Dowling Geometries of Multiary Quasigroups

Thomas Zaslavsky

Binghamton University (SUNY)

Sources:

T.A. Dowling, A class of geometric lattices based on finite groups.
Journal of Combinatorial Theory Ser. B **14** (1973), 61–86.
Erratum. *ibid.* **15** (1973), 211.

Thomas Zaslavsky, Associativity in multary quasigroups: The way of biased expansions.
Aequationes Mathematicae **83** (2012), no. 1, 1–66.

Dowling lattice/geometry/matroid (of rank r) of a group $\mathfrak{G}$:

$$Q_r(\mathfrak{G})$$

Dowling's abstract definition (lattice):

Poset of homogeneously $\mathfrak{G}$-labelled partial partitions of an r-set V.

Geometrical definition (matroid):

Set of homogeneous vectors over $\mathfrak{G}$ with 1 or 2 non-zero coordinates.

Graphical definition (matroid):

Take the complete graph K_r.
Replace each edge by a 'pencil' of parallel edges labelled invertibly by $\mathfrak{G}$ (*gains*). $\mathfrak{G}K_r$
Add a loop at each vertex labelled by a non-identity element (*gain*). $\mathfrak{G}K_r^\bullet$
Take the frame matroid $G(\mathfrak{G}K_r^\bullet)$.
Frame circuits:
- Circles with identity gain (*balanced* circles).
- Theta graphs with no balanced circle.
- Barbells and figure-eights with no balanced circle.

Advantage: minors by graph theory.

Beautiful properties

Universality, similarly to projective spaces over a skew field (Kahn–Kung).

Contractions are Dowling matroids (upper intervals are Dowling lattices).

Superbly easy computation of invariants from gain-graph coloring:

$$p_{Q_r(\mathfrak{G})}(y) \quad = \quad |\mathfrak{G}|^r \cdot \left(\frac{y-1}{|\mathfrak{G}|}\right)_r \quad = \quad |\mathfrak{G}|^r \cdot \chi_{K_r}\left(\frac{y-1}{|\mathfrak{G}|}\right)$$

Other graph expansions:

$$p_{\mathfrak{G}\Gamma^\bullet}(y) \quad = \quad |\mathfrak{G}|^r \cdot \chi_{\Gamma}\left(\frac{y-1}{|\mathfrak{G}|}\right)$$

Thus, $G(\mathfrak{G}\Gamma^\bullet)$ is a graph-based analog of Dowling matroid.

Qausigroup Dowling matroid

Quasigroup: a group without associativity: $\mathfrak{Q} = (\mathfrak{Q}, \cdot)$ such that

$$xy = z \qquad \text{has unique solvability.}$$

Analogously, rewrite a group operation as $x \cdot y = z$.
Reinterpret $Q_3(\mathfrak{G})$: A balanced circle has $xy = z$.
Apply to $\mathfrak{Q}$: you get balanced circles, a frame matroid, and the Dowling plane of $\mathfrak{Q}$:

$$Q_3(\mathfrak{Q}) \quad = \quad G(\mathfrak{Q}K_3^\bullet)$$

This has no extension to higher rank!

Question: Are there higher-rank generalized Dowling matroids?

Graphic version: Maximal biased expansion of a graph.
Replace each edge in Γ by a 'pencil' of parallel edges.
Define a class of balanced circles.
Add an unbalanced loop at each vertex.
Take the frame matroid $G(m \cdot \Gamma^\bullet)$.

Question: Are there maximal examples other than Dowling matroids of groups and quasigroups?

Multiary qausigroups

n-ary quasigroup $\mathfrak{Q}$: an n-ary operation such that

$$(x_1 x_2 \cdots x_n) = x_0 \qquad \text{has unique solvability.}$$

Example: Iterated group operation:

$$(x_1 x_2 \cdots x_n) = x_0 \text{ if } x_1 \cdot x_2 \cdot \cdots \cdot x_n.$$

Factors in all possible ways:

$$(x_1 x_2 \cdots x_n) = (x_1 \cdots x_i \; [x_{i+1} \cdots x_j] \; x_{j+1} \cdots x_n)$$

Theorem 1 (essentially: Aczél, Belousov, and Hosszú; Kahn and Kung)**.** *If $\mathfrak{Q}$ factors in all possible ways, it is an iterated group operation.*

Example: $(xy)z = (xyz) = x(yz) \implies$ group.

The opposite extreme: $\mathfrak{Q}$ is *irreducible* if it has no factorizations.

Partial factorization and the factorization graph

Draw $C_{n+1} = v_0\ e_1\ v_1\ e_2\ \cdots\ e_n\ v_n\ e_0\ v_0$. In C_{n+1}:

$$v_i \longleftrightarrow x_i.$$

$$\text{Edge } e_{ij} \longleftrightarrow \text{ factorization } (x_1 \cdots x_i\ [x_{i+1} \cdots x_j]\ x_{j+1} \cdots x_n).$$

This is the

$$\text{factorization graph } \ \Phi(\mathfrak{Q}).$$

Theorem 2 (Zaslavsky)**.**
$\mathfrak{Q}$ is an iterated group if and only if $\Phi(\mathfrak{Q})$ is 3-connected.

Proof: Intransitive combinatorial homotopy within the biased expansion graph.

Theorem 3 (Zaslavsky)**.**
Every n-ary quasigroup with $n \geq 3$ is (in a unique way) the composition of irreducible (quasigroups and) multiary quasigroups and iterated groups.

Proof: Tutte's 3-decomposition theorem.

Generalized Dowling matroids

A *generalized Dowling geometry* is a frame matroid

$$Q_r = G\big((\text{biased expansion of } \Gamma)^\bullet\big)$$

which cannot be extended to another such frame matroid, on a supergraph of Γ, on the same vertex set.

Example: $\Gamma = K_r$, biased expansion $= \mathfrak{G}K_r$ of a group.

Example: $\Gamma = K_3$, biased expansion $= \mathfrak{Q}K_3$ of a quasigroup.

Example: $\Gamma = C_r$, biased expansion $= \mathfrak{Q}C_r$ of an irreducible $r+1$-ary quasigroup.

Theorem 4 (Corollary of Theorem 3)**.**
Universal example:
(a) $\Omega =$ *2-amalgamation of complete graphs and circles, the complete graphs expanded by groups and the circles by non-group multiary quasigroups.*
(b) $Q_r = G(\Omega^\bullet)$.