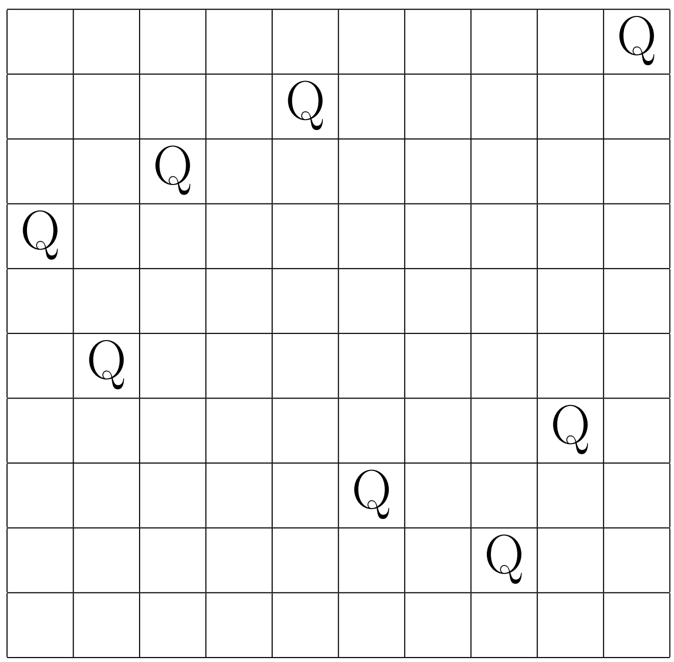
CombinaTexas 10: 26 April 2009

Lattice Points and Kindly Chess Queens

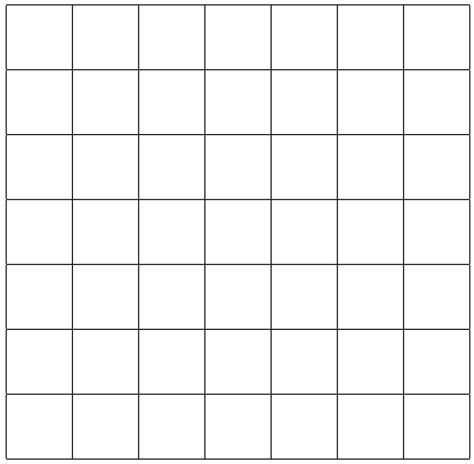
Thomas Zaslavsky
Binghamton University of SUNY

Jointly with
Matthias Beck, Seth Chaiken, and Christopher R.H. Hanusa



Board size: $n = 10$. Queens: $q = 8$.

An $n \times n$ board:



q identical chess pieces:

$$P\ P\ P\ \cdots\ P\ P$$

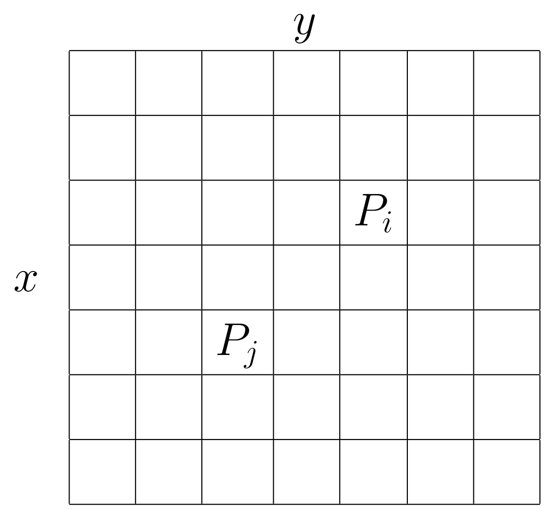
Put the pieces on the board!
The pieces are kindly and do not wish to attack each other.

The Question: How many ways are there to do this, as a function of n?

$$N_P(q; n)$$

$N_Q(n; n)$?   (The n-queens problem.)

Coordinate system:



P_i coordinates: $(x_i, y_i) \in \mathbb{Z}^2 \subseteq \mathbb{R}^2$.
Configuration: $(x_1, y_1, \ldots, x_q, y_q) \in \mathbb{R}^{2q}$.

Moves: $\alpha\mu_k$ where $\mu_k = (\mu_{k1}, \mu_{k2}) \in \mathcal{M}_P$ and $\alpha \in \mathbb{Z}$.
Attack: $(x_j, y_j) - (x_i, y_i) \in \langle \mu_k \rangle$.

Permitted configurations:

$$(x_1, y_1, \ldots, x_q, y_q) \in \{1, 2, \ldots, n\}^{2q} = (0, n+1)^{2q} \cap \mathbb{Z}^{2q}.$$

Forbidden hyperplanes:

$$H_{k,i,j} : \ [(x_j, y_j) - (x_i, y_i)] \cdot \mu_k^{\perp} = 0, \quad \text{in } \mathbb{R}^{2q}.$$

The count:

$$N_P(q; n) = \# \text{ of integer points in } (n+1)(0,1)^{2q} \setminus \bigcup_{k,i,j} H_{k,i,j}.$$

Polytopes and Ehrhart theory

Convex polytope $\mathbf{P}$ in $\mathbb{R}^\delta$ with rational vertices.

$$E_{\mathbf{P}}(t) := \#\text{ of integer points in } t\mathbf{P}, \qquad \text{for } t = 1, 2, \ldots.$$

$$d := \text{least common denominator of all vertices.}$$

Theorem 1 (Ehrhart, Macdonald)**.**
(a) $E_{\mathbf{P}}(t)$ is a quasipolynomial function of $t > 0$ with leading term $\operatorname{vol}(\mathbf{P})t^\delta$.
(b) Its period p divides d.
(c) $E_{\mathbf{P}^\circ}(t) = (-1)^\delta E_{\mathbf{P}}(-t)$. (Ehrhart reciprocity.)

Quasipolynomial $f(t)$: It is p polynomials $f_1(t), \ldots, f_p(t)$ with

$$f(t) := f_{t \bmod p}(t).$$

Its period is p.

Example:

$$\mathbf{P} = [0,1]^\delta, \quad \operatorname{vol}(\mathbf{P}) = 1, \quad p = 1.$$

(Integral vertices give a polynomial.)

Computation: `LattE` computes the number of points for fixed t.

Inside-out polytopes

Convex polytope $\mathbf{P}$ with rational vertices.
Finite set of rational hyperplanes $\mathcal{H}$ of hyperplanes, all in $\mathbb{R}^\delta$.

$$E_{\mathbf{P},\mathcal{H}}(t) := \#\text{ of integer points in } t\mathbf{P} \text{ but not in } \bigcup \mathcal{H}.$$

Theorem 2 (Beck & Zaslavsky)**.** *The Ehrhart properties (a–c) hold for $E_{\mathbf{P},\mathcal{H}}(t)$. Also:*

(d) $(-1)^\delta E^\circ_{\mathbf{P}^\circ,\mathcal{H}}(0)$ is the number of regions of $\mathbf{P}$ as dissected by $\mathcal{H}$.

Reduction to standard Ehrhart theory via

$$\mathcal{L} := \text{ the set of non-empty intersections of hyperplanes in } \mathbf{P}^\circ,$$

ordered by reverse inclusion so $\mathbf{0} = \mathbf{P}^\circ$, and

$$\mu(\mathbf{0}, u) = \text{ Möbius function of } \mathcal{L}.$$

Theorem 3 (Beck & Zaslavsky)**.**

$$E^\circ_{\mathbf{P}^\circ,\mathcal{H}}(t) = \sum_{u \in \mathcal{L}} \mu(\mathbf{0}, u) E_{\mathbf{P}^\circ \cap u}(t).$$

Example: $\mathbf{P} = [0,1]^\delta$, $\mathrm{vol}(\mathbf{P}) = 1$, period $p \gg 1$ with forbidden hyperplanes.

Graphs.

$$\begin{aligned}\chi_\Gamma(\lambda) &:= \text{ number of proper colorations of } \Gamma \text{ with colors } 1, 2, \dots, \lambda \\ &= E^\circ_{\mathbf{P}^\circ, \mathcal{H}}(\lambda+1) \quad (\text{i.e., } t = \lambda + 1),\end{aligned}$$

where $\mathbf{P} = [0,1]^{|V|}$ and $\mathcal{H} = \{x_i = x_j : \exists\, e_{ij}\}$.

Integral vertices. Denominator: 1. Period: 1.
Conclusion: One monic polynomial of degree $|V|$.

Signed graphs.
$\Sigma :=$ graph with $+$ and $-$ edges.

$$\begin{aligned}\chi_\Sigma(2k+1) &:= \text{ number of proper colorations of } \Sigma \text{ with colors } 0, \pm1, \pm2, \dots, \pm k \\ &= E^\circ_{\mathbf{P}^\circ, \mathcal{H}}(2k+2) \quad (\text{i.e., } t = 2k+2),\end{aligned}$$

$$\begin{aligned}\chi^*_\Sigma(2k) &:= \text{ number of proper colorations of } \Sigma \text{ with colors } \pm 1, \pm2, \dots, \pm k \\ &= E^\circ_{\mathbf{P}^\circ, \mathcal{H}}(2k+1) \quad (\text{i.e., } t = 2k+1),\end{aligned}$$

where $\mathbf{P} = [0,1]^{|V|}$ and $\mathcal{H} = (\frac{1}{2}, \dots, \frac{1}{2}) + \{x_i = \operatorname{sgn}(e_{ij}) x_j : \exists\, e_{ij}\}$.

Half-integral vertices. Denominator: 2. Period: 1 or 2.
Conclusion: Two monic polynomials of degree $|V|$.

With a chess piece P:

$$\delta = 2q,$$

$$\mathbf{P} = [0,1]^{2q}.$$

$$\mathcal{H} = \{H_{k,i,j} : 1 \le k \le \ \# \text{ of basic moves},\ 1 \le i < j \le q\},$$

$$N_P(n) = E^{\circ}_{\mathbf{P}^{\circ},\mathcal{H}}(n+1) \quad (\text{i.e., } t = n+1).$$

$$N_P(-1) = E^{\circ}_{\mathbf{P}^{\circ},\mathcal{H}}(0) = \text{the number of combinatorial types of configuration.}$$

The hyperplanes are given by a matrix:

$$M_P := \begin{bmatrix} \mu_1^{\perp} \\ \mu_2^{\perp} \\ \vdots \end{bmatrix},$$

one line for each basic move, and $D(K_n)$.

Period p? (Needed for computer calculation.) Hard!

A bound p' for $p \implies$ the quasipolynomial by computer calculation of all polynomial constituents of all $E_{\mathbf{P}^{\circ} \cap u}(t)$ in Theorem 3 using `LattE`.

$\therefore$ **Task:** To bound p for every q.

An upper bound is d.

$\therefore$ **Task:** To bound d for every q. Hard!

For the chess problem we need:

$$\operatorname{lcmd}(A) := \text{least common multiple of all subdeterminants of } A.$$

Kronecker product: $A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$.

Proposition 4 (Hanusa & Zaslavsky)**.** *Let A be a 2×2 matrix, not identically zero, and $q \geq 1$. The least common multiple of all square minor determinants of $A \otimes D(K_q)$ is*

$$\operatorname{lcmd}\big(A \otimes D(K_q)\big) \;=\; \operatorname{lcm}\left((\operatorname{lcmd} A)^{q-1},\ \operatorname*{LCM}_{p=2}^{\lfloor q/2 \rfloor} \Big((a_{11}a_{22})^p - (a_{12}a_{21})^p \Big)^{\lfloor q/2p \rfloor} \right).$$

For a chess piece, $B = D(K_q)$. For the bishop or queen,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \text{ from } (M_P)^T.$$

Apply Proposition 4, using $\operatorname{lcmd}(A) = 2$. We get

$$\operatorname{lcmd}\big(A \otimes D(K_q)\big) = 2^{q-1},$$

an upper bound on d, hence on the period p, for q bishops or queens.

The bishop

$$M_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \operatorname{lcmd}(M_B) = 2.$$

For two bishops,

$$N_B(2;n) \;=\; \frac{n(n-1)(3n^2-n+2)}{6} \;=\; \frac{n}{6}\big(3n^3-4n^2+3n-2\big).$$

For 3 and more bishops we haven't yet done the computer work.

The queen

$$M_Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \operatorname{lcmd}(M_Q) = 2.$$

For two queens,

$$N_Q(2;n) \;=\; \frac{n(n-1)(3n^2-7n+2)}{6} \;=\; \frac{n}{6}\big(3n^3-10n^2+9n-2\big).$$

For 3 or more queens we'll need computer work.

Ehrhart theory.

* Matthias Beck and Sinai Robins,
Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra. Undergraduate Texts in Mathematics. Springer, New York, 2007.

Matthias Beck and Thomas Zaslavsky,
Inside-out polytopes. *Advances in Math.* **205** (2006), no. 1, 134–162.

Other.

Seth Chaiken, Christopher R.H. Hanusa, and Thomas Zaslavsky,
A q-queens problem. (In preparation.)

Christopher R.H. Hanusa and Thomas Zaslavsky,
Determinants in the Kronecker product of matrices: The incidence matrix of a complete graph. (In preparation.)