THE RHODES SEMILATTICE OF A BIASED GRAPH

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Abstract. We reinterpret the Rhodes semilattices $R_n(\mathfrak{G})$ of a group $\mathfrak{G}$ in terms of gain graphs and generalize them to all gain graphs, both as sets of partition-potential pairs and as sets of subgraphs, and for the latter, further to biased graphs. Based on this we propose four different natural lattices in which the Rhodes semilattices and its generalizations are order ideals.

1. Introduction

The Rhodes semilattice $R_n(\mathfrak{G})$ of a group $\mathfrak{G}$, introduced by John Rhodes for semigroup theory (we refer to [2]), is a partial ordering of pairs consisting of a partition and a potential system on subsets of a finite set. We reinterpret the Rhodes semilattices as semilattices of subgraphs in complete link gain graphs, which are graphs with edges labeled from the group $\mathfrak{G}$. From this point of view the elements of the Rhodes semilattice are closed, balanced subgraphs and the ordering is by gain-graph inclusion. We use this reinterpretation to generalize Rhodes semilattices to semilattices of closed, balanced subgraphs of any gain graph and even more generally any biased graph. (All these concepts are defined below.) Based on our graphic interpretation we propose three natural lattices that contain the Rhodes semilattice and its generalizations as order ideals.

The graphical interpretation of $R_n(\mathfrak{G})$ was initiated by the second author in a referee's report on [2] and is mentioned in the published version; see [2, Remark 6.1]. Here we provide the full generalization suggested by that insight, with proof. Our partition-potential generalization of $R_n(\mathfrak{G})$ is newly developed by the first author.

2. A new perspective on the Rhodes semilattice

We begin with basic definitions, many of which are from [2], [3], and [5].

A *partial partition* $\pi = \{\pi_1, \dots, \pi_j\}$ of a set X is a partition of a subset of X. The *support* $\operatorname{supp} \pi$ is the union of all blocks of π. The set of partial partitions of an n-element set is denoted by $\Pi_n^\S$, which we partially order by refinement, i.e., $\tau \leq \pi$ if every block of τ is contained in a block of π. The

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smallest partial partition of X is the empty partition, $\emptyset$; the largest is the trivial partition, $\{X\}$.

For a graph Υ we define the partition of $V(\Upsilon)$ induced by Υ by $\pi(\Upsilon) = \{V(D) : D \text{ is a component of } \Upsilon\}$.

Let X be a finite set and $\mathfrak{G}$ a group. Given $I \subseteq X$, we denote by $\mathfrak{G}^I$ the collection of all functions $\theta : I \to \mathfrak{G}$; we call such a function a *potential function* on I. The group acts on the left of $\mathfrak{G}^I$ by $(g\theta)(x) = g\theta(x)$; we call an element of the quotient set $\mathfrak{G}^I/\mathfrak{G}$ a *potential* for I ([2] calls it a "cross section").

A *potential system* for a partial partition π is a collection of potentials on the blocks of π. Formally, if $\theta : Y \to \mathfrak{G}$ and $\eta : Z \to \mathfrak{G}$, where Y and Z are any subsets of X that contain $\operatorname{supp}\pi$, we write $\eta \sim_\pi \theta$ if $\eta|_{\pi_i} \in \mathfrak{G} \cdot \theta|_{\pi_i}$ for each block π_i in π; then $\sim_\pi$ is an equivalence relation. We denote the equivalence class of $\theta \in \mathfrak{G}^I$ for a partial partition π by $[\theta]_\pi$; this is a *potential system* on π.

The *Rhodes semilattice* (see [2, section 3]) is the set of all pairs $(\pi, [\theta]_\pi)$ in which $[\theta]_\pi$ is a potential system on π, which we call *partition-potential pairs*. They are partially ordered by restriction; that is, $(\tau, [\eta]_\tau) \leq (\pi, [\theta]_\pi])$ if τ refines π and $[\eta]_\tau = [\theta]_\tau$. The Rhodes semilattice has a meet operation defined by $(\pi, [\theta]_\pi) \wedge (\tau, [\eta]_\tau) = (\rho, [\theta]_\rho) = (\rho, [\eta]_\rho)$, where ρ is the partition of $\operatorname{supp}\rho = \operatorname{supp}\pi \cap \operatorname{supp}\tau$ induced by the equivalence relation $\sim$ defined on $\operatorname{supp}\rho$ in the following way: $x \sim y$ if and only if there exist blocks τ_i and π_j such that $x, y \in \tau_i \cap \pi_j$ and $\theta(x)^{-1}\theta(y) = \eta(x)^{-1}\eta(y)$. To distinguish this representation of the Rhodes semilattice from the graphical one to be defined, we call it the *partition-potential Rhodes semilattice*.

In what follows $\Gamma = (X, E)$ will denote a graph, which may have loops and multiple edges unless it is said to be simple. If $e \in E$, $\nu_\Gamma(e) = \{v, w\}$ means e has endpoints v, w (a multiset to account for loops). If $\Upsilon \subseteq \Gamma$ is a subgraph, $V(\Upsilon)$ is its vertex set and $E(\Upsilon)$ is its edge set. We will be most interested in the subgraph induced on a subset I of X, which is $\Gamma{:}I = (I, E{:}I)$ where

$$E{:}I = \{e \in E : \nu_\Gamma(e) \subseteq I\},$$

and the subgraph induced by a partial partition π of X, which is

$$\Gamma{:}\pi = \bigcup\{(\Gamma{:}\pi_i) : \pi_i \in \pi\}.$$

A *gain graph*, denoted by $\Phi = (\Gamma, \phi, \mathfrak{G})$, is a graph equipped with a *gain function* ϕ. The gain function is defined on the edges of the graph to the group $\mathfrak{G}$, such that reversing the direction of an edge inverts the gain. We write $\phi(e; v, w)$ in order to indicate the sense in which the gain is measured; thus $\phi(e; w, v) = \phi(e; v, w)^{-1}$. The gain of a path is its edge gain product; thus a path $P = v_0 e_1 v_1 e_2 \cdots e_k v_k$ has gain $\phi(P) = \phi(e_1; v_0, v_1)\phi(e_2; v_1, v_2) \cdots \phi(e_k; v_{k-1}, v_k)$. A circle is called *balanced* if its gain (considered as a closed path) is 1; this property is independent of its representation as a closed path. A subgraph Υ of a gain graph Φ is balanced if every circle in Υ is balanced. It is *closed and balanced* if it is balanced and whenever there is a balanced circle $C \subseteq \Phi$ such that $C \setminus e \in \Upsilon$, then $e \in E(\Upsilon)$.

Lemma 2.1. *In a balanced gain graph, the gain of a path depends only on its initial and final vertices.*

Proof. This is implicit in the proof of [3, Lemma 5.3]. □

A *potential function* for a balanced gain graph Φ is a function $\theta : V(\Phi) \to \mathfrak{G}$ such that $\phi(e; v, w) = \theta(v)^{-1}\theta(w)$ for each edge $e \in \Gamma$.

Lemma 2.2. *Let Φ be a gain graph. There is a potential function θ for Φ if and only if Φ is balanced. Every potential function in $[\theta]_{\pi(\Phi)}$ defines the same gains on Φ, and every potential function defining the gains of Φ is in $[\theta]_{\pi(\Phi)}$.*

Proof. Balance is clear by computing the gain of a circle.

Conversely, we can obtain a potential function for a balanced gain graph Φ by choosing a root node r for each component of Φ and defining $\theta(v) = \phi(P_{rv})$ where P_{rv} is an rv-path in a component of Φ. This is well defined because the gain function in a balanced gain graph is path independent.

Suppose $\phi(e) = \theta(v)^{-1}\theta(w)$ and η is another potential function of Φ, then (because the gain function in a balanced subgraph is path independent) for any path P_{vw} in Φ, $(\theta(v))^{-1}\theta(w) = \phi(P_{vw}) = (\eta(v))^{-1}\eta(w)$. This is true for every v, w in a block of $\pi(\Phi)$, so by definition $\eta \in [\theta]_{\pi(\Phi)}$. Conversely, if $\eta \in [\theta]_{\pi(\Phi)}$ then $(\theta(v))^{-1}\theta(w) = (\eta(v))^{-1}\eta(w)$ for any v, w in a block of $\pi(\Phi)$, so $\phi(e) = (\theta(v))^{-1}\theta(w) = (\eta(v))^{-1}\eta(w)$. □

The *group expansion* of a graph Γ is denoted by $\mathfrak{G}\cdot\Gamma = (V(K_n), \mathfrak{G} \times E(\Gamma), \phi)$, where $\nu_{\mathfrak{G}\cdot\Gamma}(g, e) = \nu_\Gamma(e)$. To define the gain of (g, e) we must take account of the endpoints, $\nu_{\mathfrak{G}\cdot\Gamma}(g, e) = \{v, w\}$. For the sake of notation, arbitrarily pick an orientation, $(e; v, w)$, and define $\phi(g, e; v, w) = g$, $\phi(g, e; w, v) = g^{-1}$ for each g. In this notation, (g, e) and (g^{-1}, e) are different edges (unless $g = g^{-1}$) whose gains are

$$\phi(g, e; v, w) = g,\ \phi(g, e; w, v) = g^{-1},$$
$$\phi(g^{-1}, e; v, w) = g,\ \phi(g^{-1}, e; w, v) = g.$$

Definition 2.3. The *graphic Rhodes semilattice of* $\mathfrak{G}\cdot K_n$, denoted by $R(\mathfrak{G}\cdot K_n)$, is the family of closed and balanced subgraphs in $\mathfrak{G}\cdot K_n$, ordered by inclusion. Its meet operation is intersection.

Theorem 2.4. *The partition-potential Rhodes semilattice $R_n(\mathfrak{G})$ is isomorphic to the graphic Rhodes semilattice $R^b(\mathfrak{G}\cdot K_n)$.*

Proof. In the natural correspondence between $R_n(\mathfrak{G})$ and $R^{\mathrm{b}}(\mathfrak{G}\cdot K_n)$, the pair $(\pi, [\theta]_\pi) \in R(\mathfrak{G})$ corresponds to the subgraph of $\mathfrak{G}\cdot K_n$ whose components are complete subgraphs in each vertex set π_i with gains given by any potential function in $[\theta]_\pi$. We correspond a closed and balanced subgraph B of $\mathfrak{G}{\cdot}K_n$ to the pair $(\pi(B), [\theta]_{\pi(B)})$, where θ is a potential function defining the gains of B. In Theorem 3.6 we prove (in more generality) that this correspondence is an isomorphism. □

Theorem 2.4 shows us how to generalize the Rhodes semilattice to gain and biased graphs, which we do in the next section.

3. Generalization to gain graphs

Let Φ be a gain graph with gain group $\mathfrak{G}$ and vertex set X. Let $\pi = \{\pi_1, \ldots, \pi_j\}$ be a partial partition of X and $[\theta]_\pi$ a potential system for π. We define a function $\mathbf{B}$ from partition-potential pairs to subgraphs of Φ by

$$\mathbf{B}(\pi, [\theta]_\pi) := (\operatorname{supp}\pi, \{e \in E(\Phi{:}\pi) : (\exists i)\ \nu_\Gamma(e) = \{v, w\} \subseteq \pi_i,\ \phi(e) = \theta^{-1}(v)\theta(w)\}).$$

This subgraph is well defined, by Lemma 2.2.

We say $(\pi, [\theta]_\pi)$ is a Φ-*connected* partition-potential pair if $\mathbf{B}(\pi, [\theta]_{\pi_i})$ is connected for each $\pi_i \in \pi$.

Definition 3.1. Let Φ be a gain graph with vertex set X and group $\mathfrak{G}$. The *partition-potential Rhodes semilattice of* Φ, denoted by $R(\Phi)$, is the set of all Φ-connected partition-potential pairs of Φ. The meet operation is the same as it is for $R_n(\mathfrak{G})$.

We can see that $R_n(\mathfrak{G}) = R(\mathfrak{G}{\cdot}K_n)$, so the partition-potential Rhodes semilattice of a gain graph is a generalization of the original Rhodes semilattice.

Definition 3.2. Let Φ be a gain graph. The *graphic Rhodes semilattice of* Φ, denoted by $R^{\mathrm{b}}(\Phi)$, is the family of closed and balanced subgraphs, ordered by inclusion. The meet operation is gain graph intersection.

Lemma 3.3. *Let Φ be a gain graph, π a partial partition of X, and $(\pi, [\theta]_\pi) \in R_n(\Phi)$. Then $\mathbf{B}(\pi, [\theta]_\pi)$ is a closed and balanced subgraph of Φ.*

Proof. $\mathbf{B}(\pi, [\theta]_\pi)$ is balanced because it has gains defined by a potential function. It is closed because if v, w are in a component of $\mathbf{B}(\pi, [\theta]_\pi)$, then the edge e with $\nu_\Gamma(\theta^{-1}(v)\theta(w), e) = \{v, w\}$ and gain $\phi(e; v, w) = \theta^{-1}(v)\theta(w)$, if it exists in Φ, is in $\mathbf{B}(\pi, [\theta]_\pi)$ by the definition of $\mathbf{B}$. □

Lemma 3.4. *Let Φ be a gain graph. $\mathbf{B}$ is a surjection onto the closed and balanced subgraphs of Φ.*

Proof. By Lemma 3.3 we know the image of $\mathbf{B}$ is contained in the set of closed and balanced subgraphs.

Let B be a closed and balanced subgraph. By Lemma 2.2 there is a potential function θ defining its gains. If $e \in B$ then its vertices are in the same block of $\pi(B)$; it follows that e is in $\mathbf{B}(\pi(B), [\theta]_{\pi(B)})$, which implies that $B \subseteq \mathbf{B}(\pi(B), [\theta]_{\pi(B)})$.

Now let e be an edge of $\mathbf{B}(\pi(B), [\theta]_{\pi(B)})$. The vertices of e are in one block of $\pi(B)$ and its gain $\phi(e; v, w) = \theta(v)^{-1}\theta(w)$. Since B is closed and θ is a potential function for B, e is in B; therefore $B = \mathbf{B}(\pi(B), [\theta]_{\pi(B)})$. □

Lemma 3.5. *Let Φ be a gain graph. If we restrict the domain of $\mathbf{B}$ to $R(\Phi)$, then $\mathbf{B}$ is injective.*

Proof. Suppose $(\tau, [\eta]_\tau) \in R(\Phi)$ and $\mathbf{B}(\tau, [\eta]) = B$. We showed in the previous proof that $B = \mathbf{B}(\pi(B), [\theta]_{\pi(B)})$. We now prove that, if $(\tau, [\eta]_\tau) \neq (\pi(B), [\theta]_{\pi(B)})$, then $(\tau, [\eta]_\tau) \notin R(\Phi)$.

First we observe that every block of $\pi(B)$ must be contained in a block of τ. Thus, $\pi(B) \leq \tau$. If two blocks $\pi_i, \pi_j \in \pi(B)$ are contained in the same block τ_k, or if some $\pi_i \subset \tau_k$, then τ_k does not induce a connected subgraph of B; such a partition-potential pair cannot be in $R(\Phi)$. Thus, the blocks of $\pi(B)$ are in distinct blocks of τ and are equal to those blocks; that is, $\tau = \pi(B)$.

Now Lemma 2.2 implies that $[\theta]_\tau = [\theta]_{\pi(B)}$, completing the proof. □

Theorem 3.6. *Let Φ be a gain graph. Then $R(\Phi)$ is isomorphic to $R^b(\Phi)$.*

Proof. We have shown $\mathbf{B}$ is a bijection. Now we prove it preserves order. By the definition $(\tau, [\eta]_\tau) \leq (\pi, [\theta]_\pi)$ if and only if each block $\tau_i \in \tau$ is contained in some block of π and $\mathfrak{G}\cdot\theta|_{\tau_i} = \mathfrak{G}\cdot\eta|_{\tau_i}$. Equivalently, each component of $\mathbf{B}(\tau, [\eta]_\tau)$ is contained in a component of $\mathbf{B}(\pi, [\theta]_\pi)$; that is, $\mathbf{B}(\tau, [\eta]_\tau) \subseteq \mathbf{B}(\pi, [\theta]_\pi)$. □

Example 3.7 (Group expansions)**.** If Φ is a group expansion we can simplify the description of the graphic Rhodes semilattice. Suppose Γ is a simple graph and $\mathfrak{G}$ is a group. A subgraph $\Psi \subseteq \Gamma$ is *closed in* Γ if, whenever e is an edge of Γ such that $\Psi \cup \{e\}$ contains a circle that contains e, then e is in Ψ.

For a subgraph B of $\mathfrak{G}\cdot\Gamma$, by $p(B)$ we mean the projection of B onto the underlying graph Γ. If B is closed and balanced subgraph, then $p(B)$ is a closed subgraph in Γ. Conversely, if C is closed in Γ and $[\theta]_{\pi(C)}$ is a potential system on $\pi(C)$, then $\mathbf{B}(\pi(C)), [\theta]_{\pi(C)})$ is a closed and balanced subgraph of $\mathfrak{G}\cdot\Gamma$.

This shows that the properties of closure and balance for elements of $R^{\mathrm{b}}(\mathfrak{G}\cdot\Gamma)$ can be split into closure of the underlying base graph in Γ and an arbitrary choice of potential for that base graph. If in particular $\Gamma = K_n$, the closed subgraphs correspond bijectively to the partial partitions of the vertex set of K_n.

4. Generalization to biased graphs

A *biased graph* is a graph together with a class of circles (edge sets or graphs of simple closed paths), called *balanced circles*, such that no theta subgraph contains exactly two balanced circles. We denote the graph along with the set of balanced circles by $\Omega = (\Gamma, \mathscr{B})$. We define closed and balanced subgraphs of a biased graph exactly the same as we did for gain graphs. A subgraph Υ of a biased graph Ω is balanced if every circle in Υ is balanced and is *closed and balanced* if in addition whenever there is a balanced circle $C \in \Omega$ such that $C \setminus e \in \Upsilon$, C is in Υ. A gain graph Φ with underlying graph Γ gives rise to the biased graph $\langle\Phi\rangle = (\Gamma, \mathscr{B}(\Phi))$ [3, Section 5].

The definition of the graphic Rhodes semilattice of a gain graph depends on the subgraphs and the balanced circles in the subgraph. Since the balanced circles of a gain graph define a biased graph, we can readily generalize the definition of the graphic Rhodes semilattice to biased graphs.

Definition 4.1. Let Ω be a biased graph. The *Rhodes semilattice of* Ω, denoted by $R^{\mathrm{b}}(\Omega)$, is the family of closed and balanced subgraphs in Ω ordered by inclusion.

A difference between bias and gains is that we cannot state a partition-potential description of balanced subgraphs of a biased graph. This is not a trivial difference, since not all biased graphs can be given gains; see [3, Example 5.8].

5. The four lattices

In this section we have a biased graph Ω. We treat a gain graph Φ as the biased graph $\langle\Phi\rangle$. We propose to embed the Rhodes semilattice as an order ideal of a lattice. The simplest such lattice is the first.

Definition 5.1. The *trivial Rhodes lattice* of Ω, denoted by $\widehat{R}(\Omega)$, is $R(\Omega)$ with an added top element $\hat{1}$. $\widehat{R}(\mathfrak{G}{\cdot}K_n)$ is the Rhodes lattice $\widehat{R}_n(\mathfrak{G})$ defined in [2]. (This lattice is trivial only from the viewpoint of partially ordered sets; we do not mean it is useless.)

With the benefit of our subgraph interpretation we propose more substantial kinds of Rhodes lattices.

The frame matroid of Ω is a matroid $\mathbf{F}(\Omega)$ on the edge set $E(\Omega)$ [4, Section 2]. We regard each edge set S as the spanning subgraph $(V(\Omega), S)$. From this viewpoint the flats of Ω become frame-closed spanning subgraphs of Ω.

Definition 5.2. The *frame Rhodes lattice* of Ω, denoted by $R^{\mathbf{F}}(\Omega)$, is the family of all frame-closed subgraphs of Ω; that is, the set of all frame-closed spanning subgraphs of all induced subgraphs of Ω.

The lift matroid of Ω is a matroid $\mathbf{L}(\Omega)$ on the set $E(\Omega)$ [4, Section 4]. We regard each edge set S as a spanning subgraph of Ω, $(V(\Omega), S)$. From this viewpoint the flats of Ω become spanning subgraphs of Ω whose edge sets are closed in $\mathbf{L}(\Omega)$. There is a natural one-point extension of the lift matroid, called the *complete* or *extended lift matroid*, whose balanced flats are the same as those of $\mathbf{L}(\Omega)$; the following definition extends to it but we omit the formal definition.

Definition 5.3. The *lift Rhodes lattice*, denoted by $R^{\mathbf{L}}(\Omega)$, is the family of all lift-closed subgraphs of Ω; that is, the set of all lift-closed spanning subgraphs of all induced subgraphs of Ω.

A subgraph Υ is called *balance-closed* (which does not mean balanced and does not mean closed in a matroid) if, whenever there is a balanced circle $C \in \Omega$ such that $C \setminus e \in \Upsilon$, then $C \subseteq \Upsilon$.

Definition 5.4. The *balance-closed Rhodes lattice*, denoted by $R^{\mathbf{BC}}(\Omega)$, is the family of balance-closed subgraphs of all induced subgraphs of Ω, ordered by inclusion.

The graphic Rhodes semilattice is the order ideal of balanced subgraphs in each of these proposed Rhodes lattices.

We plan to study these lattices in order to draw conclusions about the structure of the original Rhodes semilattice and the generalizations to gain and biased graphs.

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