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Biased Graphs Whose Matroids Are Special Binary Matroids

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Abstract. A *biased graph* is a graph together with a class of polygons such that no theta subgraph contains exactly two members of the class. To a biased graph Ω are naturally associated three edge matroids: $G(\Omega)$, $L(\Omega)$, $L_0(\Omega)$. We determine all biased graphs for which any of these matroids is isomorphic to the Fano plane, the polygon matroid of K_4, K_5, or $K_{3,3}$, any of their duals, Bixby's regular matroid R_{10}, or the polygon matroid of K_m for $m > 5$. In each case the bias is derived from edge signs. We conclude by finding the biased graphs Ω for which $L_0(\Omega)$ is not a graphic [or, regular] matroid but every proper contraction is.

Introduction

A *biased graph* $\Omega = (\Gamma, \mathscr{B})$ consists of an underlying graph $\Gamma = \|\Omega\| = (V, E)$ and a class of polygons $\mathscr{B} = \mathscr{B}(\Omega)$ such that, if the union of C_1, $C_2 \in \mathscr{B}$ is a theta graph, then the third polygon in $C_1 \cup C_2$ is in $\mathscr{B}$. We call a subgraph (or edge set) S of $\|\Omega\|$ *balanced* if every polygon in S belongs to $\mathscr{B}$, *contrabalanced* if none does. A biased graph has three naturally associated matroids which we call the "bias", "lift", and "complete lift" matroids. The *bias matroid* $G(\Omega)$ has E for point set; its circuits are the balanced polygons and the minimal contrabalanced, connected edge sets with cyclomatic number two. The *lift matroid* $L(\Omega)$ has point set E and circuits the balanced polygons and the minimal contrabalanced edge sets of cyclomatic number two (not necessarily connected). The *complete lift matroid* $L_0(\Omega)$ has point set $E_0 = E \cup \{e_0\}$, where the *extra point* e_0 is not in Γ; its circuits are those of $L(\Omega)$ and the sets of the form $C \cup \{e_0\}$ where C is an unbalanced polygon. Biased graphs and the bias matroid were introduced in [13; 14]. The lift and complete lift are special cases of the general matroid lift construction, dual to elementary strong maps and one-point extension respectively. An important kind of biased graph is a *sign-biased* graph, whose bias is obtained by labelling the edges of Γ with signs and letting $\mathscr{B}$ consist of the polygons whose edge sign product is positive. In particular a biased

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graph is *antibalanced* if it is all-negatively sign-biased; then its balanced polygon class consists of the even polygons in Γ.

The purpose of this article is to find the biased graphs having a matroid G, L, or L_0 isomorphic to any of the special binary matroids of importance in characterizing regular, graphic, planar-graphic, and series-parallel-network matroids. These matroids are the Fano plane F_7 and its dual $F_7^{\perp}$, Bixby's regular matroid R_{10}, the polygon matroids $G(K_{3,3})$ and $G(K_5)$ and their duals, and the polygon matroid $G(K_4)$. We also find the biased graphs having a matroid isomorphic to $G(K_m)$ where $m > 5$. It is noteworthy that for each matroid all the biased graphs are sign-biased.

These results should help to give new proofs of characterizations, recently found using a different method by Lovász *et al.* [2, 3], of sign-biased graphs whose lift matroid is regular, graphic, or cographic and of sign-biased graphs having no two vertex-disjoint unbalanced polygons or no subdivision of an antibalanced K_4. The first of these characterizations to be found was that of sign-biased graphs whose lift matroid is graphic; this is due to C.-H. Shih [5, Theorem 1], whose approach seems to be quite different from that initiated herein.

The corresponding problems for the bias matroid and for biased graphs belong to the future, but a problem we do solve here is to find all Ω for which $L_0(\Omega)$ is not a graphic matroid but every proper contraction is. This is of interest in connection with the isomorphism problem for complete lift matroids of biased graphs [17]. We also develop corollaries that display useful properties of the special graphs and which take small steps towards characterizing biased graphs having particular types of matroids. Results of this article were also used in [16] to help prove that $G(\Gamma, \varnothing)$, which is the *bicircular matroid* of Simões-Pereira, is a series-parallel matroid if it is binary, thus slightly improving a result of Matthews.

1. Definitions and General Lemmas

This section begins with the definitions of the objects and operations we need, continues with Useful Lemmas, and concludes with a brief outline of the general approach we take in the examples.

The *sum* (that is, symmetric difference) of sets is written $S + T$.

Our graphs are finite and undirected; loops and multiple edges are allowed (except where explicitly excluded). A *link* is an edge with two distinct endpoints; a *loop* has coincident endpoints. For the vertex and edge sets of a graph Δ we write $V(\Delta)$ and $E(\Delta)$; or we say "let $\Delta = (W, F)$ be a graph" to mean $V(\Delta) = W$, $E(\Delta) = F$. We always let Γ be the graph (V, E) and denote the order $|V|$ by n. The vertex set V equals $\{v_1, v_2, \ldots, v_n\}$ unless we say otherwise. We use several notations for edges, in part to differentiate those of different graphs; thus $e_{uv} = e_{vu} = uv = vu$ is an edge between u and v, e_{ij} joins v_i and v_j (or x_i and x_j, if the vertex set of the graph is $\{x_1, x_2, \ldots\}$). A special notation ij for edges of a bipartite graph is described (for $K_{3,3}$) in Section 4. A *bond* in a graph is a minimal edge cutset.

By $m\Delta$, where m is a positive integer, we mean Δ with each link replaced by m parallel copies. Particular graphs are the familiar K_n and $K_{l,m}$ (with *left set* of l vertices and *right set* of m vertices), the n-edge polygon graph C_n, the *wheel* $W_n =$

$C_n + v$ (where $\Gamma + v$ means Γ with an additional *hub* vertex v, simply adjacent to every vertex of Γ). By *subdividing* an edge e_{uv} we mean replacing it by two edges in series. A *subdivision of* Γ is any graph obtained from Γ by subdividing edges, including Γ itself as a trivial subdivision. A *theta graph* is a subdivision of $3K_2$. A *tight handcuff* is a subdivision of two loops at a vertex. A *loose handcuff* is a subdivision of K_2 with a loop at each vertex. The *restriction* $\Gamma|S$ of Γ to $S \subseteq E$ is the spanning subgraph (V, S); the *contraction* Γ/S of Γ by S is obtained by shrinking to a point every edge in S. A *minor* of Γ is any contraction of a subgraph of Γ, including Γ itself, which is the *improper* minor. If $W \subseteq V$, $\Gamma \backslash W$ denotes $(V \backslash W, S)$ where S consists of all edges with both ends in $V \backslash W$. We write $\Gamma \backslash v = \Gamma \backslash \{v\}$.

If $W \subseteq V$, a *bridge* of W is a maximal subgraph Δ of Γ which is connected through vertices not in W. For instance, an edge whose endpoints lie in W is a bridge. A *cutpair* in a 2-connected graph is a pair of vertices having at least three bridges or two bridges each larger than a single edge.

A *signed graph* Σ consists of an underlying graph $\|\Sigma\|$ together with a sign labelling of its edges. Some types of signed graphs are $+\Delta$ and $-\Delta$, which are Δ with all edges positive or negative, respectively, and $\pm\Delta$, whose underlying graph is 2Δ with one edge of each pair labelled positive and the other negative. We indicate the sign of an edge by a superscript: e.g., e_{ij}^+ and e_{ij}^- are distinct edges with the same endpoints and the indicated signs. The biased graph $(\|\Sigma\|, \mathscr{B}(\Sigma))$ derived from Σ by setting $\mathscr{B}(\Sigma) =$ the set of polygons having positive sign product is denoted by $[\Sigma]$. (We call any biased graph of this form *sign-biased*.) In particular we write $[\Gamma] = [+\Gamma]$; also, the antibalanced graph equals $[-\Gamma]$. *Switching* Σ means choosing an $X \subseteq V$ and changing the signs of the edges crossing between X and its complement. Clearly, switching leaves $[\Sigma]$ invariant; conversely one can easily prove that, if Σ_1 and Σ_2 have $[\Sigma_1] = [\Sigma_2]$, then Σ_1 switches to Σ_2. The rule for *contracting* an edge in Σ is this: if e is a link, switch so it is positive, then coalesce its vertices and delete e; if e is a positive loop, delete it; if e is a negative loop at v, delete it and v and any other loops at v, then change each link uv to a negative loop at u. The contraction Σ/e is well-defined up to switching (that is, $[\Sigma/e]$ is well-defined). A *minor* of Σ is any contraction of a subgraph. We abbreviate $G([\Sigma])$, etc., sometimes by $G(\Sigma)$, etc.

Here is the first Useful Lemma. A set $\mathscr{D}$ of polygons in Σ *spans* if every polygon can be written as the symmetric difference of polygons in $\mathscr{D}$.

Lemma 1A. *Let Γ be a graph and $\mathscr{D}$ a spanning set of polygons. Let $\mathscr{A} \subseteq \mathscr{D}$ be given such that, if $\mathscr{D}' \subseteq \mathscr{D}$ has symmetric difference equal to the null set, then $\mathscr{A} \cap \mathscr{D}$ is even. Then there exists a unique sign-biased graph $(\Gamma, \mathscr{B})$ such that $\mathscr{B} \cap \mathscr{D} = \mathscr{A}$.*

Proof. This is part of [11, Theorem 2]. □

We let Ω always denote the biased graph $(\Gamma, \mathscr{B})$, where $\Gamma = (V, E)$ and $n = |V|$. A *subdivision* of Ω is a subdivision of Γ with the obvious bias. The *restriction* $\Omega|S$ of Ω to $S \subseteq E$ is $\Gamma|S$ with the subgraph bias $\mathscr{B}(\Omega|S) = \{C \in \mathscr{B}: C \subseteq S\}$. The *contraction* Ω/e (if e is a link) is Γ/e with $\mathscr{B}(\Omega/e) = \{C \in \mathscr{B}(\Omega \backslash \{e\}): C$ remains a polygon in $\Gamma/e\} \cup \{C \backslash \{e\}: e \in C \in \mathscr{B}(\Omega)\}$. If e is a balanced loop, $\Omega/e = \Omega \backslash \{e\}$. If e is an unbalanced loop at v, $\Omega/e = (\Omega \backslash v) \cup \{e_{uv} \in E$, regarded as an unbalanced loop at u, for

$u \neq v\}$. (This definition is simplified from [13].) A *minor* of Ω is any result of contracting edges of a subgraph. We have $[\Sigma/e] = [\Sigma]/e$ and $[\Sigma|S] = [\Sigma]|S$, so biased and signed graph minors are consistent with each other.

If $\Omega_1, \ldots, \Omega_k$ are biased graphs, each with an ordered pair (u_i, v_i) of distinguished vertices, their *unbalanced parallel connection* $\Omega = \hat{P}(\Omega_1, \ldots, \Omega_k)$ is obtained by identifying all u_i to u and all v_i to v and taking $\mathscr{B} = \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_k$. (We implicitly assume the Ω_i are pairwise disjoint.)

We assume acquaintance with relevant parts of matroid theory as for instance in Welsh [8]. Our matroids are all finite. We write rk for rank. The *n-point line* L_n, for $n \geq 2$, is the n-point uniform matroid of rank two. The *Fano plane* F_7 is the matroid of the projective plane of order 2. The *Bixby matroid* R_{10} (called "Type R" by Bixby [1], R_{10} by Seymour [4], who also states the definition given here [4, p. 328]) is the linear dependence matroid of the ten vectors in $GF(2)^5$ having exactly three non-zero coordinates. We write $M|S$ for the submatroid of a matroid M on a point subset S, called the *restriction* of M to S, $M \backslash S$ for M restricted to the complement of S, and M/S for the *contraction* by S, which is also called the contraction of M to the complementary point subset. A *minor* of M is any contraction of a restriction, including the *improper minor* M itself. We sometimes write $M_1 \geq M_2$ to mean M_1 has a minor isomorphic to M_2. If P is a property of matroids, we call M *minimally non-P* if M lacks P but every proper minor has P. Major properties are binarity, regularity, graphicity, cographicity (see [8] for definitions), and being the matroid of a planar graph or of a series-parallel network. A very Useful Lemma indeed and the basis for this paper is the following:

Lemma 1B. *Let M be a matroid.*
(1) *M is binary $\Leftrightarrow M \ngeq L_4$.*
(2) *M is regular $\Leftrightarrow M \ngeq L_4, F_7, F_7^\perp$.*
(3) *M is graphic $\Leftrightarrow M \ngeq L_4, F_7, F_7^\perp, G^\perp(K_{3,3}), G^\perp(K_5)$.*
(4) *M is planar-graphic $\Leftrightarrow M \ngeq L_4, F_7, G(K_{3,3}), G(K_5)$, and their duals.*
(5) *M is series-parallel $\Leftrightarrow M \ngeq L_4, G(K_4)$.*
Each matroid listed on the right is minimally non-P for P = the property on the left.

Proof. These are due to Tutte (see [7]) except for (5), which is due in essence to Duffin (see [8, p. 237, Theorem 14.2.2]). □

The bias matroid $G(\Omega)$ has for circuits the balanced polygons and the contrabalanced thetas and handcuffs. Its rank function is $\mathrm{rk}_G(S) = n - b(S)$, where $b(S)$ = the number of balanced connected components of (V, S). It has the properties that $G(\Omega)|S = G(\Omega|S)$ and $G(\Omega)/S = G(\Omega/S)$ with perhaps some matroid loops deleted. (See [14].)

The lift matroid $L(\Omega)$ has for circuits the balanced polygons and the contrabalanced thetas, tight handcuffs, and pairs of vertex-disjoint polygons. Its rank function is $\mathrm{rk}_L(S) = n - c(S)$ if S is balanced, $= n + 1 - c(S)$ if S is unbalanced, where $c(S)$ = the number of connected components of (V, S). A copoint of $L(\Omega)$ is either a copoint of $G(\Omega)$, that is to say the complement of a bond of Γ, or is a maximal balanced edge set (if Ω is not balanced; if Ω is balanced, there are no copoints of the latter kind). The lift has the properties that $L(\Omega)|S = L(\Omega|S)$ and that $L(\Omega)/S = L(\Omega/S)$ if S is balanced, $= G(\Gamma/S)$ if S is unbalanced. (See [14].)

The complete lift $L_0(\Omega)$ has the circuits of $L(\Omega)$ and also those of the form $C \cup \{e_0\}$ where C is an unbalanced polygon. For $S \subseteq E$, $\mathrm{rk}(S \cup \{e_0\}) = n + 1 - c(S)$. A copoint of $L_0(\Omega)$ is a maximal balanced edge set or a set $F \cup \{e_0\}$ where F is a copoint of $G(\Gamma)$ that is not balanced in Ω. (See [14].) L_0 has the properties that $L_0(\Omega)|S = L(\Omega|S)$ and $L_0(\Omega)|(S \cup \{e_0\}) = L_0(\Omega|S)$ if $S \subseteq E$ and that $L_0(\Omega)/S = L_0(\Omega/S)$ if S is a balanced edge set, $= G(\Gamma/(S\backslash e_0))$ if S is any other subset of E_0.

Consider the operations on a biased graph Ω of (1) attaching two connected components at a vertex, or splitting a cut vertex in two so the new vertices are in different components, and (2) separating a connected component into two disconnected parts Ω_1 and Ω_2 by splitting two vertices u and v (that form a cutpair) into $u_i, v_i \in V(\Omega_i)$ and reconnecting them in reverse by identifying u_1 with v_2 and u_2 with v_1. The class $\mathscr{B}$ remains the same as it was. We call Ω and Ω' *2-isomorphic* if one is obtained from the other by operations of types (1) and (2). Whitney [9] observed that 2-isomorphic graphs have the same polygon matroid and he proved that, conversely, graphs with the same matroid are 2-isomorphic. The first (but not the second) statement generalizes to lift matroids of biased graphs.

Lemma 1C. *If Ω and Ω' are 2-isomorphic biased graphs, then $L_0(\Omega) = L_0(\Omega')$ with e_0 corresponding to e_0'.*

Proof. Obvious from the definition of L_0. □

Several more Useful Lemmas are the following. By $G(\Gamma) \oplus e_0$ we mean the direct sum of $G(\Gamma)$ with a matroid isthmus e_0.

Lemma 1D. *If Ω is balanced, then $G(\Omega) = L(\Omega) = G(\Gamma)$ and $L_0(\Omega) = G(\Gamma) \oplus e_0$ are graphic matroids. Conversely, if e_0 is a direct summand of $L_0(\Omega)$, then Ω is balanced.*

Proof. Easy from the definitions. □

Lemma 1E. *We have $L(\Omega) = G(\Omega)$ if and only if Ω has no two vertex-disjoint unbalanced polygons.*

Proof. If there are no such pairs of polygons, all circuits in $L(\Omega)$ are connected. If there are two such polygons, they form a circuit in $L(\Omega)$ but are independent in $G(\Omega)$. □

Lemma 1F. *Let Γ be a graph and e an edge on vertices of L. Let $\Sigma = +\Gamma \cup \{e^-\}$. Then $G(\Sigma/e^-) = L(\Sigma/e^-) = G(\Gamma)$. If e is a link, then $L_0(\Sigma/e^-) = G(\Sigma \uplus \{e\})$, where $\Gamma \uplus \{e\}$ means Γ with e added (even if that creates a multiple edge).*

Proof. Since e is in no circuit of $L(\Sigma)$ or $G(\Sigma)$, it is an isthmus and $G(\Sigma) = L(\Sigma) = G(\Gamma) \oplus e$.

In $(+\Gamma \cup \{e^+, e^-\})/e^-$, e^+ becomes an unbalanced loop. Taking the lift matroid, e^+ functions like e_0 in $L_0(\Sigma/e^-)$. □

A *balancing vertex* of Ω is a vertex v such that $\Omega\backslash v$ is balanced, but Ω is not.

Lemma 1G. *If Ω is sign-biased and is either balanced or has a balancing vertex, then $G(\Omega) = L(\Omega)$ and $L_0(\Omega)$ are all graphic.*

Proof. From [15, Corollary 1] we know that Ω has the form $[+\Gamma \cup \{e^-\}]/e^-$ for some graph Γ and edge e^-. Then we apply Lemma 1F. □

Lemma 1H. *Let M denote any of F_7, R_{10}, $G(K_{3,3})$, $G(K_5)$, $G(K_4)$, and their duals.*

If $L(\Omega)$ or $L_0(\Omega) \cong M$, then Ω is sign-biased and Γ is edge-2-connected and has no cutpoint. Moreover, if $L(\Omega) \cong M$ then Γ is edge-3-connected.

We have $G(\Omega) \cong M$ if and only if $L(\Omega) \cong M$ and Ω has no two vertex-disjoint unbalanced polygons.

Proof. See [16, Corollary 4.3] and Lemma 1E. The edge connectedness follows from the 3-connectedness of every M. □

Here is an outline of our general approach. We are trying to determine all Ω for which $L_0(\Omega)$, $L(\Omega)$, or $G(\Omega) \cong M$, where M is one of the matroids listed in Lemma 1H. That lemma reduces the problem for $G(\Omega)$ to that for $L(\Omega)$ and shows that we need only consider sign-biased Ω. In most cases this is not enough to determine Ω. If we have $L_0(\Omega) \cong M$, then $G(\Gamma) \cong M/e_0$ gives information about Γ. In general, we may assume Ω is unbalanced (since otherwise the solution is known) and, by Lemma 1H, connected, so $n = \operatorname{rk} M$. We then argue individually for each different M to determine Ω, making use of symmetries of M to simplify the argument. Finally, we have to verify the actual isomorphism of M with the appropriate matroid of Ω. Since Ω is sign-biased, both M and $L_0(\Omega)$, hence also $L(\Omega)$, are binary; therefore we can employ either the basis or the circuit (or copoint) method stated in the final two lemmas.

Lemma 1I. *If M_1 and M_2 are binary matroids on the same set A and B is a basis of both, and if for every $e \in A \backslash B$ the (unique) circuit $B(e)$ contained in $B \cup \{e\}$ is the same in M_1 and M_2, then $M_1 = M_2$.*

Proof. (This result is well known.) We can choose coordinates for M_1 and M_2 in $GF(2)^{|B|}$ so B consists of the unit basis vectors. Then $B(e)$ completely determines the vector representing e. □

Lemma 1J. *If M_1 and M_2 are binary matroids of the same rank on the same set A and every circuit [or, copoint] of M_1 is a circuit [copoint] of M_2, then $M_1 = M_2$.*

Proof for circuits. Since both have n points and rank r, the cycle space of each (the space spanned by the circuits under symmetric difference) has the same dimension. Hence, the cycle space $\mathscr{C}_1$ of M_1 spans $\mathscr{C}_2$, the cycle space of M_2. So each circuit C_2 of M_2 has a representation $C_2 = C_{11} + C_{12} + \cdots + C_{1m}$ where the C_{1i} are circuits of M_1. The latter sum is a disjoint union of circuits of M_1 because M_1 is binary (Whitney [10, pp. 530–533]; or see [8, Theorem 10.1.3]). But each such circuit is also a circuit of M_2; since it is contained in C_2, it equals C_2. So C_2 is a circuit of M_1.

Proof for copoints. Dualize the result for circuits. □

2. Series-Parallel: the Matroid $G(K_4)$

Proposition 2A. *For a biased graph Ω we have*

$$G(\Omega) \cong G(K_4) \Leftrightarrow \Omega = [K_4] \text{ or } [\pm K_3] \text{ or } [(\pm K_3 \cup e_{33}^-)\backslash e_{12}^-].$$

If Ω has no loops,

$$L(\Omega) \cong G(K_4) \Leftrightarrow \Omega = [K_4] \text{ or } [\pm K_3],$$

$$L_0(\Omega) \cong G(K_4) \Leftrightarrow \Omega = [\pm K_3]\backslash edge.$$

Proof. For balanced Ω the results are obvious. Let Ω be unbalanced; thus $n = 3$.

Suppose $L(\Omega) \cong G(K_4)$. Since Ω has six edges and is sign-biased, it is $[\pm K_3]$. It is easy to verify by inspection that $e_{ij}^+ \to e_{ij}$, $e_{ij}^- \to e_{k4}$ (where $k \neq i, j$) is an isomorphism of $L(\Omega)$ with $G(K_4)$.

Suppose $L_0(\Omega) \cong G(K_4)$. Then $G(\Gamma) \cong L_0(\Omega)/e_0 \cong G(K_4/e_{34})$, say. Thus, Γ is K_3 with two edges (e_{13} and e_{23}) doubled; whence $\Omega = [\pm K_3]\backslash$edge. Tracing the isomorphisms, $e_{ij}^+ \to e_{ij}$ and $e_{i3}^- \to e_{i4}$, also $e_0 \to e_{34}$. To prove we have the right matroid, let $\Gamma = K_4$ and $e = e_{34}$ and take $L(\Sigma/e^-)$ in Lemma 1F.

Suppose that $G(\Omega) \cong G(K_4)$. Then Ω is one of the L examples, neither of which has vertex-disjoint unbalanced polygons, or is the L_0 example with a negative loop at v_3. □

3. Regularity: the Matroids F_7 and $F_7^\perp$

Proposition 3A. *If Ω is a biased graph, then $G(\Omega) \cong F_7$ or $F_7^\perp$ is impossible. If Ω has no loops, then*

$$L_0(\Omega) \cong F_7 \Leftrightarrow \Omega = [\pm K_3],$$

$$L_0(\Omega) \cong F_7^\perp \Leftrightarrow \Omega = [-K_4];$$

also, $L(\Omega) \cong F_7$ is impossible and

$$L(\Omega) \cong F_7^\perp \Leftrightarrow \Omega = [\pm C_4]\backslash edge.$$

Proof. Since F_7 and $F_7^\perp$ are not graphic, Ω is unbalanced. Let the points of F_7 be p_1, p_2, p_3 (collinear), q, and r_i collinear with $p_i q$.

In the case of F_7, $n = 3$. Since seven edges are required, $L(\Omega)$ cannot be F_7. With seven edges, Ω must have vertex-disjoint unbalanced polygons. This implies that $G(\Omega) \cong F_7$ is not possible. The one possible case is $L_0(\pm K_3) \cong F_7$. To verify that this is an example, let $e_0 \to q$, $e_{ij}^+ \to p_k$, and $e_{ij}^- \to r_k$ (where $k \neq i, j$). We easily confirm that $L_0(\pm K_3)$ has the seven lines $p_1p_2p_3$, p_iqr_i, and $p_ir_jr_k$ of F_7 and no more.

In the case of $F_7^\perp$, $n = 4$ and there are no 3-point circuits. Suppose $L(\Omega) \cong F_7^\perp$. Since $G(\Omega)$ cannot equal $L(\Omega)$ (by [16, Corollary 4.2]), Ω must contain vertex-disjoint unbalanced polygons. It therefore contains two opposite digons, say $e_{12}^\pm$ and $e_{34}^\pm$, which form a circuit; we may fix the isomorphism to be $e_{12}^- \to p_1$, $e_{12}^+ \to r_1$, $e_{34}^- \to p_3$, $e_{34}^+ \to r_3$. Since $p_1r_1p_2r_2$ and $p_2r_2p_3r_3$ are circuits, p_2r_2 must correspond to another digon in Ω, say $e_{23}^- \to p_2$ and $e_{23}^+ \to r_2$. In order for Ω to be edge-3-

connected, the last edge must be e_{14}^+ (or e_{14}^-), corresponding to q. Thus, $\Omega = [\pm C_4]\backslash$edge. To complete the labelled isomorphism we have been constructing, we must choose e_{14}^+; then the circuit $qr_1r_2r_3$ is an all-positive C_4 and $p_ip_jqr_k$ is a C_4 with two negative edges.

Let Ω be loopless such that $L_0(\Omega) \cong F_7^\perp$. Then $G(\Gamma) \cong F_7^\perp/q \cong G(K_4)$ so $\Gamma = K_4$. Since $F_7^\perp$ has no 3-element circuits, $\Omega = [-K_4]$. Examining the isomorphism $F_7^\perp/q \cong G(K_4)$, we see that $r_1r_2r_3$ form a triangle and p_i is independent of r_i. One can easily verify that the seven circuits of $F_7^\perp$ are indeed circuits of $L_0(-K_4)$. □

Corollary 3A. *If $L(\Omega) \cong F_7^\perp$, then $L_0(\Omega)/e \cong F_7$ for some edge e.* □

Let a *cycle of unbalanced polygons* be any biased graph Ω consisting of $k \geq 2$ unbalanced polygons $C^1, \ldots, C^k$ and k pairwise vertex-disjoint simple connecting paths P^i for $1 \leq i \leq k$, where P^i meets C^i and C^{i+1} at its endpoints and meets no other C^j. (Superscripts are modulo k.) Each two polygons C^i and C^j are vertex-disjoint unless $j = i + 1$ and P^i has length zero (and then they have a single common vertex).

Corollary 3B. *Let Ω be a loop-free biased graph. $L_0(\Omega)$ is regular if and only if Ω is sign-biased and has no subgraph that is a subdivision of $[-K_4]$ or a cycle of three unbalanced polygons.*

Proof. We know by Tutte's characterization that $L_0(\Omega)$ is regular if and only if it is binary and $\not\geq F_7, F_7^\perp$. Thus Ω is sign-biased and has no subgraph contracting to $[-K_4]$, $[\pm K_3]$, or $[\pm C_4]\backslash$edge. Since the latter contracts to $[\pm K_3]$, we may ignore it. Since K_4 is cubic, a subgraph contracting to $[-K_4]$ is a subdivision of it.

A subgraph Ψ such that Ψ contracts to $[\pm K_3]$, if not a cycle of three unbalanced polygons, must contract to $[\Sigma]$ where Σ is $+K_4$ with e_{12}^+ doubled by e_{12}^- and with e_{14} and e_{24} made negative. ($[\Sigma]$ is obtained from $[\pm K_3]$ by splitting a vertex so as not to form a cycle of unbalanced polygons.) But $[\Sigma]\backslash e_{12}^+ = [-K_4]$, so $[\Sigma]$ need not be specifically excluded. □

Corollary 3C. *Let Ω be a loopless biased graph. For $L(\Omega)$ to be regular it is necessary and sufficient that Ω be either (1) an unbalanced triple parallel connection of balanced graphs or else (2) sign-biased containing no subgraph contracting to $[\pm K_3] \cup \{$unbalanced loop$\}$ or $[-K_4] \cup \{$unbalanced loop$\}$ and containing no cycle of three unbalanced polygons unless it be a subdivision of $[\pm K_3]$.*

Proof. This follows from results of [16] and Proposition 3A, except that we have to prove it suffices to exclude any cycle of three unbalanced polygons having a nontrivial path P^1 rather than having to exclude any subgraph contracting to $\Psi = [\pm C_4]\backslash$edge. Suppose some subgraph, not a cycle of three unbalanced polygons with nontrivial P^1, contracts to Ψ. Then it can be contracted to Υ, obtained from $[-K_4]$ by replacing one edge by a path of length 2 with one edge e^+ doubled by e^- having opposite sign. (This is the result of splitting a tetravalent vertex of Ψ so as not to have a cycle of polygons.) But Υ contracted by e^+ or e^- is $[-K_4] \cup \{$unbalanced loop$\}$. □

Corollary 3D. *If a biased graph Ω has no two vertex-disjoint unbalanced polygons, then either $L(\Omega)$ is regular or else it is not binary and Ω is not sign-biased.*

Proof. If $L(\Omega)$ is binary and Ω is not sign-biased, then $L(\Omega)$ is graphic (from [16]).

If Ω is sign-biased but $L(\Omega)$ is not regular, then $L(\Omega)$ has a minor $L(\Psi)$ that is minimally non-regular. We apply Proposition 3A to Ψ to deduce that it has two vertex-disjoint unbalanced polygons, one of which might be a loop. They can only be contractions of vertex-disjoint unbalanced polygons in Ω, contrary to hypothesis. □

The sign-biased case of Corollary 3D was observed first by Lovász and Schrijver [2].

4. Graphicity: the Matroids $G^{\perp}(K_{3,3})$ and $G^{\perp}(K_5)$

We need some special graphs for this section. The graph Δ_6 is the *prism*: two vertex-disjoint triangles joined by three independent edges. The signed graph Σ_4 is derived from $+K_4$ by doubling two opositive edges with negative edges and negating the sign of one simple edge. By $\pm K_4 \backslash E(+K_3)$ we mean $\pm K_4$ with a balanced triangle removed; this equals the contraction $(-K_5)/e$, where e is any edge. Two six-vertex signed graphs are Σ_{6a}, which is $+K_{3,3}$ (whose edges we label ij, meaning from left vertex x_i to right vertex y_j) with the edges 13 and 31 made negative and a negative edge 11^- doubling 11^+, and Σ_{6b}, which is obtained from Σ_{6a} through replacing 31^- by an edge 33^- doubling 33^+.

Proposition 4A. *Let Ω be a biased graph. We have*

$$G(\Omega) \cong G^{\perp}(K_{3,3}) \Leftrightarrow \Omega = [\pm K_4 \backslash E(+K_3)],$$
$$G(\Omega) \cong G^{\perp}(K_5) \Leftrightarrow \Omega = [-W_5] \text{ or } [\Sigma_{6a}].$$

If Ω is loop free, then

$$L(\Omega) \cong G^{\perp}(K_{3,3}) \Leftrightarrow \Omega = [\pm K_4 \backslash E(+K_3)],$$
$$L(\Omega) = G^{\perp}(K_5) \Leftrightarrow \Omega = [-W_5] \text{ or } [\Sigma_{6a}] \text{ or } [\Sigma_{6b}],$$

and for the complete lift,

$$L_0(\Omega) \cong G^{\perp}(K_{3,3}) \Leftrightarrow \Omega = [\Sigma_4],$$
$$L_0(\Omega) \cong G^{\perp}(K_5) \Leftrightarrow \Omega = [-\Delta_6].$$

Proof. In no case can Ω be balanced.

Consider the case of $G^{\perp}(K_{3,3})$. Then $n = 4$. Suppose first that $L_0(\Omega) \cong G^{\perp}(K_{3,3})$ with $e_0 \to 33$, say. Then $G(\Gamma) = L_0(\Omega)/e_0 \cong G^{\perp}(K_{3,3} \backslash 33)$ implies by planar duality that Γ is K_4 with two opposite edges doubled in parallel. The parallel pairs correspond to $\{13, 23\}$ and to $\{31, 32\}$. The remaining edges form a circuit in Γ in the order 12, 22, 21, 11. The vertex stars of $K_{3,3}$ give six circuits of $L_0(\Omega)$ which imply that the edges corresponding to $\{13, 23\}$ and $\{31, 32\}$ form unbalanced digons and those corresponding to {11, 21, 31}, {11, 12, 13}, {12, 22, 32}, and {21, 22, 23} form balanced triangles. These six polygons can be drawn as the faces of a plane embedding of Γ; hence they span the cycle space. Since their balance properties are compatible with Lemma 1A (they have one linear relation under symmetric difference: they sum to $\varnothing$; and an even number are unbalanced), they determine the bias of Ω. In fact, $\Omega = [\Sigma_4]$.

To verify that $L_0(\Sigma_4) \cong G^{\perp}(K_{3,3})$ we note that, for the basis $B = \{11, 21, 12, 22\}$ of the latter, whose corresponding edges are a basis of $L_0(\Sigma_4)$, the fundamental circuit of any edge is the same in both matroids. Let $\bar{B}(e)$ be the bond of $K_{3,3}$ contained in $E(K_{3,3})\backslash B\backslash e$, for $e \notin B$. The verification that, for each $f = e_0$ or $f \in E(\Sigma_4)\backslash B$, its fundamental circuit in $L_0(\Sigma_4)$ is $\bar{B}(e)$, is an easy matter of inspection that we leave to the reader.

Suppose that $L(\Omega) \cong G^{\perp}(K_{3,3})$. A bond of Γ, if its complement is unbalanced (and therefore a copoint of $L(\Omega)$), corresponds to a circuit of $K_{3,3}$, which is even. Therefore, Γ cannot contain a subgraph $2K_3$ or a pair of vertex-disjoint double edges. It follows that Ω is K_4 with a triangle removed, say $+K_3$ or $-K_3$ on vertices $v_1v_2v_3$. There are six balanced triangles at v_4 in Ω, corresponding to the six vertex stars of $K_{3,3}$; since $G^{\perp}(K_{3,3})$ has no other 3-circuits, it must be $+K_3$ that is removed. The three left vertex stars $S(x_i)$ of $K_{3,3}$, being disjoint, correspond to disjoint balanced triangles in Ω; similarly for the right vertex stars $S(y_i)$. Without loss of generality we may say $S(x_i) \leftrightarrow \{e^-_{i,i+1}, e^+_{i,4}, e^-_{i+1,4}\}$ and $S(y_i) \leftrightarrow \{e^-_{i,i+1}, e^-_{i,4}, e^+_{i+1,4}\}$. Thus we have the edge correspondence $ii \leftrightarrow e^-_{i,i+1}$, $(i, i-1) \leftrightarrow e^+_{i4}$, $(i-1, i) \leftrightarrow e^-_{i4}$.

We must show this really is an isomorphism. It suffices to show that the complement of a polygon in $K_{3,3}$ is a plane (a copoint) in $L(\Omega)$. The complement of a hexagon is a perfect matching. This could be {11, 22, 33}, which corresponds to the induced subgraph of Ω on $v_1v_2v_3$, which is an unbalanced plane; or {12, 23, 31} [or {21, 32, 13}], corresponding to the negative [or positive] star at v_4, which is a maximal balanced set and therefore a plane. The complement of a quadrilateral is the undeleted neighborhood of an edge, $N_{ij} = S(x_i) \cup S(y_j)$. N_{ii} corresponds to the induced subgraph of Ω on $v_iv_{i+1}v_4$, which is an unbalanced plane. $N_{i,i-1}$ corresponds to the balanced plane $\{e^-_{i-1,i}, e^-_{i,i+1}, e^-_{i-1,4}, e^+_{i4}, e^-_{i+1,4}\}$ and $N_{i-1,i}$ is similar, with the signs of the last three edges reversed. Thus $L(\Omega)$ is isomorphic to $G^{\perp}(K_{3,3})$ by our edge correspondence.

If $G(\Omega) \cong G^{\perp}(K_{3,3})$, where Ω may have loops, there is only the example $\Omega = [\pm K_4\backslash E(+K_3)]$, because Σ_4 already (without the extra loop needed for $G(\Omega)$) has vertex-disjoint unbalanced polygons.

Now we treat $G^{\perp}(K_5)$. Since it is not graphic, Ω is unbalanced and $n = 6$. An important fact is that $G^{\perp}(K_5)$ has no 3-circuits. Its circuits are the vertex stars S_i and the bonds $D_{ij} = \{e_{ik}, e_{jk}: k \neq i, j\}$, having respectively four and six elements.

If $L_0(\Omega) \cong G^{\perp}(K_5)$, e_0 corresponding (let us say) to the edge e_{12} of K_5, then $G(\Gamma) \cong G^{\perp}(K_5\backslash e_{12})$, which by planar duality gives $\Gamma = \Delta_6$. Let $f: E(K_5) \to E(\Delta_6) \cup \{e_0\}$ be the edge correspondence. The triangles of Δ_6 correspond to e_{13}, e_{14}, e_{15} and e_{23}, e_{24}, e_{25} in K_5. The connecting edge $f(e_{ij})$, where $ij = 34$, 35, or 45, is adjacent to $f(e_{1i})$, $f(e_{2i})$, $f(e_{1j})$, $f(e_{2j})$. Both triangles in Δ_6 are unbalanced; with e_0 they correspond to S_1 and S_2. The three quadrilaterals of Δ_6 correspond to S_3, S_4, and S_5; consequently, they are balanced. We now know everything about balance in Ω: it is indeed $[-\Delta_6]$. To verify the isomorphism we check that each $f(D_{ij})$ is a circuit in $L_0(\Omega)$. D_{12} corresponds to the two triangles, which are a circuit. $f(D_{13})$ is a pentagon of Ω (unbalanced) together with e_0. $f(D_{34})$ is a hexagon in Ω, balanced, hence a circuit. These three represent all cases. So the isomorphism does hold.

Suppose $L(\Omega) \cong G^{\perp}(K_5)$. Since Γ is edge-3-connected, its minimum degree is three and it has, besides trivalent vertices, either one pentavalent or two tetravalent vertices. If it has a double edge, it can have no triangles, for if it did, there would be an odd circuit: either a balanced triangle (this must happen with a triangle that shares one of the double edges), or a 5-circuit composed of the double edge and the triangle. Let the vertices of Γ be $v_1, v_2, \ldots$ on the left and $w_1, w_2, \ldots$ on the right.

There cannot be three double edges in Γ, because any two edge pairs would form a circuit corresponding to a vertex star S_i and two of these circuits would meet in two edges, which is impossible in $G^{\perp}(K_5)$.

Suppose Γ has two double edges. If we contract each pair we get a graph with 4 vertices and 6 edges that is edge-3-connected. This can only be K_4. Thus Γ must have one edge between each two of the two double edge pairs and remaining two vertices. Since Γ is triangle-free, Γ must be $\|\Sigma_{6b}\|$. Let the two double edges constitute the circuit $f(S_1)$, say. The other 4-circuits $f(S_i)$ must be balanced quadrilaterals in Ω. There are only two quadrilaterals on each edge in $f(S_1)$. If, say, $f(e_{12})$ and $f(e_{13})$ are parallel in Γ, we must assign one quadrilateral on $f(e_{12})$ to be $f(S_2)$ and the unique edge-disjoint quadrilateral on $f(e_{13})$ to be $f(S_3)$. Similarly we assign $f(S_4)$ and $f(S_5)$. The choices here, which do not sacrifice generality, determine the edge labellings and the balance of all polygons. One such choice gives the edge correspondence f defined by

$$e_{12} \to 11^+,\ e_{13} \to 11^-,\ e_{14} \to 33^+,\ e_{15} \to 33^-,\ e_{23} \to 21^+,$$
$$e_{24} \to 22^+,\ e_{25} \to 12^+,\ e_{34} \to 23^+,\ e_{35} \to 32^+,\ e_{45} \to 13^-.$$

This implies that $\Omega = [\Sigma_{6b}]$.

We must prove this is an isomorphism of $G^{\perp}(K_5)$ with $L(\Sigma_{6b})$. For this purpose we choose the basis $B = E \backslash S_1$ of $G^{\perp}(K_5)$; clearly $f(B)$ is a basis of $L(\Sigma_{6b})$. We have to verify that the circuit in $f(B) \cup \{f(1i)\}$ is the set corresponding to S_i. This is immediate by inspection. Thus, we have one example of Ω.

If Γ has just one double edge, contracting it gives a graph Γ' with 5 vertices and 8 edges, none doubled, and which is edge-3-connected. Since the minimum degree is three, the two edges of K_5 not present in Γ' are non-adjacent. We conclude that $\Gamma' = W_4$. If the vertex representing the contracted double edge were not the hub, Γ would have a triangle. It follows that $\Gamma = \|\Sigma_{6a}\|$. Of the two quadrilaterals on $v_1 v_2 w_1 w_2$ in Γ, one must be balanced. Let us sign it all positive and arbitrarily call it $f(S_1)$, specifically letting $f(e_{12}) = 11^+$, $f(e_{13}) = 12^+$, $f(e_{14}) = 22^+$, $f(e_{15}) = 21^+$. The only other quadrilateral on 22^+ that can be balanced, that is the one on $v_2 v_3 w_2 w_3$, must therefore be $f(S_4)$. We may declare it all positive. At w_2 we have $f(e_{13}) = 12^+$, $f(e_{14}) = 22^+$; the quadrilateral $f(S_3)$ must therefore continue at w_2 with the edge 32^+ which consequently $= f(e_{34})$. Similarly $f(e_{45}) = 23^+$, leaving $f(e_{24}) = 33^+$. To complete $f(S_3)$ we can only use 12^- and the edge, necessarily negative for balance of $f(S_3)$, 31^-. Similarly 12^-, $13^- \in f(S_5)$. So we have $f(e_{35}) = 12^-$, hence $f(e_{23}) = 31^-$, $f(e_{25}) = 13^-$. This determines f and consequently $f(S_2)$, which is fortunately a balanced quadrilateral. We now have $\Omega = [\Sigma_{6a}]$.

In order to verify that f is an isomorphism we choose the same basis $B = E \backslash S_1$ of $G^\perp(K_5)$. It corresponds to a basis of $L(\Sigma_{6a})$. We should verify that the circuit in $f(B) \cup \{f(e_{1i})\}$ is $f(S_i)$; the slight labor we leave to the reader.

Suppose now that Γ has no double edges. If it has two tetravalent vertices which are nonadjacent, it must have the form of two independent vertices v_1, v_2 joined to all vertices of a two-edge matching. Then every quadrilateral passes through v_1 and v_2; every two quadrilaterals meet in 0 or 2 edges. But this is impossible for balanced quadrilaterals, if Ω exists. So there is no possible Ω here. If Γ has two tetravalent vertices v_1, v_2 that are adjacent, they may have the same neighbors. Then Γ is $K_{3,3}$ with an added edge v_1v_2. The added edge lies in no quadrilateral, hence it cannot be in a 4-circuit of $L(\Omega)$. This is therefore another impossible form of Γ. But suppose v_1, v_2 do not have the same neighbors. Then Γ consists of a path $w_1w_2w_3w_4$ and v_1 adjacent to all but w_3, v_2 adjacent to all but w_2. Here edge w_1w_2 lies in only one quadrilateral so it cannot be in two 4-edge circuits. Thus no Ω can exist on this Γ either.

Therefore Γ has a vertex of degree 5 and equals the wheel $W_5 = C_5 + v$. There are exactly five quadrilaterals. We arbitrarily choose $f(S_i)$ to be the set $\{v_{i-1}v_i, v_iv_{i+1}, vv_{i-1}, vv_{i+1}\}$. This determines f on edges: it is given by

$$f(i, i+1) = v_iv_{i+1}, \quad f(i-1, i+1) = vv_i.$$

Since all triangles are unbalanced, $\Omega = [-W_5]$.

To establish isomorphism we note that the 4-edge circuits of $G^\perp(K_5)$ are balanced quadrilaterals and the 6-edge circuits D_{ij} are balanced hexagons in Ω.

Suppose $G(\Omega) \cong G^\perp(K_5)$. If Ω has no loop, then $L(\Omega)$ must equal $G(\Omega)$. This occurs only for $\Omega = [-W_5]$ and $[\Sigma_{6a}]$. If Ω has a loop, Ω without its loop must have $L_0 \cong G^\perp(K_5)$ and must also have no two vertex-disjoint unbalanced triangles. Our results show this to be impossible. □

Corollary 4A. *If Ψ is a biased graph, without loops or balanced digons, having $[\Sigma_4]$ or $[-\Delta_6]$ as a spanning proper subgraph, then $L_0(\Psi)$ is not regular.*

Proof. Suppose $\Psi \supseteq [\Sigma_4]$. Adding any edge that keeps the graph sign-biased (so its matroid remains binary) gives a subgraph $[\pm C_4] \backslash$edge, which contracts to $[\pm K_3]$.

If $\Psi \supseteq [-\Delta_6]$ and Ψ is sign-biased (so $L_0(\Psi)$ is binary) there are three cases. Let the triangles of Δ_6 have vertex sets $v_1v_2v_3$ and $w_1w_2w_3$ and let v_i, w_i be adjacent for each i. Doubling an existing edge (with opposite sign) creates a subgraph that is a subdivision of $[\pm C_4] \backslash$edge. Adding a simple positive edge, say $v_1w_2{}^+$, and deleting $v_1w_1{}^-$ gives a graph that contracts to $[\pm C_4] \backslash$edge. Adding a negative simple edge, say $v_1w_2{}^-$, and contracting by $v_1v_2{}^-$ and $v_1w_2{}^-$ creates a subgraph $[\pm C_4] \backslash$edge. In each case the enlarged graph $\Psi \cup \{\text{edge}\}$ has non-regular complete lift matroid. □

Corollary 4B. (1) *If $L(\Omega) \cong G^\perp(K_{3,3})$, then $L_0(\Omega)$ contains a submatroid isomorphic to $F_7^\perp$ and $L_0(\Omega)/e$, for some edge e, contains a submatroid isomorphic to F_7.*

(2) *If* $L(\Omega) \cong G^{\perp}(K_5)$ *and, for every edge* e, $L_0(\Omega)/e$ *is regular, then* $\Omega \cong [\Sigma_{6b}]$. $L_0(\Sigma_{6b})$ *itself is regular but not every contraction of it by an edge is graphic.*

Proof. (1) The first part is obvious. For the second, contract an edge of the simple triangle in Ω to get a graph containing $[\pm K_3]$.

(2) If $\Omega = [-W_5]$, then deleting two consecutive spokes and contracting by two of the three resulting series edges gives $[-K_4]$.

If $\Omega = [\Sigma_{6a}]$, contracting by the edge 33^+ gives a graph containing $[\pm K_3]$.

If $\Omega = [\Sigma_{6b}]$, contracting by 32^+ and 21^+ gives $[\Sigma_4]$, whose L_0 is regular but not graphic. No contraction of $[\Sigma_{6b}]$ by a balanced edge set contains either $[\pm K_3]$ or $[-K_4]$; thus $L_0(\Sigma_{6b})$ is regular. To prove this, we first consider $[-K_4]$. To find a subdivision of K_4 in $\|\Sigma_{6b}\|$ we can ignore the parallel edges; the simplified $\|\Sigma_{6b}\|$ is a subdivision of K_4 and this is the only such subdivision in $\|\Sigma_{6b}\|$. By choosing which of the parallel edges to drop we can choose the signs of two edges in the simplified Σ_{6b}, but we cannot choose them so as to make all (subdivided) triangles unbalanced.

If $[\Sigma_{6b}]$ contained a subdivision of $[\pm K_3]$, it would contain three edge-disjoint unbalanced polygons. The only such polygons in Σ_{6b} are the two digons and $\{12^+, 22^+, 23^+, 13^-\}$. But these are joined in an open chain, not a cycle, by their vertices of pairwise intersection. So there is no subdivided $[\pm K_3]$. If a subgraph Ψ of $[\Sigma_{6b}]$ contracted to $[\pm K_3]$ but were not a subdivision, it would contract (perhaps trivially) to either $[\pm C_4]\backslash$edge or $[-K_4 \cup e^+]$ (where e^+ is any positive edge). We have shown the latter to be impossible. The former is impossible because it would imply $G^{\perp}(K_5) \cong L(\Sigma_{6b}) \geq L(\pm C_4\backslash\text{edge}) \cong F_7^{\perp}$. □

Corollary 4C. *Let* Ψ *be a biased graph without balanced loops or digons. If* Ψ *contains a spanning proper subgraph* Ω *with* $G(\Omega)$ *or* $L(\Omega)$ *isomorphic to* $G^{\perp}(K_{3,3})$, *then* $G(\Psi)$ *and* $L(\Psi)$ *are not regular.*

Proof. If Ψ is not sign-biased, then $L(\Psi)$ and $G(\Psi)$ are not binary, by the theorems of [16]. If it is sign-biased then adding an edge to Ω creates a $[\pm C_4]\backslash$edge. Then $G(\Psi)$ is not binary and $L(\Psi)$ is binary but not regular. □

The analogous proposition for $G(\Omega)$ or $L(\Omega) \cong G^{\perp}(K_5)$ is false. A counterexample is $[\Psi_5]$, where $\Psi_m = \pm W_m \backslash E(+C_m)$, that is, a negative wheel with all spokes doubled by positive edges. Any contraction of Ψ_m by a spoke edge has a balancing vertex. Thus by Lemma 1G the L and G of this contraction are graphic. The contraction of $[\Psi_m]$ by a rim edge is $[\Psi_{m-1}]$, if we simplify balanced digons to single edges. Since $G(\Psi_3) = L(\Psi_3) = G^{\perp}(K_{3,3})$ by Proposition 4A, an argument by induction demonstrates that all $L(\Psi_m) = G(\Psi_m)$ are regular.

5. Cographicity: the Matroids $G(K_{3,3})$ and $G(K_m)$, $m \geq 5$

For this section we need some more special signed graphs. If $W_4 = C_4 + v$ is the wheel with hub v, let $W_4^{(v)}$ be the wheel with a loop at v. By Φ'_{m-1} we denote $(+K_m \cup \{e_{1m}^-\})/e_{1m}^-$; this has the single loop e_{1m}^+ (which is negative), and by Φ_{m-1} we mean $\Phi'_{m-1}\backslash\{e_{1m}^+\}$. Another description of Φ_{m-1} is as $\pm K_{m-1}\backslash E(-K_{m-2})$, that is,

$\pm K_{m-1}$ with the edges of a negative K_{m-2} removed (on vertices $v_2, v_3, \ldots, v_{m-1}$, to be precise); then Φ'_{m-1} is Φ_{m-1} with a negative loop adjoined to v_1. By Ψ_5 we mean $\pm K_{2,3}$ with the three negative edges at one left vertex removed; equivalently, $\Psi_5 = (+K_{3,3} \cup \{v_1 v_2^-\})/v_1 v_2^-$. Let v be the vertex of Ψ_5 corresponding to $\{v_1, v_2\}$ in $K_{3,3}$. Then $\{v, v_3\}$ is a cutpair in Ψ_5. If we reverse one of its bridges we get Ψ_{5t} (a "twisted" Ψ_5).

Proposition 5A. *Let Ω be a biased graph and let $m \geq 5$. Then*

$$G(\Omega) \cong G(K_{3,3}) \Leftrightarrow \Omega = [K_{3,3}], [-K_5]\backslash \textit{edge}, [\Psi_5], \textit{ or } [-W_4^{(v)}],$$
$$G(\Omega) \cong G(K_m) \Leftrightarrow \Omega = [K_m] \textit{ or } [\Phi'_{m-1}].$$

If Ω has no loops, then

$$L(\Omega) \cong G(K_{3,3}) \Leftrightarrow \Omega = [K_{3,3}], [-K_5]\backslash \textit{edge}, [\Psi_5], \textit{ or } [\Psi_{5t}],$$
$$L(\Omega) = G(K_m) \Leftrightarrow \Omega = [K_m],$$

and also

$$L_0(\Omega) \cong G(K_{3,3}) \Leftrightarrow \Omega = [-W_4],$$
$$L_0(\Omega) \cong G(K_m) \Leftrightarrow \Omega = [\Phi_m].$$

Remark. Comparing Propositions 4A and 5A shows some coincidences. The example giving $L_0(\Omega) \cong G(K_5)$ is the negation of that giving $L(\Omega) \cong G^{\perp}(K_{3,3})$. Replacing the simple triangle in either by a claw gives $[\Psi_5]$, whose lift matroid is $G(K_{3,3})$. I do not know whether there is any systematic reason for these similarities.

Proof. We may as well assume Ω is unbalanced, for if it is balanced then $L(\Omega) = G(\|\Omega\|)$ implies $\Omega = [K_m]$ or $[K_{3,3}]$ and $L_0(\Omega) = G(\|\Omega\|) \oplus e_0$ shows there is no possible Ω.

Consider the case of $G(K_m)$. Then we have $n = m - 1$. If $L_0(\Omega) \cong G(K_m)$, then $G(\Gamma) \cong L_0(\Omega)/e_0 \cong G(K_m/e_{1m})$ (where we take $e_0 \leftrightarrow e_{1m}$ in K_m), whence Γ is K_{m-1} with all edges at v_1 doubled. The simple K_{m-2} in Γ corresponds to a complete subgraph in K_m, so it is balanced. Thus, $\Omega = [\Phi_{m-1}]$. To establish isomorphism we apply Lemma 1F to $\Gamma = K_m\backslash\{e_{1m}\}$ and $e = e_{1m}$.

If $L(\Omega) \cong G(K_5)$, we have 10 edges on 4 vertices in Ω so there are at least four double edges. One can easily see that Γ must therefore contain $2C_4\backslash$edge. Since $G(K_5)$ is regular, Ω cannot contain $[\pm C_4]\backslash$edge (by Proposition 3A). But this is an inconsistency, so no Ω exists.

Suppose f is an isomorphism $G(K_m) \cong L(\Omega)$ where $m > 5$ and Ω is unbalanced. Let $V(K_m) = \{x_1, x_2, \ldots, x_m\}$. By induction on m every subgraph of Ω corresponding to $E_i = E(K_m\backslash x_i)$ is balanced. For example, the edges in $f(E_m)$ constitute a balanced K_{m-1}; let us say (by switching) it is $+K_{m-1}$ on vertex set $\{v_1, \ldots, v_{m-1}\} = V(\Omega)$ and let $f(e_{ij}) = e_{ij}^+$. The edge set $f(E_1)$ is also a balanced K_{m-1} meeting $+K_{m-1}$ in the edges of $+K_{m-1}\backslash v_1$; hence their union is $E(\pm K_{m-1})\backslash E(-K_{m-2})$, where $-K_{m-2}$ is on the vertices $v_2, \ldots, v_{m-1}$. But we have a contradiction: $f(E_3)$ contains the unbalanced edge pair $e_{12}^+ = f(e_{12}), e_{12}^- = f(e_{2m})$. So Ω could not have been unbalanced.

In the case of $G(K_{3,3})$ we have $n = 5$. All circuits have four or six edges. If $L_0(\Omega) \cong G(K_{3,3})$, let e_0 correspond to 33 in $K_{3,3}$; then $G(\Gamma) = L_0(\Omega)/e_0 \cong$

$G(K_{3,3}/33)$, so $\Gamma \cong K_{3,3}/33 \cong W_4$. Since all triangles are unbalanced, $\Omega = [-W_4]$. To prove isomorphism we apply Lemma 1F to $(+K_{3,3} \cup \{33^-\})/33^-$.

If $L(\Omega) \cong G(K_{3,3})$, we have 9 edges on 5 vertices. Suppose Γ is simple. Then $\Gamma = K_5 \backslash$edge and $\Omega = -\Gamma$ becuase there are no 3-edge circuits. To prove isomorphism we let the edge of K_5 not in Γ be e_{45} and we define $f: E(K_{3,3}) \to E(\Gamma)$ by

$$f(i1) = e_{i4}, \quad f(i2) = e_{i5},$$
$$f(13) = e_{23}, \quad f(23) = e_{13}, \quad f(33) = e_{12}.$$

The set $B = \{11, 12, 13, 21, 31\}$ is a basis for $G(K_{3,3})$ and $f(B)$ is one for $L(\Omega)$. It is easy to verify by inspection that the fundamental circuits $B(e)$ of $e \notin B$ with respect to B are preserved by f.

Suppose Γ is not simple. In that case Ω is bipartite; for if it had a balanced odd polygon it would have an odd circuit, while an unbalanced odd polgyon C and a double edge $\{e^+, e^-\}$ give either a balanced polygon if, say, $e^+ \in C$, or an odd circuit $C \cup \{e^+, e^-\}$. Edge-3-connectedess of Ω obliges the bipartition of $V(\Omega)$ to have left set of order two, say $\{v_1, v_2\}$, and right set $\{w_1, w_2, w_3\}$. There cannot be four double edges because that would imply a quadrilateral in $K_{3,3}$ having the same two edges in common with two other quadrilaterals, which is impossible. So Γ contains a complete bipartite graph $K_{2,3}$ and three double edges. If the double edges are distributed one to each bridge of $\{v_1, v_2\}$, we have $\Omega = [\Psi_5]$ or $[\Psi_{5t}]$. Otherwise, we would have two double edges, say $e_{11}^+e_{11}^-$ and $e_{21}^+e_{21}^-$, at w_1, one at w_2 (say $e_{12}^+e_{12}^-$), and no more. Now again we have a quadrilateral sharing the same two edges with two other quadrilaterals: it is $C_1 = \{e_{i1}^+, e_{i1}^-: i = 1, 2\}$. By switching we may assume the simple edges are all positive. Then $C_2 = \{e_{ij}^+: i = 1, 2; j = 1, 2\}$ and $C_3 = \{e_{ij}^+: i = 1, 2; j = 1, 3\}$ are the two other quadrilaterals. So we have found all the possible Ω.

We still have to prove $L(\Psi_5)$ and $L(\Psi_{5t})$ isomorphic to $G(K_{3,3})$. For Ψ_5 we apply Lemma 1F to $K_{3,3}$ and $e =$ an edge within the left side of $K_{3,3}$. As for Ψ_{5t}, it is 2-isomorphic to Ψ_5, so their matroids are isomorphic by Lemma 1C. □

6. The Matroid R_{10}

In $GF(2)^5$ let $\mathbf{b}_i$ denote the i-th basis vector, $\mathbf{j}$ the all ones vector, and $\mathbf{x}_{ij} = \mathbf{j} + \mathbf{b}_i + \mathbf{b}_j$. R_{10} by definition is the linear dependence matroid of $\{\mathbf{x}_{ij}: 1 \le i < j \le 5\}$.

Proposition 6A. *For a biased graph Ω we have*

$$G(\Omega) \cong R_{10} \Leftrightarrow L(\Omega) \cong R_{10} \Leftrightarrow \Omega = [-K_5]$$

and $L_0(\Omega) \cong R_{10}$ is impossible.

We begin with a Lemma demonstrating that $-K_5$ is an example.

Lemma 6A. *We have $G(-K_5) \cong R_{10}$.*

Proof. We show that the circuits in R_{10} are the same as those of $G(-K_5)$. We identify $\mathbf{x}_{ij}$ with the edge ij in K_5; thus each subset of R_{10} is an edge subset of K_5. Suppose

$S \subseteq E(K_5)$ gives a linear relation

$$\sum_{ij \in S} \mathbf{x}_{ij} = \mathbf{0}.$$

Equivalently,

$$\sum_{ij \in S} \mathbf{j} + \sum_{i} \mathbf{b}_i \left(\sum_{j: ij \in S} 1 \right) = \mathbf{0}.$$

Either S is even and has all even degrees, or S is odd and has all odd degrees. Since K_5 has odd order, the latter is not possible. If S has degrees 0 and 2 only, it is a polygon, necessarily of length four. If it has a vertex of degree greater than 2, say v_5, it has the form $E(T + v_5)$ where T is a subgraph of K_4 of odd degree. Thus T is a pair of independent edges and S consists of two triangles joined at a vertex, or $T = K_4$ and $S = E(K_5)$. We see that the circuits of R_{10} are exactly those of $G(-K_5)$. □

Proof of Proposition 6A. We use the fact that, if e is any point of R_{10}, then $R_{10} \backslash e \cong G(K_{3,3})$ and $R_{10}/e \cong G^{\perp}(K_{3,3})$ (see [4]; these facts are also corollaries of Lemma 6A and Propositions 4A and 5A). Note that $R_{10} \backslash e$ spans R_{10} and all circuits in R_{10} have even size 4 or 6.

If $L_0(\Omega) \cong R_{10}$, then $G(\|\Omega\|) \cong L_0(\Omega)/e_0 \cong R_{10}/e_0 \cong G^{\perp}(K_{3,3})$, a contradiction because $G^{\perp}(K_{3,3})$ is not graphic.

Let $L(\Omega) \cong R_{10}$ where Ω is loopless. Since R_{10} is not graphic, Ω is unbalanced and $n = \operatorname{rk} R_{10} = 5$. By Proposition 5A, for each $e \in E(\Omega)$ we have $\Omega \backslash e \cong [-K_5] \backslash \text{edge}$, or $[\Psi_5]$, or $[\Psi_{5t}]$. If Ω has no double edge, $\Omega = [-K_5]$ is the only possibility. Suppose Ω has a double edge $\{e^+, e^-\}$. Then some $\Omega \backslash e$ has a double edge, hence three double edges; consequently Ω is bipartite with left set of order two and has four double edges. But then some edge deletion has four double edges and is therefore not isomorphic to $[\Psi_5]$ or $[\Psi_{5t}]$, a contradiction. □

7. If Every Contraction of the Complete Lift is Graphic

For the isomorphism theory of the complete lift [17] we want to know all pairs (Ω, e) such that $e \in E$ and $L_0(\Omega)/e$ is graphic. The foregoing propositions do not solve that problem but they do suffice to decide which Ω have the property that *every* $L_0(\Omega)/e$ is graphic.

Proposition 7A. *Let Ω be a biased graph without loops or balanced digons.*
(1) *Every contraction $L_0(\Omega)/e$, $e \in E$, is graphic if and only if $L_0(\Omega)$ is graphic or $\Omega = [\pm K_3], [-K_4], [-\Delta_6], [\Sigma_4]$, or $(mK_2, \varnothing)$, $m \geq 3$.*
(2) *Every contraction $L_0(\Omega)/e$, $e \in E$, is regular if and only if $L_0(\Omega)$ is regular or $\Omega = [\pm K_3], [-K_4]$, or $(mK_2, \varnothing)$, $m \geq 3$.*
(3) *Every contraction $L_0(\Omega)/e$, $e \in E$, is binary if and only if $L_0(\Omega)$ is binary or $\Omega = (mK_2, \varnothing)$, $m \geq 3$.*

Proof. (3) is obvious. We therefore assume that Ω is sign-biased so $L_0(\Omega)$ is binary. It is also obvious, since $L_0(\Omega)$ is a minimal non-graphic [or, non-regular] matroid, that each example listed is an example.

Suppose Ω is non-graphic [or, non-regular] but every proper contraction is graphic [regular]. Then, by Proposition 3A (on $L(\Omega) \cong F_7$) and Corollaries 3A and 4B, Ω contains one of the listed examples as a spanning subgraph Ψ. If $\Psi = [\pm K_3]$, there is no room for more edges, so $\Omega = \Psi$. If $\Psi = [-K_4]$, then adding a positive edge to $-K_4$ allows contracting to give $\pm K_3$, so $\Omega = \Psi$. This settles (2).

For $\Psi = [\Sigma_4]$ or $[-\Delta_6]$ the result follows from Corollary 4A, noting that $L_0(\Omega)$ has rank greater than $4 = \operatorname{rk} F_7^\perp > \operatorname{rk} F_7$. □

The corresponding problem for the bias or lift matroid seems (from partial calculations we do not reproduce here) to have a much more complicated solution.

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