X,$$

$$\tau_x(e, y) = \tau(e, y), \ \forall \ y \in V - X,$$

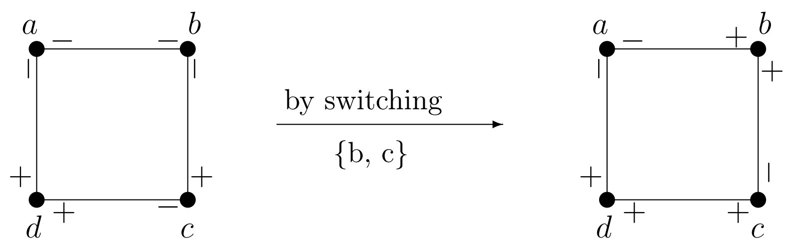
for any edge $e \in E$. We say that the biorientation τ_X and the bidirected graph G_{τ_X} are obtained respectively from τ and G_τ, by *switching* X. If $X = \{x\}$ where $x \in V$, we write τ_x for simplicity. The definition of switching a signed graph is similar. Let G_σ be a signed graph and $X \subseteq V$. The sign function σ *switched by* X is σ_X defined as follows:

$$\sigma_X(e) = \begin{cases} \sigma(e), & \text{if } x, y \in X \text{ or } x, y \in V - X, \\ -\sigma(e), & \text{otherwise,} \end{cases}$$

where x and y are the ends of the edge e.

We note that switching X is a self-inverse operation. It also follows from the definitions that the following result holds:

Proposition 2.11 *Let G_τ be a bidirected graph and σ the signature determined by τ. Let $X \subseteq V$. Then τ_X determines the signature σ_X.*



Proposition 2.12 (i) [12] *The result of switching a balanced signed graph is balanced.*

(ii) [7,12] *A signed graph is balanced if, and only if, there is a subset X of vertices such that switching X produces a signed graph in which all edges are positive.*

(iii) [8,12] *A signed graph is antibalanced if, and only if, there is a subset X of vertices such that switching X produces a signed graph in which all edges are negative.*

Proof. (i) Switching does not change the sign of any cycle.

(ii) Harary [7] has shown that the set of negative edges of a balanced signed graph, if it is not empty, constitutes a cocycle of G_σ. The cocycle divides V into two sets, X and $V - X$, such that F consists of all edges with one end in each set. Thus by switching X, we obtain $\sigma(e) = +1 \ \forall \ e \in F$ in the new graph G_{σ_X} and the other edge signs remain positive. Thus G_{σ_X} is all positive.

Conversely, if there exists $X \subseteq V$ such that G_{σ_X} is all positive, then G_{σ_X} is balanced, so G_σ is balanced by part (i).

(iii) G_σ is antibalanced $\Leftrightarrow$ $G_{-\sigma}$ is balanced $\Leftrightarrow$ $\exists \ X \subseteq V$ such that $G_{(-\sigma)_X} = G_{-\sigma_X}$ is all positive $\Leftrightarrow$ $\exists \ X \subseteq V$ such that G_{σ_X} is all negative. ■

Proposition 2.12 applies to bidirected graphs (cf. [3]) because of Definition 2.8 and Proposition 2.11. Similarly, all propositions about signed graphs G_σ apply to bidirected graphs G_τ through the signature σ determined by τ.

2.2 Characterization of b-Paths in Bidirected Graphs

Definition 2.13 [3] Let G_τ be a bidirected graph, and let P be a chain connecting x and y in G_τ:

$$P : xe_1x_1 \dots e_ix_ie_{i+1} \dots x_{k-1}e_ky,$$

where $x, x_1, \dots, y$ are vertices of G and $e_1, e_2, \dots, e_k$ are edges of G. We define

$$W_P(x_i) = \tau(e_i, x_i) + \tau(e_{i+1}, x_i) \text{ for every } x_i \in V(P),\ i = 1, \dots, k-1.$$

(We note that $W_P(x_i)$ and $W_P(x_j)$ may differ when $i \neq j$, even if $x_i = x_j$.) Let $\tau(e_1, x) = \alpha$ and $\tau(e_k, y) = \beta$; then we write

$$P = P_{(\alpha,\beta)}(x, y) : x^\alpha e_1x_1 \dots e_ix_{i+1}e_{i+1} \dots x_{k-1}e_ky^\beta.$$

We call $P_{(\alpha,\beta)}(x, y)$ an (α, β) *b-path* (short for *bidirected path*) from x to y, or a *b-path from x^α to y^β*, if:

(i) $k \geq 1$.
(ii) $\tau(e_1, x) = \alpha$, and $\tau(e_k, y) = \beta$.
(iii) $W_P(x_i) = 0$, $\forall i = 1, \dots, k-1$ (if $k > 1$).
(iv) $P_{(\alpha,\beta)}(x, y)$ is minimal for the properties (i)–(iii), given x^α and y^β.

If $P_{(\alpha,\beta)}(x, y)$ satisfies (i)–(iii), we call it a *b-walk* from x^α to y^β. Thus a b-path is a minimal b-walk (in the sense of (iv)).

In the notation for a b-path P, we define $x_0 = x$ and $x_k = y$. Then edge e_i has vertices x_{i-1} and x_i, $\forall i = 1, 2, \dots, k$.

If $P_{(\alpha,\beta)}(x, y)$ is a b-path from x^α to y^β, then

$$P_{(\beta,\alpha)}(y, x) : y^\beta e_kx_{k-1} \dots e_{i+1}x_ie_i \dots x_1e_1x^\alpha$$

is also a b-path, from y^β to x^α.

Remark 2.14 No two consecutive edges e_i, e_{i+1} can be equal, because then by cutting out $e_ix_ie_{i+1}$ we obtain a shorter b-path, which is absurd.

Proposition 2.15 The sign of a b-walk $P_{(\alpha,\beta)}(x, y)$ is $\sigma(P) = -\alpha\beta$.

Proof. Let $P_{(\alpha,\beta)}(x,y) : x^\alpha e_1 x_1 \ldots x_{k-1} e_k y^\beta$ be a b-walk from x^α to y^β. The sign of this b-walk is given by

$$\begin{aligned}\sigma(P_{(\alpha,\beta)}(x,y)) &= \prod_{e\in P_{(\alpha,\beta)}(x,y)} \sigma(e) \\ &= [-\tau(e_1,x)\tau(e_1,x_1)][-\tau(e_2,x_1)\tau(e_2,x_2)]\ldots \\ &\quad [-\tau(e_{k-2},x_{k-2})\tau(e_{k-1},x_{k-1})][-\tau(e_k,x_{k-1})\tau(e_k,y)] \\ &= -\tau(e_1,x)[-\tau(e_1,x_1)\tau(e_2,x_1)]\ldots \\ &\quad [-\tau(e_{k-1},x_{k-1})\tau(e_k,x_{k-1})]\tau(e_k,y).\end{aligned}$$

According to the definition of b-walks we have

$$W_P(x_i) = \tau(e_i,x_i) + \tau(e_{i+1},x_i) = 0,$$

therefore $\tau(e_i,x_i)\tau(e_{i+1},x_i) = -1 \ \forall\ i = 1,\ldots,k-1$. Thus,

$$\sigma(P_{(\alpha,\beta)}) = -\tau(e_1,x)\tau(e_k,y) = -\alpha\beta,$$

which proves the result. ■

Proposition 2.16 *Let G_τ be a bidirected graph, and let*

$$P : x_0 e_1 x_1 \ldots e_i x_i e_{i+1} \ldots x_{k-1} e_k x_k,$$

where $k \geq 1$ and $e_i = \{x_{i-1}^{\alpha_{i-1}}, x_i^{\beta_i}\}$ for $i = 1,\ldots,k$, be a chain in G_τ. Then P is a b-path if, and only if,

(a) $\alpha_i = -\beta_i$ *for* $i = 1,\ldots,k-1$ *and*
(b) $x_i^{\alpha_i} \neq x_j^{\alpha_j}$ *when* $i < j$ *and* $(i,j) \neq (0,k)$*;*

and then it is an (α_0,β_k) b-path from $x_0^{\alpha_0}$ to $x_k^{\beta_k}$.

Proof. Let $x = x_0$, $y = x_k$, $\alpha = \alpha_0$, $\beta = \alpha_k$. Since $W_P(x_i) = \beta_i + \alpha_i$ for $i = 1,\ldots,k-1$, condition (iii) is equivalent to condition (a).

Assume P is an (α,β) b-path from x to y. Therefore, P is a chain

$$x^{\alpha_0} e_1 x_1 \ldots e_i x_i e_{i+1} \ldots x_{k-1} e_k y^{\beta_k}$$

from $x_0^{\alpha_0}$ to $x_k^{\beta_k}$ that satisfies (i)–(iii) in Definition 2.13. If $x_i^{\alpha_i} = x_j^{\alpha_j}$ for some $i < j$, then by cutting out $e_{i+1} \ldots e_j$ we get a shorter chain with the same properties (i)–(iii), unless $(i,j) = (0,k)$. We conclude that if $x_i = x_j$ ($i < j$ and $(i,j) \neq (0,k)$), then $\alpha_i \neq \alpha_j$. It follows that P satisfies (b).

Assume P satisfies (a) and (b). Then it satisfies (i)–(iii). Suppose P were not minimal with those properties. Then there is an (α,β) b-path Q from x to y whose edges are some of the edges of P in the same order as in P. If Q begins with edge e_{i+1}, then it begins at $x_i^{\alpha_i}$ and $x_i^{\alpha_i} = x^\alpha = x_0^{\alpha_0}$, therefore $i = 0$ by (b). Similarly, Q ends at edge e_k and vertex $x_k^{\beta_k}$. If Q includes edges e_i and e_{j+1} with $i < j$ but not edges $e_{i+1}, \dots, e_j$, then $x_i^{\alpha_i} = x_i^{-\beta_i} = x_j^{\alpha_j}$, contrary to (b). Therefore, Q cannot omit any edges of P. It follows that P is minimal satisfying (i)–(iii), so P is an (α, β) b-path from x to y. ■

Corollary 2.17 *If P is a b-path that contains a positive cycle C, then $P = C$.*

Examples of b-paths can be seen in figure 3.

We now give the different types of b-path which have a unique cycle which is negative.

Definition 2.18 A *purely cyclic b-path* at a vertex x in a bidirected graph is a b-path C whose chain is a cycle. We say C is *on* the vertex x. The sign of C is the sign of its chain.

We note that in a purely cyclic negative b-path C on x, x is the unique vertex in $V(C)$ such that $W_C(x) = \pm 2$.

Definition 2.19 A *cyclic b-path* P connecting two vertices x and y (not necessarily distinct) in a bidirected graph G_τ, is a b-path from x to y which contains a unique purely cyclic b-path, which is negative. Figure 3 shows the three possible cases. We note that $\alpha, \beta, \gamma, \lambda \in \{-1, +1\}$. If $x = y$ in type (a), the cyclic b-path is purely cyclic.

Lemma 2.20 *A cyclic b-path must have one of the forms in figure 3.*

Proof. Let P be a cyclic b-path from x to y, C the purely cyclic b-path in P, and v the vertex at which $W_C(v) = \pm 2$. The graph of P must consist of C and trees attached to C at a vertex, and it can have at most two vertices with degree 1 because P has only two ends. We may assume $y \neq v$. P must enter C at v (unless $x = v$) and leave it at v to get to y. Therefore, there must be a tree attached to v. There cannot be a tree attached to any vertex z of C other than v, because P would have to enter the tree from C at z^γ and retrace its path back to z^γ in C, which would oblige P to contradict Remark 2.14. Therefore x and y are both in the tree T attached to v, possibly with $x = v$. If $x \neq v$, then x must be a vertex of degree 1 in T, or P would contradict Remark 2.14. Similarly, y must be a vertex of degree 1 in T. As T must be the union of the paths in T from x and y to v, P can only be one of the types in figure 3. ■

Definition 2.21 [3] Let G_τ be a bidirected graph, let $\alpha, \beta \in \{-1, +1\}$, and let C be a b-path $C : x^\alpha\ e_1\ x_1\ \dots\ x_{k-1}\ e_k\ x^\beta$. If $\alpha = -\beta$, we say that C is a *b-circuit* of G_τ.

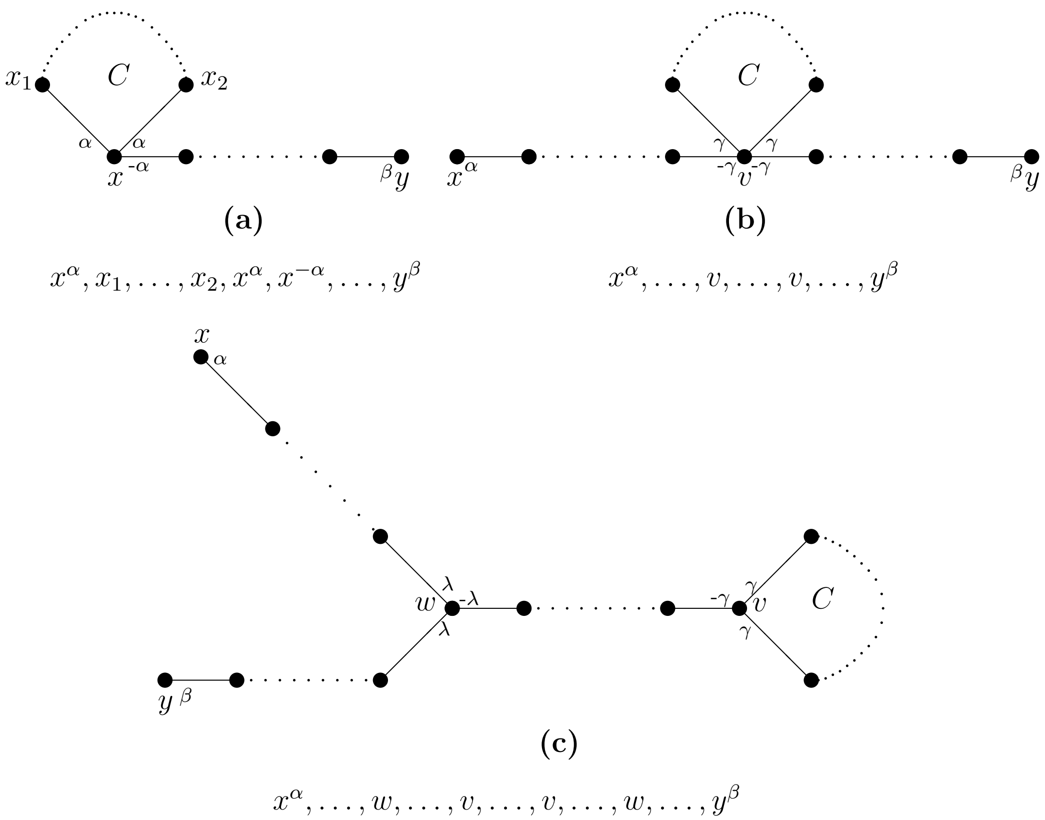


Fig. 3. The three types of cyclic b-path. In (a), $x = y$ is possible. In (c), x or y or both may equal w, but $w \neq v$.

Not all b-circuits trace out a matroid circuit. The b-circuits have been classified in a paper by Chen, Wang, and Zaslavsky [5].

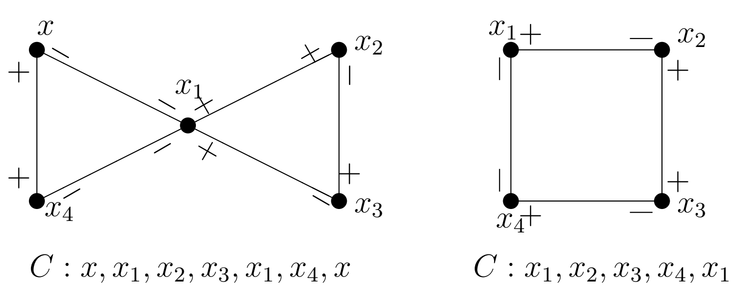


Fig. 4. Two kinds of b-circuit

3 Transitive Closure in Bidirected Graphs

Definition 3.1 *Let G_τ be a bidirected graph. G_τ is* transitive *if, for any vertices x and y (not necessarily distinct) such that there is an (α,β) b-path from x^α to y^β in G_τ, there is an edge $\{x^\alpha, y^\beta\}$ in G_τ.*

Definition 3.2 Let G_τ be a bidirected graph. The *transitive closure* of G_τ is the graph, notated $\mathrm{Ft}(G_\tau) = (V, \mathrm{Ft}(E); \tau)$, such that $\{x^\alpha, y^\beta\} \in \mathrm{Ft}(E)$ if there is a b-path $P_{(\alpha,\beta)}(x, y)$ from x^α to y^β in G_τ (x and y are not necessarily distinct).

Remark 3.3 We see that $E \subseteq \mathrm{Ft}(E)$. If $\{x^\alpha, y^\beta\} \in E$, then $\{x^\alpha, y^\beta\}$ is the edge of a b-path of length 1, so $\{x^\alpha, y^\beta\} \in \mathrm{Ft}(E)$.

Remark 3.4 If there is a b-path $P_{(\alpha,\beta)}(x, x)$ from x^α to x^β in G_τ, then there is a loop $\{x^\alpha, x^\beta\}$ with sign $-\alpha\beta$ in $\mathrm{Ft}(G_\tau)$.

Remark 3.5 If G_τ contains a b-circuit C, then $\mathrm{Ft}(G_\tau)$ contains all the edges $\{x^{-\alpha}, y^{-\beta}\}$ such that $\{x^\alpha, y^\beta\} \in E(C)$. In other words, the transitive closure contains the opposite orientation of every edge that lies in a b-circuit in G_τ. For an example see Figure 8.

Proposition 3.6 Ft *is an abstract closure operator; that is:*

(i) *G_τ is a partial graph of* $\mathrm{Ft}(G_\tau)$.
(ii) *If H_τ is a partial graph of G_τ, then* $\mathrm{Ft}(H_\tau)$ *is a partial graph of* $\mathrm{Ft}(G_\tau)$.
(iii) $\mathrm{Ft}(\mathrm{Ft}(G_\tau)) = \mathrm{Ft}(G_\tau)$.

Proof. (i) and (ii) are obvious from the definition.

(iii) We prove that $\mathrm{Ft}(G_\tau)$ is transitive. Let $P : xe_1x_1 \dots e_ix_ie_{i+1} \dots x_{k-1}e_ky$ be a b-path in $\mathrm{Ft}(G_\tau)$, with $e_i = \{x_{i-1}^{\alpha_{i-1}}, x_i^{\beta_i}\}$ for $i = 1, \dots, k$. By Proposition 2.16 $\alpha_i = -\beta_i$ for $i = 1, \dots, k-1$ and P is an (α_0, β_k) b-path. For each edge e_i there is an (α_{i-1}, β_i) b-path $Q_i(x_{i-1}, x_i)$ in G_τ (which may be e_i itself). Let $R = x_0Q_1x_1Q_2 \dots Q_kx_k$, the concatenation of $Q_1, \dots, Q_k$. At each intermediate vertex y of any Q_i we have $W_R(y) = W_{Q_i}(y) = 0$ by property (iii) of Definition 2.13. At each x_i, $i = 1, \dots, k-1$ we have $W_R(x_i) = \beta_i + \alpha_i = 0$ by Proposition 2.16. Therefore, R is an (α_0, β_k) b-path from x to y in G_τ. We deduce that the edge $\{x^{\alpha_0}, y^{\beta_k}\}$ is in the transitive closure of G_τ. Thus, $\mathrm{Ft}(G_\tau)$ is transitive and its transitive closure is itself. ■

Define

$$\overline{W}(P_{(\alpha,\beta)}(x, y)) = \sum_{e \in P_{(\alpha,\beta)}(x,y)} \overline{W}(e).$$

Theorem 3.7 *Given a bidirected graph G_τ and its transitive closure* $\mathrm{Ft}(G_\tau) =$

$(V, \mathrm{Ft}(E); \tau)$. *If* $e = \{x^\alpha, y^\beta\}$ *is the edge in* $\mathrm{Ft}(G_\tau)$ *implied by transitive closure of the b-path* $P_{(\alpha,\beta)}(x, y)$ *from* x^α *to* y^β *in* G_τ, *then*

$$\overline{W}(P_{(\alpha,\beta)}(x, y)) = \overline{W}(e).$$

Proof. Let $P_{(\alpha,\beta)}(x, y) : x^\alpha e_1 x_1 \ldots x_{k-1} e_k y^\beta$ be a b-path from x^α to y^β in G_τ. According to Definition 2.3 we have

$$\overline{W}(P_{(\alpha,\beta)}(x, y)) = \tau(e_1, x) + W(x_1) + W(x_2) + \ldots + W(x_{k-1}) + \tau(e_k, y),$$

and by Definition 2.13 we obtain

$$\overline{W}(P_{(\alpha,\beta)}(x, y)) = \tau(e_1, x) + \tau(e_k, y) = \alpha + \beta.$$

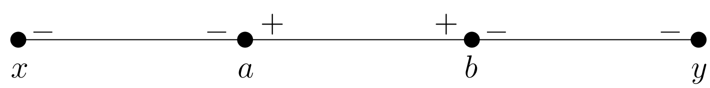
According to Definition 2.3, $\overline{W}(e) = \alpha + \beta = \overline{W}(P_{(\alpha,\beta)})$. ■

We recall that $\sigma(P)$ designates the sign of a chain P (Definition 2.5).

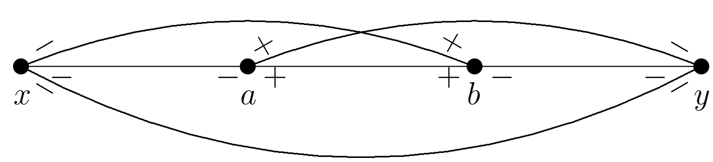
Corollary 3.8 *Given a bidirected graph* G_τ *and its transitive closure* $\mathrm{Ft}(G_\tau) = (V, \mathrm{Ft}(E); \tau)$. *If* $e = \{x^\alpha, y^\beta\}$ *is the edge implied by transitive closure of the b-path* $P_{(\alpha,\beta)}(x, y)$ *from* x^α *to* y^β, *then*

$$\sigma(P_{(\alpha,\beta)}(x, y)) = \sigma(e).$$

Proof. The sign of the b-path is $\sigma(P_{(\alpha,\beta)}(x, y)) = -\alpha\beta$ by Proposition 2.15. We have $\sigma(e) = -\tau(e, x)\tau(e, y) = -\alpha\beta$, from which the result follows. ■



$P_{(-,-)}(x, y) : x^-, a, b, y^-$



$\mathrm{Ft}(P_{(-,-)}(x, y))$

Fig. 5. An example of transitive closure of a bidirected graph.

Lemma 3.9 *Let* G_τ *be a bidirected graph and* $X \subseteq V$. *Then* $\mathrm{Ft}(G_{\tau_X})$ *is the result of switching* $\mathrm{Ft}(G_\tau)$ *by* X.

Proof. We observe that a b-path in G_τ remains a b-path after switching G_τ. For a set $X \subseteq V$, define $\iota(v) = +1$ if $v \notin X$ and -1 if $v \in X$.

Assume that $e = \{x^\alpha, y^\beta\}$ is an edge of $\mathrm{Ft}(G_\tau)$, not in $E(G_\tau)$, that is implied by a b-path $P_{(\alpha,\beta)}(x, y)$ in G_τ. Switch G_τ and $\mathrm{Ft}(G_\tau)$ by X. Then e becomes $\{x^{\alpha\iota(x)}, y^{\beta\iota(y)}\}$ and $P_{(\alpha,\beta)}(x, y)$ becomes $P_{(\alpha\iota(x),\beta\iota(y))}(x, y)$. Therefore e is implied by the b-path $P_{(\alpha\iota(x),\beta\iota(y))}(x, y)$ in $\mathrm{Ft}(G_{\tau_X})$, so e is an edge in $\mathrm{Ft}(G_{\tau_X})$. This proves that $\mathrm{Ft}(G_\tau)$ switched by X is a partial graph of $\mathrm{Ft}(G_{\tau_X})$.

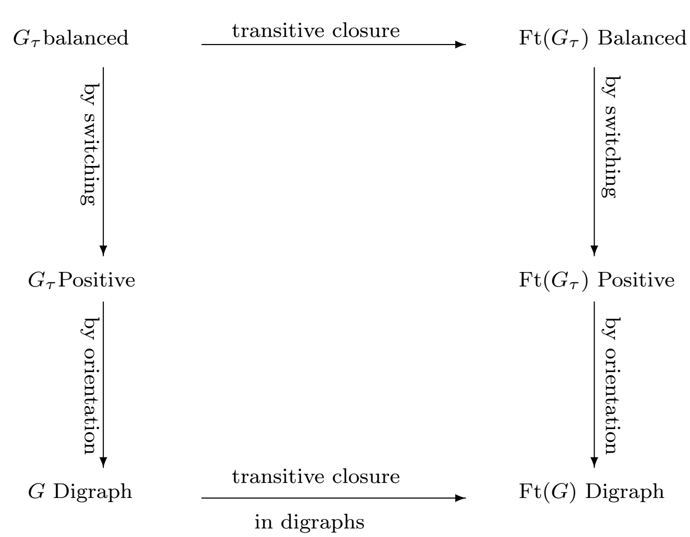
By similar reasoning, if $e = \{x^\alpha, y^\beta\}$ is an edge of $\mathrm{Ft}(G_{\tau_X})$ not in $E(G_{\tau_X})$, it is implied by a b-path $P_{(\alpha,\beta)}(x, y)$ in G_{τ_X}. Switching by X, the edge $\{x^{\alpha\iota(x)}, y^{\beta\iota(y)}\}$ is implied by $P_{(\alpha\iota(x),\beta\iota(y))}(x, y)$, which is a b-path in G_{τ_X}, so that $\{x^{\alpha\iota(x)}, y^{\beta\iota(y)}\}$ is an edge of G_{τ_X} switched by X, which is G_τ. Therefore $\mathrm{Ft}(G_{\tau_X})$ switched by X is a partial graph of $\mathrm{Ft}(G_\tau)$. The result follows. ■

Proposition 3.10 *The transitive closure of an all-positive bidirected graph is all positive. The transitive closure of a balanced bidirected graph is balanced.*

Proof. Assume G_τ is all positive. Let $P_{(\alpha,\beta)}(x, y) : x^\alpha e_1 x_1 \ldots x_{k-1} e_k y^\beta$ be a b-path from x^α to y^β; we close this b-path by the positive edge $e = \{x^\alpha, y^\beta\}$. Since $\overline{W}(e_i) = 0$ for a positive edge e_i, $\overline{W}(P_{(\alpha,\beta)}(x, y)) = 0$. By Theorem 3.7, $\overline{W}(e) = 0$, which means that $\beta = -\alpha$. Thus, e is positive.

Assume G_τ is balanced. By Proposition 2.11 there is a vertex set $X \subseteq V$ such that G_{τ_X} is all positive. By the first part, $\mathrm{Ft}(G_{\tau_X})$ is all positive, therefore balanced, and by Lemma 3.9 it equals $\mathrm{Ft}(G_\tau)$ switched by X. Therefore $\mathrm{Ft}(G_\tau)$ equals $\mathrm{Ft}(G_{\tau_X})$ switched by X, which is balanced by Proposition 2.12. ■

The diagram below, obtained from the results above, shows that the classical notion of transitive closure for directed graphs is a particular case of that found for bidirected graphs.



4 Transitive Reduction in Bidirected Graphs

4.1 Definition and Basic Results

Definition 4.1 Let $G_\tau = (V, E; \tau)$ be a bidirected graph. Define $\text{ft}(G_\tau; H_\tau) = \text{ft}(H_\tau) =$ the transitive closure of H_τ in G_τ, where H_τ is a partial graph of G_τ. (We can write only H_τ when the larger graph, here G_τ, is obvious).

Proposition 4.2 *Let H_τ be a partial graph of G_τ. Then*

$$\text{ft}(G_\tau; H_\tau) = \text{Ft}(H_\tau) \cap G_\tau.$$

Proof. The definition implies that $\text{Ft}(H_\tau) \cap G_\tau \subseteq \text{ft}(G_\tau; H_\tau)$.

Let e be an edge of $\text{ft}(G_\tau; H_\tau)$ not in H_τ. The edge e is induced by a b-path P in H_τ. Thus, $e \in \text{Ft}(E(H_\tau))$. It follows that e is an edge of $\text{Ft}(H_\tau)$. Since also $e \in E(G_\tau)$, we conclude that $\text{ft}(G_\tau; H_\tau) \subseteq \text{Ft}(H_\tau) \cap G_\tau$. ■

Definition 4.3 Let $G_\tau = (V, E; \tau)$ be a bidirected graph. A *transitive reduction* of G_τ is a minimal generating set under ft. Thus we define $\text{Rt}(G_\tau) = (V, \text{Rt}(E); \tau)$ to be a minimal partial graph of G_τ with the property that $\text{ft}(G_\tau; \text{Rt}(G_\tau)) = G_\tau$. We note that $\text{Rt}(G_\tau)$ may not be unique; see Remark 4.10.

The definitions and Proposition 3.6 immediately imply that

$$\text{ft}(\text{Ft}(G_\tau); \text{Rt}(G_\tau)) = \text{ft}(\text{Ft}(G_\tau); G_\tau) = \text{Ft}(G_\tau).$$

Proposition 4.4 *Let G_τ be a bidirected graph and $\text{Rt}(G_\tau)$ a transitive reduction. Then $\text{Ft}(\text{Rt}(G_\tau)) = \text{Ft}(G_\tau)$.*

Proof. $\text{Ft}(\text{Rt}(G_\tau)) = \text{ft}(\text{Ft}(G_\tau); \text{Rt}(G_\tau)) = \text{Ft}(G_\tau)$. ■

Proposition 4.5 *Let G_τ be a bidirected graph. A partial graph H_τ of G_τ is a transitive reduction $\text{Rt}(G_\tau)$ if, and only if, it is minimal such that $G_\tau \subseteq \text{Ft}(H_\tau)$.*

Proof. It follows from Proposition 4.2 that for H_τ to be a transitive reduction of G_τ it is necessary that $G_\tau \subseteq \text{Ft}(H_\tau)$. It follows that H_τ is a transitive reduction of $G_\tau \Leftrightarrow H_\tau$ is minimal such that $G_\tau \subseteq \text{Ft}(H_\tau)$. ■

Proposition 4.6 *If G_τ is a connected bidirected graph, then $\text{Rt}(G_\tau)$ is also connected.*

Proof. The operator Ft does not change the connected components of a graph. ■

Corollary 4.7 *Let G_τ be a bidirected graph without positive loops. If it has no b-path of length greater than 1, then* $\mathrm{Rt}(G_\tau) = G_\tau$ *and every vertex is a source or a sink.*

Proof. Since $\mathrm{Rt}(G_\tau)$ is a partial graph of G_τ, it is enough to prove that each edge e in G_τ is in $\mathrm{Rt}(G_\tau)$. Assume that there exists an edge $e = \{x^\alpha, y^\beta\} \in G_\tau - \mathrm{Rt}(G_\tau)$. According to the definitions of transitive reduction and transitive closure, there exists a b-path from x^α to y^β in $G_\tau - \{e\}$ with length $k \geq 2$, which is absurd.

If a vertex x is neither a source nor a sink, it has incident half-edges (e, x) and (f, x) with $\tau(e, x) = +1$ and $\tau(f, x) = -1$. Then ef is a b-path of length 2, which is absurd, or $e = f$, which implies that e is a positive loop, which is also absurd. ■

We can characterize the graphs in Corollary 4.7 as follows (see figure 6):

- $G_\tau = (V_{+1} \cup V_{-1}, E; \tau)$. ($V_{+1}$ and V_{-1} are the sets of sources and sinks; see Definition 2.2.)
- G_τ is antibalanced (Definition 2.6). (Thus, if G_τ is balanced, then it is bipartite.) The edges connecting a vertex of V_{+1} to a vertex of V_{-1} are positive. The edges connecting two vertices of the same set are negative.

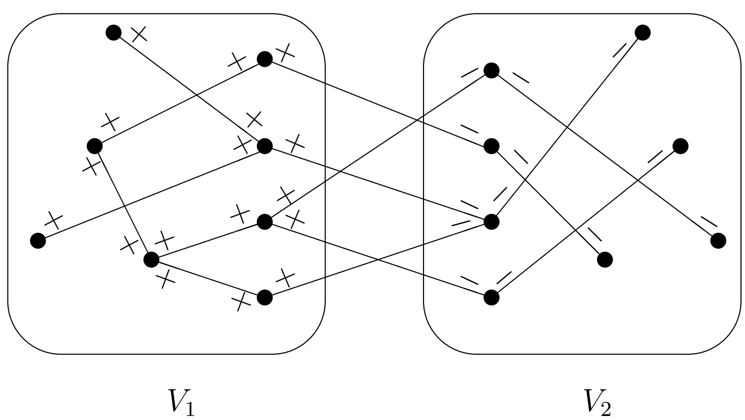
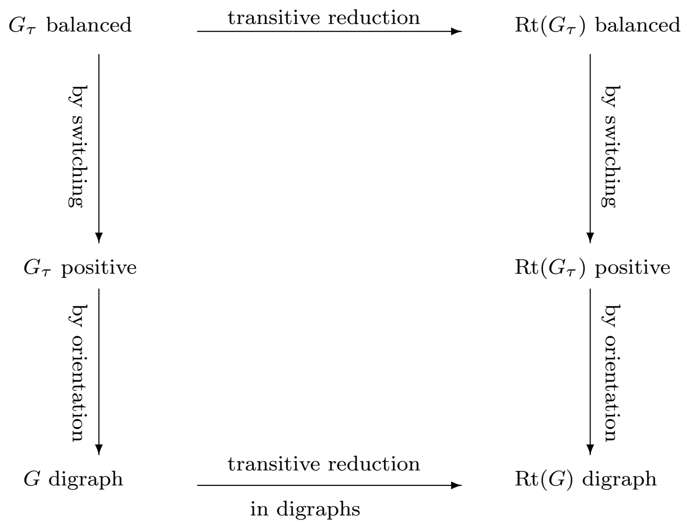


Fig. 6. The form of a graph that satisfies the hypotheses of Corollary 4.7.

Theorem 4.8 (i) *The transitive reduction* $\mathrm{Rt}(G_\tau)$ *is balanced if, and only if, G_τ is balanced.*
(ii) $\mathrm{Rt}(G_\tau)$ *is all positive if, and only if, G_τ is all positive.*

Proof. We conclude from Proposition 3.10 that $\mathrm{Rt}(G_\tau)$ is all positive (resp., balanced) $\Leftrightarrow$ $\mathrm{Ft}(\mathrm{Rt}(G_\tau))$ is all positive (resp., balanced) and that G_τ is all positive (resp., balanced) $\Leftrightarrow$ $\mathrm{Ft}(G_\tau)$ is all positive (resp., balanced). By Proposition 4.4, $\mathrm{Ft}(\mathrm{Rt}(G_\tau)) = \mathrm{Ft}(G_\tau)$. The result follows. ■

Theorem 4.8(ii) is important because all-positive bidirected graphs are the usual directed graphs. Thus, it says that the transitive reduction of a directed graph, in our definition of transitive reduction, is a directed graph. The diagram below, obtained from the results above, shows the stronger statement that the classical notion of transitive reduction for directed graphs is a particular case of our notion for bidirected graphs.



Proposition 4.9 *For a bidirected graph without b-circuits, the graph* $\mathrm{Rt}(G_\tau)$ *is unique. It is obtained from* G_τ *by removing the edges which are in the transitive closure of a b-path.*

Proof. Let $R(G_\tau)$ be the set of edges which are in the transitive closure of a b-path. We note that if H_τ is a partial graph of G_τ, then $R(H_\tau) \subseteq R(G_\tau)$.

We prove first that, if $e, f \in R(G_\tau)$, then $f \in R(G_\tau - \{e\})$. Suppose $e = \{x^\alpha, y^\beta\}$ is implied by a b-path $P_0 = P_{(\alpha,\beta)}(x, y)$ and $f = \{z^\gamma, w^\delta\}$ is implied by a b-path $Q_0 = Q_{(\gamma,\delta)}(z, w)$.

If $e \notin Q_0$, then Q_0 is a b-path in $G_\tau - \{e\}$ that implies f.

If $e \in Q_0$ but $f \notin P_0$, then $Q_0 = Q_1 e Q_2$ where, by choice of notation, e appears as (x^α, y^β) in that order, so that $Q_1 = P_{(\gamma,-\alpha)}(z, x)$ and $Q_2 = P_{-\beta,\delta)}(y, w)$. Replace Q_0 by $Q_1 P_0 Q_2$. This is a b-walk from z^γ to w^δ so it contains a b-path P from z^γ to w^δ, in $G_\tau - \{e\}$, and P implies f.

If $e \in Q_0$ and $f \in P_0$, then $Q_0 = Q_1 e Q_2$ where e, Q_1 and Q_2 are as in the previous case, and $P_0 = P_1 f P_2$ where f appears in P_0 as either (z^γ, w^δ) or (w^δ, z^γ). Suppose the first possibility. Then $Q_1 = P_{(\alpha,-\gamma)}(x, z)$ so $P_1 Q_1$ is a b-walk from x^α to x^α; therefore $P_1 Q_1$ contains a b-circuit, which is impossible. Now suppose the second possibility and let P^* denote the reverse of the path

P. Then $P_1 Q_1 P_2^* Q_2^*$ is a b-walk from x^α to x^α; therefore it contains a b-circuit, which is impossible. Therefore, this case cannot occur.

We conclude that $f \in R(G_\tau - \{e\})$ for every edge $f \in R(G_\tau)$, $f \neq e$. Therefore, $R(G_\tau - \{e\}) \supseteq R(G_\tau) - \{e\}$. Since $G_\tau) - \{e\}$ is a partial graph of G_τ, $R(G_\tau) - \{e\}) \subseteq R(G_\tau)$ so $R(G_\tau - \{e\}) = R(G_\tau) - \{e\}$. By induction, $R(G_\tau - R(G_\tau)) = R(G_\tau) - R(G_\tau) = \emptyset$. We also conclude that $f \in \mathrm{Ft}(G_\tau - \{e\})$ and by induction that $R(G_\tau) \subseteq \mathrm{Ft}(G_\tau - R(G_\tau))$. Therefore, $\mathrm{Rt}(G_\tau) = G_\tau - R(G_\tau)$, so $G_\tau = \mathrm{ft}(G_\tau; \mathrm{Rt}(G_\tau))$ by Propositions 4.2 and 4.4. ■

We can say that these edges are *redundant edges* in G_τ. Figure 7 shows that the transitive closure of the b-path $P_{(-,-)}(2,3) : 2^-, 1, 3^-$ contains the edge $\{2^-, 3^-\}$ which is a redundant edge.

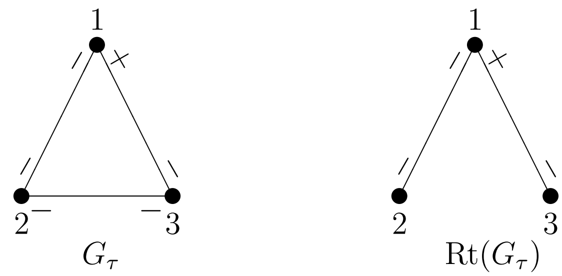


Fig. 7. $\{2^-, 3^-\}$ is a redundant edge.

Remark 4.10 We show an example in which the transitive reduction is unique, and an example in which it is not unique. If C_1 and C_2 are two symmetrical b-circuits, that is, $\{x^\alpha, y^\beta\} \in E(C_1) \iff \{x^{-\alpha}, y^{-\beta}\} \in E(C_2))$, then $\mathrm{Ft}(C_1) = \mathrm{Ft}(C_2) = \mathrm{Ft}(G_\tau)$. Hence in figure 8 G_τ has only one transitive reduction, C_1, but $\mathrm{Ft}(G_\tau)$ has both C_1 and C_2 as transitive reductions.

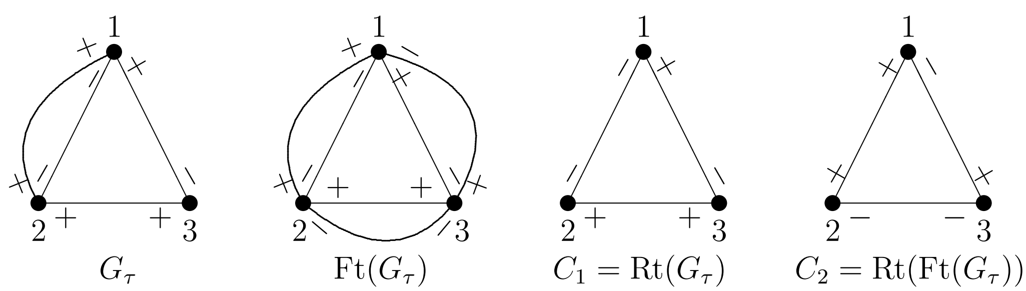


Fig. 8. Example for Remark 4.10. C_1 is an $\mathrm{Rt}(G_\tau)$ and an $\mathrm{Rt}(\mathrm{Ft}(G_\tau))$. C_2 is an $\mathrm{Rt}(\mathrm{Ft}(G_\tau))$, but not an $\mathrm{Rt}(G_\tau)$ because it is not contained in G_τ. For legibility, in $\mathrm{Ft}(G_\tau)$ we do not show the positive loops that exist at every vertex.

Let G_τ be a bidirected graph and $E = \{e_1, e_2, \ldots, e_n\}$ its set of edges. Assume E is linearly ordered by a linear ordering $<$ in index order, i.e., $e_i < e_j \Leftrightarrow i < j$.

Let $(G_\tau)_i$ be a family of graphs constructed from G_τ as follows:

$$(G_\tau)_0 = G_\tau \text{ and}$$
$$(G_\tau)_i = \begin{cases} (G_\tau)_{i-1} - e_i & \text{if } e_i = \{x^\alpha, y^\beta\} \text{ is implied by transitive closure} \\ & \text{of a b-path } P_{(\alpha,\beta)}(x,y) \text{ in } (G_\tau)_{i-1} - \{e_i\}, \\ (G_\tau)_{i-1} & \text{otherwise.} \end{cases}$$

We put

$$S_<(G_\tau) = \{e_i \in E : e_i \in (G_\tau)_{i-1} \text{ and } e_i \notin (G_\tau)_i\}.$$

Proposition 4.11 *Let G_τ be a bidirected graph. For each linear ordering $<$ of E, $G_\tau - S_<(G_\tau)$ is a transitive reduction of G_τ. Conversely, if $\mathrm{Rt}(G_\tau) = (V, \mathrm{Rt}(E); \tau)$ is a transitive reduction of G_τ, then $\mathrm{Rt}(G_\tau) = G_\tau - S_<(G_\tau)$ for some linear ordering $<$ of E.*

Proof. Assume a linear ordering $<$ of E. By construction, if $e_i \in S_<(G_\tau)$, then $e_i \in \mathrm{Ft}((G_\tau)_i)$. Let $m = |E|$; then $\mathrm{Ft}(G_\tau = \mathrm{Ft}^m(G_\tau - S_<(G_\tau)$ (the m-times iterate of ft) $= \mathrm{Ft}(G_\tau - S_<(G_\tau)$ by Proposition 3.6. By Proposition 4.2, $\mathrm{ft}(G_\tau; G_\tau - S_<(G_\tau) = G_\tau$ since $G_\tau \subseteq \mathrm{Ft}(G_\tau = \mathrm{Ft}(G_\tau - S_<(G_\tau)$. If $G_\tau - S_<(G_\tau)$ were not a minimal partial graph that generates G_τ under Ft, then there would be an edge $e_j \in E - S_<(G_\tau)$ such that e_j is implied by a b-path $P_{(\alpha,\beta)}(x,y)$ in $G_\tau - S_<(G_\tau) - e_j$, where $e_j = \{x^\alpha, y^\beta\}$. This b-path is in $(G_\tau)_{j-1} - e_j$ so by construction $e_j \notin (G_\tau)_j$, therefore $e_j \in S_<(G_\tau)$, which is absurd. Therefore $G_\tau - S_<(G_\tau)$ is a transitive reduction of G_τ.

Suppose $\mathrm{Rt}(G_\tau)$ is a transitive reduction of G_τ. Let $S = E - \mathrm{Rt}(E)$. Every edge in S is implied by a b-path in $\mathrm{Rt}(G_\tau)$. Linearly order E by $<$ so that $S = \{e_, \ldots, e_k\}$ is initial in the ordering. Then at step $i \leq k$ of the construction of $S_<(G_\tau)$, edge e_i is implied by transitive closure of a b-path in $(G_\tau)_{i-1} - \{e_i\}$ so $e_i \in S_<(G_\tau)$; but at step $i > k$, $(G_\tau)_i = \mathrm{Rt}(G_\tau)$, which has no such b-path because of minimality of $\mathrm{Rt}(G_\tau)$. ■

Corollary 4.12 *Let G_τ be a bidirected graph. If $S_<(G_\tau) = \emptyset$, then $\mathrm{Rt}(G_\tau) = G_\tau$.*

Corollary 4.13 *If $P_{(\alpha,\beta)}(x,y)$ is a b-path, then $\mathrm{Rt}(P_{(\alpha,\beta)}(x,y)) = P_{(\alpha,\beta)}(x,y)$.*

Proof. For the graph $P_{(\alpha,\beta)}(x,y)$, we have $S_<(G_\tau) = \emptyset$. ■

4.2 Transitive Closure – Transitive Reduction

In this section we study the relationship between transitive closure and transitive reduction.

Proposition 4.14 *Let G_τ be a bidirected graph. Then every transitive reduction of G_τ is a transitive reduction of* $\mathrm{Ft}(G_\tau)$.

Proof. We apply Proposition 4.11. Let $\mathrm{Rt}(G_\tau)$ be a transitive reduction of G_τ. Choose a linear ordering of $E(\mathrm{Ft}(G_\tau)$ in which the edges of $\mathrm{Ft}(G_\tau) - E(G_\tau)$ are initial and the edges of $\mathrm{Rt}(G_\tau)$ are final. By the definition of Ft, the m edges of $\mathrm{Ft}(G_\tau) - E(G_\tau)$ are in $S_<(\mathrm{Ft}(G_\tau) - E(G_\tau))$ and $(\mathrm{Ft}(G_\tau))_m = G_\tau$. The proposition follows. ■

It may not be true that every transitive reduction of $\mathrm{Ft}(G_\tau)$ is an $\mathrm{Rt}(G_\tau)$. Let H_τ be a bidirected graph that has more than one transitive reduction, and let $G_\tau = \mathrm{Rt}(H_\tau)$. Then $G_\tau = \mathrm{Rt}(G_\tau)$. Since $H_\tau \subseteq \mathrm{Ft}(H_\tau) = \mathrm{Ft}(G_\tau)$, every transitive reduction of H_τ is a transitive reduction of $\mathrm{Ft}(G_\tau)$, but only one of those transitive reductions can be $G_\tau = \mathrm{Rt}(G_\tau)$. That cannot happen if G_τ has no b-circuit. We prove a lemma first.

Lemma 4.15 *Let G_τ be a bidirected graph without a b-circuit. Then* $\mathrm{Ft}(G_\tau)$ *has no b-circuit.*

Proof. Suppose $e = \{x^\alpha, y^\beta\} \in \mathrm{Ft}(G_\tau) - E(G_\tau)$. That means there is a b-path $P_{(\alpha,\beta)}(x, y)$ in G_τ. Now suppose there is a b-circuit C from z^γ to z^γ in $G_\tau \cup \{e\}$, $C = C_1 e C_2$, where we may assume C_1 ends at x^α and C_2 begins at y^β. Then $C_1 P C_2$ is a closed b-walk from z^γ to z^γ in G_τ, so it contains a b-circuit, but that is absurd. Therefore, $G_\tau \cup \{e\}$ contains no b-circuit. The proof follows by induction on the number of edges in $\mathrm{Ft}(G_\tau) - E(G_\tau)$. ■

Proposition 4.16 *Let G_τ be a bidirected graph without a b-circuit. Then* $\mathrm{Rt}(\mathrm{Ft}(G_\tau)) = \mathrm{Rt}(G_\tau)$. *That is, the unique transitive reduction of* $\mathrm{Ft}(G_\tau)$ *is the (unique) transitive reduction of G_τ.*

Proof. By Lemma 4.15, $\mathrm{Ft}(G_\tau)$ has no b-circuit. According to Proposition 4.9, $\mathrm{Ft}(G_\tau)$ has a unique transitive reduction. $\mathrm{Rt}(G_\tau)$ is such a transitive reduction. Therefore, $\mathrm{Rt}(\mathrm{Ft}(G_\tau)) = \mathrm{Rt}(G_\tau)$. ■

Corollary 4.17 *Let G_τ be a bidirected graph without a b-circuit. If $S_<(G_\tau) = \emptyset$, then* $\mathrm{Rt}(\mathrm{Ft}(G_\tau)) = G_\tau$.

Proof. If $S_<(G_\tau) = \emptyset$, then according to Corollary 4.12 we have $\mathrm{Rt}(G_\tau) = G_\tau$. Moreover, it follows from Proposition 4.16 that $\mathrm{Rt}(\mathrm{Ft}(G_\tau)) = \mathrm{Rt}(G_\tau)$, from which the result follows. ■

Proposition 4.18 *Let G_τ be a bidirected graph without a b-circuit. Then* $\mathrm{Ft}(\mathrm{Rt}(\mathrm{Ft}(G_\tau))) = \mathrm{Ft}(G_\tau)$.

Proof. By Proposition 4.16, $\mathrm{Rt}(\mathrm{Ft}(G_\tau)) = \mathrm{Rt}(G_\tau) \Rightarrow \mathrm{Ft}(\mathrm{Rt}(\mathrm{Ft}(G_\tau))) = \mathrm{Ft}(\mathrm{Rt}(G_\tau)) \Rightarrow \mathrm{Ft}(\mathrm{Rt}(\mathrm{Ft}(G_\tau))) = \mathrm{Ft}(G_\tau)$ by Proposition 4.4. ■

5 The Matroid of a Bidirected Graph

We indicate by $b(G_\tau)$ the number of balanced connected components of G_τ.

Theorem 5.1 [12] *Given a signed graph G_σ, there is a matroid $M(G_\sigma)$ associated to G_σ, such that a subset F of the edge set E is a circuit of $M(G_\sigma)$ if, and only if, either*

- Type (i) *F is a positive cycle, or*
- Type (ii) *F is the union of two negative cycles, having exactly one common vertex, or*
- Type (iii) *F is the union of two vertex-disjoint negative cycles and an elementary chain which is internally disjoint from both cycles.*

See figure 9, where we represent a positive (resp., negative) cycle by a quadrilateral (resp., triangle).

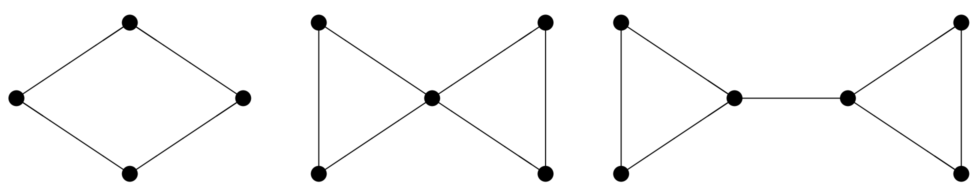


Fig. 9.

The matroid associated to the bidirected graph is the matroid associated to its signed graph (given by Definition 2.8). The rank function is $r(M(G_\tau)) = |V| - b(G_\tau)$ [12].

Definition 5.2 [9] A signed graph G_σ is called *quasibalanced* if it does not admit circuits of types (ii) and (iii). We have the same definition for bidirected graphs.

Proposition 5.3 *A connected signed graph G_σ is quasibalanced if, and only if, for any two negative cycles C and $\acute{C}$ we have $|V(C) \cap V(\acute{C})| \geq 2$.*

Proof. Sufficiency results from Definition 5.2 and Theorem 5.1.

To prove necessity, suppose that G_σ admits two negative cycles C and $\acute{C}$. Suppose $|V(C) \cap V(\acute{C})| = 0$. Since G_σ is connected, there exists a chain connecting a vertex of C with a vertex of $\acute{C}$, therefore there exists a circuit of type (iii) which contains C and $\acute{C}$. Suppose $|V(C) \cap V(\acute{C})| = 1$. Then $C \cup \acute{C}$ is a circuit of type (ii). Both cases are impossible; therefore $|V(C) \cap V(\acute{C})| > 1$. ■

Problem 5.4 Describe all quasibalanced signed graphs, i.e., signed graphs in which every pair of negative cycles has at least two common vertices.

Proposition 5.5 *If G_τ is a quasibalanced bidirected graph, then* $\mathrm{Rt}(G_\tau)$ *is quasibalanced.*

Proof. As $M(G_\tau)$ is without circuits of type (ii) and (iii), and since $\mathrm{Rt}(G_\tau)$ is a partial graph of G_τ, the result follows. ■

Proposition 5.6 *Let G_τ be a bidirected graph such that* $\mathrm{Rt}(G_\tau)$ *is quasibalanced. Let $<$ be a linear ordering of E. If G_τ is quasibalanced, then for every edge e belonging to the set $S_<(G_\tau)$, e is not in the transitive closure of any cyclic b-path in G_τ.*

Proof. Let G_σ be the signed graph corresponding to G_τ.

Assume that there is an edge $e = \{x^\alpha, y^\beta\}$, belonging to the set $S_<(G_\tau)$, which is in the transitive closure of the cyclic b-path $P = P_{(\alpha,\beta)}(x, y)$, containing the negative cycle C. This implies that $P \cup \{e\}$ is a circuit of the type (ii) or (iii) of G_σ, if the cycle $\acute{C}$ of $P \cup \{e\}$ that contains e is negative. By Proposition 2.15 the sign of P is $-\alpha\beta$, which is also the sign of e. Therefore, the sign of the closed walk Pe is $+$. This sign equals $\sigma(C)\sigma(\acute{C})$, so $\sigma(\acute{C}) = \sigma(C) = -$. Thus G_τ is not quasibalanced, which is absurd. ■

We do not have a sufficient condition for quasibalance. The converse of Proposition 5.6 is false. Consider G_τ with $V = \{1, 2, 3, 4, 5, 6, 7\}$ and edges

$$e = 1^-2^+,\ 2^+3^+,\ 3^+4^+,\ 4^+5^+,\ 1^-5^+,\ 5^-6^+,\ 6^-7^+,\ 5^+7^+,\ 7^-2^+,$$

linearly ordered in that order. I claim that e is removable using the path P : 15672, and that no other edge is removable. There is no matroid circuit of type (ii) or (iii) in $G_\tau - e$. But $G_\tau - 7^-2^+$ is a matroid circuit of type (ii). Therefore, G_τ is not quasibalanced, but $\mathrm{Rt}(G_\tau) = G_\tau - e$ is quasibalanced. However, e is not in the transitive closure of any cyclic b-path. P and e are the only b-paths from 1 to 2.

Let $\overline{F}$ denote the closure of F in a matroid.

Lemma 5.7 *Let G_τ be a bidirected graph with edge set E. If $e \in E - \mathrm{Rt}(E)$, then e belongs to the closure $\overline{\mathrm{Rt}(E)}$ in $M(G_\tau)$. If $e \in E(\mathrm{Ft}(G_\tau)) - E$, then e belongs to the closure $\overline{E}$ in $M(\mathrm{Ft}(G_\tau))$.*

Proof. Let P be a b-path in $\mathrm{Rt}(G_\tau)$ which induces $e \in E$. Then $P \cup \{e\}$ is a matroid circuit of type (i), (ii) or (iii). Thus, $e \in \overline{\mathrm{Rt}(E)}$ in $M(G_\tau)$.

The second statement follows from the first because in $M(\mathrm{Ft}(G_\tau))$, $\overline{E}$ is the closure of $\overline{\mathrm{Rt}(E)}$, which equals $\overline{\mathrm{Rt}(E)}$. ■

Theorem 5.8 *Let G_τ be a bidirected graph. Then*

$$r(M(\mathrm{Ft}(G_\tau))) = r(M(G_\tau)) = r(M(\mathrm{Rt}(G_\tau))).$$

Proof. For $r(M(\mathrm{Ft}(G_\tau))) = r(M(G_\tau))$, it is enough to use Lemma 5.7.

For $r(M(\mathrm{Rt}(G_\tau))) = r(M(\mathrm{Ft}(G_\tau)))$, it is enough to cite Proposition 4.4 and replace G_τ in the previous case by $\mathrm{Rt}(G_\tau)$. ■

The definitions of a connected matroid in [11] apply to the matroids of signed graphs. In particular:

Definition 5.9 Let $G_\sigma = (V, E; \sigma)$ be a signed graph. The matroid $M(G_\sigma)$ is *connected* if each pair of distinct edges e and $\acute{e}$ from G_σ, is contained in a circuit C of $\mathrm{M}(G_\sigma)$.

Theorem 5.10 *Let $G_\tau = (V, E; \tau)$ be a bidirected graph and let $\mathrm{Rt}(G_\tau)$ be any transitive reduction of G_τ. If $M(\mathrm{Rt}(G_\tau))$ is connected, then $M(G_\tau)$ is connected.*

Proof. Theorem 5.8 implies that $\overline{E(\mathrm{Rt}(G_\tau))} = \overline{E(G_\tau)} = E(\mathrm{Ft}(G_\tau)$ in $M(\mathrm{Ft}(G_\tau)$. It follows by standard matroid theory, since $M(\mathrm{Rt}(G_\tau))$ is connected, that $M(G_\tau)$ and $M(\mathrm{Ft}(G_\tau)$ are connected. ■

We note that the converse is false. For example, let $P_{(\alpha,\beta)}(x, y)$ be a b-path of length not less than 2, whose graph is an elementary chain, and let e be the edge $\{x^\alpha, y^\beta\}$. Let $G_\tau = P_{(\alpha,\beta)}(x, y) \cup \{e\}$. Then $P_{(\alpha,\beta)}(x, y) = \mathrm{Rt}(G_\tau)$, but $M(P_{(\alpha,\beta)}(x, y))$ is disconnected while $M(G_\tau)$ is connected (since the corresponding signed graph is a positive cycle).

References

[1] Gautam Appa and Balazs Kotnyek, A bidirected generalization of network matrices, Networks 47 (2006), 185–198.

[2] Claude Berge, *Théorie des graphes et ses applications,* Collection Universitaire de Mathématiques, II, Dunod, Paris, 1958.

[3] O. Bessouf, Théorème de Menger dans les graphes biorientés, Thèse de magister, U. S. T H. B. Faculté de Mathematiques (U.S.T.H.B), Alger, Mars 1999.

[4] Ethan D. Bolker and Thomas Zaslavsky, A simple algorithm that proves half integrality of bidirected network programming, Networks 48 (2006), 36–38.

[5] Beifang Chen, Jue Wang and Thomas Zaslavsky, Resolution of irreducible integral flows on a signed graph, submitted.

[6] Curtis Greene and Thomas Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, Trans. Amer. Math. Soc. 280 (1983), 97–126.

[7] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. 3 (1953–1954), 143–146.

[8] F. Harary, Structural duality, Behavioral Sci. 2 (1957), 255–265.

[9] A. Khelladi, Propriétés algebriques de structures combinatoires, Thèse de doctorat d'état, Institut de Mathématiques (U.S.T.H.B), Alger, Mai 1985.

[10] A. Khelladi, Nowhere-zero integral chains and flows in bidirected graphs, J. Combin. Theory Ser. B 43 (1987), 95–115.

[11] James G. Oxley, Matroid Theory, Oxford Univ. Press, Oxford, 1992. Second ed., 2011.

[12] Thomas Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982), 47–74. Erratum, Discrete Appl. Math. 5 (1983), 248.

[13] Thomas Zaslavsky, Orientation of signed graphs, European J. Comb. 12 (1991), 361–375.