ADVANCED PROBLEMS

6661. *Proposed by Jeffrey C. Lagarias, AT & T Bell Laboratories, Murray Hill, NJ, and Thomas Zaslavsky, SUNY at Binghamton.*

A curious property of $\frac{1}{7}$ is that to two decimal places it equals $.02 \times 7$. Add $.02^2 \times 7$ and you obtain $\frac{1}{7}$ to four decimal places. Add $.02^3 \times 7$ and you obtain it to six places (with an error of 1 in the last place), and so on. In fact $\frac{1}{7}$ equals 7 times the sum of a geometric series whose ratio has a terminating decimal expansion:

$$\frac{1}{7} = 7 \times \sum_{i=1}^{\infty} (.02)^i.$$

Which positive integers N have a similar representation,

$$\frac{1}{N} = N \sum_{i=1}^{\infty} r^i,$$

where r is a terminating decimal?

6662. *Proposed by F. S. Cater and John Erdman, Portland State University, Oregon.*

(a) Let I be the unit interval $[0,1]$ and let $I \times I$ be the unit square. Let $\mathscr{A}$ denote the smallest σ-algebra of subsets of $I \times I$ containing all rectangles of the form $U \times V$ where either U or $I \setminus U$ is a first category set, and either V or $I \setminus V$ is a first category set. Prove that the diagonal $D = \{(x,x)\colon x \in I\}$ does not lie in $\mathscr{A}$.

(b) Is this true when "first category set" is replaced by "set of measure zero"?

(c) Let a and b be cardinal numbers such that $a > b \geqslant \aleph_0$, and let S be a set with cardinality $|S| = a$. Let $\mathscr{B}$ denote the smallest σ-algebra of subsets of $S \times S$ containing all rectangles of the form $U \times V$ where either U or $I \setminus U$ has cardinality $\leqslant b$, and either V or $I \setminus V$ has cardinality $\leqslant b$. Prove that the diagonal $D = \{(x,x)\colon x \in S\}$ does not lie in $\mathscr{B}$.

6663. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and the editors.*

Show that

$$\sum_{j=1}^{N} \left(\frac{1 + x + x^2 + \cdots + x^{j-1}}{j} \right)^2 < (4 \log 2)(1 + x^2 + x^4 + \cdots + x^{2N-2})$$

for $0 < x < 1$ and all positive integers N; also show that the constant $4 \log 2$ is best possible. (If we drop the factor $\log 2$, we have a special case of Hardy's inequality; see Hardy, Littlewood, and Pólya, *Inequalities*, pp. 239–242).