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Line Graphs of Switching Classes

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A switching class (of signed graphs) is an equivalence class of signed graphs under the switching relation: $\Sigma_1 \sim \Sigma_2 \iff V(\Sigma_1) = V(\Sigma_1) = V$, $E(\Sigma_1) = E(\Sigma_2)$, and $(\exists X \subseteq V)$ an edge has the same sign in Σ_1 and Σ_2 iff both ends are in X or both in $V \backslash X$. A switching class $[\Sigma]$ is characterized by the underlying graph (V, E) and its "balant" $\mathcal{B} = \{C : C$ is a circuit of Σ whose edge signs have positive product$\}$. The triple $S(\Sigma) = (V, E, \mathcal{B})$ is a "sign-biased graph"; it is equivalent to $[\Sigma]$. A circuit is "balanced" if it is in $\mathcal{B}$. Our graphs will be finite and loopless.

Let S be a sign-biased graph and let

$$V_L = E(S),$$

$$E'_L = \{\{v, e, f\} : \ e, f \in E(S) \ \text{ and } \ v \ \text{ is a vertex of } \ e \ \text{ and } \ f\}.$$

Then (V_L, E'_L) is the ordinary line graph $L(V(S), E(S))$. A circuit in it is "derived" (from C_0) if its vertices are the edges of a circuit C_0 of (V, E), a "vertex triangle" if it is a triangle whose vertices are edges at a common vertex in (V, E). Let

$$\begin{aligned}\mathcal{B}'_L = \{C : C \ &\text{ is a circuit of } \ (V_L, E'_L) \ \text{ and its edge set is a sum of} \\ &\text{derived circuits and vertex triangles, all but an even number} \\ &\text{being derived circuits from } \ C_0 \ \text{ in } \ \mathcal{B}(S)\}.\end{aligned}$$

The *unreduced line graph* $L'(S)$ is $(V_L, E'_L, \mathcal{B}'_L)$.

From $L'(S)$ remove pairs of edges forming unbalanced digons until this is no longer possible. The result of this "reduction" is the *reduced line graph* $L(S)$.

Example 1. Given a graph $G = (V, E)$, let $\mathcal{B}_e = \{$circuits of even length$\}$. Then $S_e(G)$ is a sign-biased graph and $L(S_e(G)) = S_e(L(G))$. That is, ordinary line graphs are an example of our definition.

Example 2. Given also an integral weight $m(v) \geq 0$ for each vertex, let $S_e(G; m)$ consist of $S_e(G)$ with $m(v)$ new vertices doubly adjacent to each original vertex v, each edge pair forming an unbalanced digon. Then $L(S_e(G; m)) = S_e(L(G; m))$, where $L(G; m)$ is Hoffman's generalized line graph. Thus the latter are, in our system, simply line graphs. Conversely, if $L(S)$ has the form $S_e(G)$, then G is a generalized line graph.

Example 3. Let D be an orientation of G. A circuit is "semicoherent" if an even number of its vertices have both arcs head or both tail. Let $S(D) = (V, E, \mathcal{B})$ where $\mathcal{B} = \{$semicoherent circuits$\}$. Then $L(S(D))$ is a line graph of the digraph D.

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An adjacency [or incidence] matrix $A(S)$ [or $M(S)$] is an adjacency [incidence] matrix of any Σ in the switching class of S. The eigenvalues of $A(S)$ are well-defined. Since $A(L(S)) = 2I - M(S)M(S)^T$, we have

Theorem 1. *The eigenvalues of $A(L(S))$ are all ≤ 2.*

The usual root system arguments yield

Theorem 2. *With finitely many exceptions, any reduced sign-biased graph with all eigenvalues ≤ 2 is a reduced line graph.*

For sign-biased graphs there is an extension (and explanation) of Whitney's theorem on line isomorphisms. A line isomorphism is an isomorphism of line graphs. By $\pm K_4$ we mean the signed graph of order 4 with all twelve positive and negative edges. A subgraph of a sign-biased graph is understood to have the inherited balant.

Theorem 3. *A line isomorphism between sign-biased graphs is induced by an isomorphism, unless the graphs are subgraphs of $S(\pm K_4)$.*

The line automorphisms of $S(\pm K_4)$ form a degree 2 extension of the automorphism group; the extra automorphisms are due to the "triality" of the root system D_4.

Theorem 4. *A reduced sign-biased graph is a reduced line graph if, and only if, all its induced subgraphs of orders up to 6 are.*

This characterization implies a similar one for ordinary graphs that are generalized line graphs.