RESOLUTION OF IRREDUCIBLE INTEGRAL FLOWS ON A SIGNED GRAPH

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Abstract. We completely describe the structure of irreducible integral flows on a signed graph by lifting them to the signed double covering graph.

A (real-valued) *flow* (sometimes also called a *circulation*) on a graph or a signed graph (a graph with signed edges) is a real-valued function on oriented edges, $f : \vec{E} \to \mathbb{R}$, such that the net inflow to any vertex is zero. An *integral flow* is a flow whose values are integers. There are many reasons to be interested in flows on graphs; an important one is their relationship to graph structure through the analysis of *irreducible flows*, that is, integral flows that cannot be decomposed as the sum of other flows of lesser value. It is well known, and an important observation in the thoery of integral network flows, that the irreducible flows are identical to the *circuit flows*, which have value 1 on the edges of a graph circuit (that is, a cycle) and 0 on all other edges. Extending the theory of irreducible integral flows to signed graphs, which was one of the topics of the doctoral dissertation of Wang [4], led to the remarkable discovery that there are, besides the anticipated circuit flows (which in signed graphs are already more complicated than in unsigned graphs), also many 'strange' irreducible flows with elaborate structure not describable by circuits. In this article we characterize that structure by lifting it to a simple cycle in the signed covering graph. (Indeed, this was how we discovered the correct characterization, though we were also guided by the partial result in Wang's thesis.)

We like to think of lifting as a combinatorial analog of resolution of singularities in continuous mathematics. The strange irreducible flows are singular phenomena, which we resolve by lifting them to ordinary cycle flows in a covering graph. This is not a precise statement but a philosophy that we believe will be fruitful.

1. Graphs and signed graphs

Graphs. A graph is (V, E), with vertex set V and edge set E. There may be loops and multiple edges. An edge e with endpoints v and w has two *ends*, which we symbolize by (v, e) and (w, e). A tricky technical point is that this notation does not distinguish the two ends of a loop; we take an easy way out by treating (v, e) and (w, e) as different ends even when $v = w$. (There are more technically correct means of distinguishing the ends but they make the notation very complicated.)

A *walk* is a sequence $W = v_0e_1v_1e_2 \cdots e_lv_l$ of vertices and edges such that the endpoints of e_i are v_{i-1} and v_i. A walk is *closed* if $l > 0$ and $v_0 = v_l$ and *open* otherwise. A *segment* of W is a consecutive subwalk, i.e., $v_ie_{i+1} \cdots e_jv_j$. When W is closed we allow $j > l$, interpreting indices modulo l; thus, a segment may pass through v_0. A *circle* is the edge set of a simple closed walk, i.e., there is no repeated edge or vertex other than that $v_0 = v_l$. A *graph circuit*

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is a circle; the name comes from the fact that the circles of a graph form the circuits of the well known graphic matroid whose elements are the edges of the graph.

A *isthmus* is an edge whose deletion increases the number of connected components. A *cutpoint* is a vertex whose deletion, with all incident edges, increases the number of components, or that supports a loop and is incident with at least one other edge. A *block* is a maximal subgraph without cutpoints. Thus, a loop or isthmus or isolated vertex is a (trivial) block. We call blocks *adjacent* if they have a common vertex (which is necessarily a cutpoint). An *end block* is a block adjacent to exactly one other block.

Signed graphs. A *signed graph* $\Sigma = (V, E, \sigma)$ consists of a graph (V, E) and a *signature* $\sigma : E \to \{+1, -1\}$. (Signs multiply; they do not add.) A walk in Σ has a sign $\sigma(W) := \sigma(e_1)\sigma(e_2) \cdots \sigma(e_l)$. In particular, a circle has a sign (which is the sign of any walk that goes once around the circle), so it is either positive or negative. A subgraph or edge set is *balanced* if every circle in it is positive.

A *signed circuit* in a signed graph is a subgraph (or its edge set) of one of the following three types:

(I) a positive circle (that is, a balanced circle);
(II) a pair of negative circles whose intersection is one vertex (sometimes called a *contrabalanced tight handcuff*); and
(III) a pair of vertex-disjoint negative circles together with a connecting simple path (called the *circuit path*) that is internally disjoint from the circles (this type is sometimes called a *contrabalanced loose handcuff*).

The signed circuits are the circuits of a matroid on the edge set of the signed graph [5].

An ordinary, unsigned graph can be treated as an all-positive signed graph. When all edges are positive, every circle is positive and the only signed circuits are those of Type I, i.e., the graph circuits.

Orientation. A *bidirection* of a graph (a concept introduced by Edmonds [3]) is a function from the edge ends to the sign group. On thinks of an end with sign $+1$ as having an arrow directed away from the vertex and an end with sign -1 as having an arrow directed toward the vertex (or vice versa; see [6]); thus a bidirected graph has two arrows, one at each end. Formally, we write a bidirection as a function $\varepsilon : V \times E \to \{+1, -1\}$ such that $\varepsilon(v, e) = 0$ if and only if v is not an endpoint of e. (For loops this formalism is technically incorrect, but we trust the reader will be willing to understand $\varepsilon(v, e)$ and $\varepsilon(w, e)$ as independent values even when e is a loop so $v = w$. Otherwise we are forced into technical complications.)

An *orientation* of a signed graph Σ is a bidirection of its edges such that $\sigma(e) = -\nu(v, e)\nu(w, e)$ [6]. Thus, a positive edge has two arrows that are consistent and give e a direction, just as in an ordinary directed graph. A negative edge has arrows that both point towards, or both away from, the endpoints.

A *source* in an oriented signed graph is a vertex v at which all edges are directed outwards; that is, $\varepsilon(v, e) = +1$ for all edges at e. Conversely, if all edges point into v, v is a *sink*.

A walk $W = v_0 e_1 v_1 e_2 \cdots e_l v_l$ of an oriented signed graph is called *coherent* at v_i if $\varepsilon(v_i, e_i) = -\varepsilon(e_{i+1})$; that is, if the walk has a consistent direction at v_i. We apply this definition to v_0, if W is closed, by taking subscripts modulo l. A *directed walk* is coherent at every vertex except v_0 and v_l. A *directed closed walk* is a closed walk that is coherent at every vertex with the possible exception of v_0. Although each vertex has a direction, W itself need not have an overall direction, since at every negative edge the arrows reverse. A positive (or negative) directed closed walk must be coherent (or incoherent, respectively) at v_0, by the following lemma.

Lemma 1. *The sign of a closed walk equals $(-1)^k$, where k is the number of incoherent vertices in the walk, including the final vertex if it is incoherent.*

Proof. We perform a short calculation that applies to open as well as closed walks. Note that if W is open, the final vertex cannot be incoherent.

$$\begin{aligned}
\sigma(W) &= \prod_{i=1}^{l} \sigma(e_i) = \prod_{i=1}^{l} [-\varepsilon(v_{i-1}, e_i)\varepsilon(v_i, e_i)] \\
&= \prod_{j=1}^{l-1} [-\varepsilon(v_j, e_j)\varepsilon(v_j, e_{j+1})] \;\cdot\; [-\varepsilon(v_0, e_1)\varepsilon(v_l, e_l)] \qquad (1)\\
&= \begin{cases} (-1)^k & \text{if } W \text{ is closed,} \\ -(-1)^k \varepsilon(v_0, e_1)\varepsilon(v_l, e_l) & \text{if } W \text{ is open.} \end{cases}
\end{aligned}$$

□

Reorienting $S \subseteq E$ means reversing the orientations of the edges in S but not those outside S. Thus, ε changes to ε_S defined by

$$\varepsilon_S(v, e) = \begin{cases} -\varepsilon(v, e) \text{ if} & e \in S, \\ \varepsilon(v, e) \text{ if} & e \notin S. \end{cases}$$

The signed covering graph. Let $\tilde{V} := V \times \{+1, -1\}$ and let $\tilde{E} := E \times \mathbb{Z}_2$, the union of two disjoint copies of E. For brevity we write (v, α) as v^α. If an edge e of Σ has endpoints v and w, then one copy of e in $\tilde{\Sigma}$ has endpoints v^{+1} and $w^{\sigma(e)}$ while the other copy has endpoints v^{-1} and $w^{-\sigma(e)}$. This defines $\tilde{\Sigma}$, the *signed covering graph* of Σ, which is a graph with unsigned edges and signed vertices. $\tilde{\Sigma}$ has a canonical involutory automorphism * defined by $(v^\alpha)^* := v^{-\alpha}$ and $(\tilde{e}, k)^* := (\tilde{e}, k+1)$ and projects to Σ by the mapping $p(v^\alpha) = v$ and $p(e, k) = k$, which is a 2-to-1 graph homomorphism. We write $\tilde{e}$ for an edge in the covering graph that projects to e and, following the notation of the canonical involution, $p^{-1}(e) = \{\tilde{e}, \tilde{e}^*\}$. When e is a negative loop at v, $\tilde{e}$ and $\tilde{e}^*$ are two parallel edges in $\tilde{\Sigma}$ with the endpoints v^{+1} and v^{-1}.

We can think of $\tilde{V}$ as having a positive level, $V \times \{+1\}$, and a negative level, $V \times \{-1\}$. However, it is not possible to assign all edges to levels. A positive edge lifts to an edge that stays within a level, but a negative edge crosses between levels.

Suppose $W = v_0 e_1 v_1 e_2 \cdots e_l v_l$ is a walk in Σ. We can *lift* W to a walk $\tilde{W}$ in $\tilde{\Sigma}$. First, we choose a vertex $v_0^{\alpha_0}$ that projects to v_0. Then we follow edges that cover the edges of W, getting a walk $\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \tilde{e}_2 \cdots v_{l-1}^{\alpha_{l-1}} \tilde{e}_l v_l^{\alpha_l}$ such that $\alpha_i = \alpha_{i-1}\sigma(e_i)$. We call such a walk a *lift* of W. There are at least two such lifts since the choice of α_0 is arbitrary, and there are only two lifts if W contains no loops.

Lemma 2. *Let W be a closed walk in Σ and $\tilde{W}$ a lift of it in $\tilde{\Sigma}$. The sign of W is $\sigma(W) = +1$ if $\tilde{W}$ is closed and -1 if $\tilde{W}$ is open.*

Proof. Let $W = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$ where $v_l = v_0$, lifted to $\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 \cdots \tilde{e}_l v_l^{\alpha_l}$. Then

$$\sigma(W) = \prod_{i=1}^{l} \sigma(e_i) = \prod_{i=1}^{l} \alpha_{i-1}\alpha_i = \alpha_0 \alpha_l.$$

This equals $+1$ if and only if $\tilde{W}$ is closed. $\square$

An orientation ε of Σ lifts to $\tilde{\Sigma}$ by the rule

$$\tilde{\varepsilon}(v^{\alpha}, \tilde{e}) = \alpha\varepsilon(v, e),$$

where $\tilde{e}$ denotes the lift of e that is incident with v^{α}. By this definition, the other lift edge, $\tilde{e}^*$, is incident with $v^{-\alpha}$ and has the opposite orientation, $\tilde{\varepsilon}(v^{-\alpha}, \tilde{e}^*) = -\alpha\varepsilon(v, e)$. It is easy to see that $\tilde{\varepsilon}$ orients $\tilde{\Sigma}$ as an all-positive signed graph; that is, $(\tilde{\Sigma}, \tilde{\varepsilon})$ is an ordinary directed graph. For a positive edge, one lift lies in the positive level and has the same direction as e while the other lies in the negative level and has the direction opposite to that of e. For a negative edge, both lifts are directed from the positive to the negative level or the reverse.

2. Flows

Flows on a signed graph. An *integral flow* on an oriented signed graph (Σ, ε) (or one could say, following [1], on a bidirected graph) is a function $f : E \to \mathbb{Z}$ which is *conservative* at every vertex v, meaning that

$$\partial f(v) := \sum_{e \in E} \varepsilon(v, e) f(e) = 0.$$

(We assume that a loop appears twice in this sum, once for each end.) The set of all integral flows on (Σ, ε) forms a $\mathbb{Z}$-module, called the *flow lattice* by Chen and Wang, who developed its basic theory in [2]. One can define flows with values in any abelian group, such as the additive reals; many of the following remarks are applicable in general (so we omit the word 'integral').

The theory of flows depends essentially on the graph and signature but it is not really oriented, since when one reorients an edge set S one gets a natural isomorphism of flow lattices by taking a flow f on (Σ, ε) to the flow f' on (Σ, ε_S) that is defined by $f'(e) = -f(e)$ for $e \in S$ while $f'(e) = f(e)$ for $e \notin S$.

The *support* of a function $f : E \to \mathbb{Z}$ is the set $\operatorname{supp} f := E \setminus f^(- 1)(0)$. The flow that is zero on all edges is the *trivial flow*. A *circuit flow* has support that is signed circuit; on a directed circuit it takes value 1 on edges in a circle of the circuit and, in Type III, 2 on edges in the circuit path, and one gets a circuit flow on an arbitrary oriented circuit by reorienting it to be a directed circuit, applying the definition for directed circuits, and reorienting back to the original orientation while applying the natural flow-lattice isomorphism; i.e., negating the flow values on the reoriented edges.

The usual theory of flows on graphs is simply the all-positive case. There a circuit flow is ± 1 on the edges of a circle and zero elsewhere; we omit further details.

We say an integral flow f' *conforms to the sign pattern of* f if $\operatorname{supp} f' \subseteq \operatorname{supp} f$ and, whenever $f'(e)$ is nonzero, it has the same sign as $f(e)$.

An integral flow f on (Σ, ε) lifts to a flow on the oriented signed covering graph, possibly in more than one way. The best way to see this is through the correspondence between flows and walks, which exists if (and only if) the support is connected.

A positive, directed closed walk W on (Σ, ε) implies a unique *corresponding flow*, which is the integral flow defined by taking $f_W(e)$ (with e in an arbitrary fixed orientation) to be the number of times W traverses e in the fixed orientation less the number of times e appears in the opposite orientation. To prove f_W is a flow, consider the contribution to f of a pair of consecutive edges, $e_i v_i e_{i+1}$, at the intervening vertex. Because W is coherent at v_i, the contribution of these edges to $\partial f(v_i)$ is 0. This same argument applies to the initial vertex if we take subscripts modulo the length of W. We can apply the same construction of a function $f_W : E \to \mathbb{Z}$ to any walk, but the result may not be a flow.

Lemma 3. *For a directed walk $W = v_0 e_1 v_1 e_2 \cdots v_{l-1} e_l v_l$, the function f_W satisfies $\partial f(u) = 0$ if $u \neq v_0, v_l$, and*

$$\partial f(v_0) = \begin{cases} 0 & \text{if } W \text{ is closed and positive,} \\ \pm 2 & \text{if } W \text{ is closed and negative;} \end{cases}$$

and if W is open, then $\partial f(v_0) = -\sigma(W)\partial f(v_l) = \pm 1$.

Proof. If W is closed and negative, it is incoherent at v_0 and therefore e_1 and e_l contribute the same value ± 1 to $\partial f(v_0)$.

If W is open, then e_1 contributes $\varepsilon(v_0, e_1) = \pm 1$ to $\partial f(v_0)$ and $\varepsilon(v_l, e_l) = \pm 1$ to $\partial f(v_l)$. These values are related by $\varepsilon(v_l, e_l) = -\sigma(W)\varepsilon(v_0, e_1)$, by Equation (1), because W is coherent. □

In the other direction, the construction of a *corresponding walk* (which is not usually unique) from a flow f is just like the usual one of an Eulerian tour of a connected digraph with equal in- and out-degrees, but more complicated because of negative edges.

Proposition 4. *Let f be a nonnegative, nontrivial integral flow on (Σ, ε). Then there is a positive, directed closed walk W such that $f_W = f$.*

Proof. We apply induction on $\|f\| := \sum_e f(e)$. Choose a vertex of $\operatorname{supp} f$, call it v_0, and an edge $e_1 \in \operatorname{supp} f$ that is incident with v_0; call its other endpoint v_1. This gives a walk $W_1 = v_0 e_1 v_1$ of length 1.

Now, suppose we have selected a partial walk, $W_k = v_0 e_1 v_1 \cdots e_k v_k$.

If W_k is open or negative, then the lemma applied to W_k, together with $f \geq f_{W_k} \geq 0$, shows that there must be an edge $e_{k+1} \in \operatorname{supp} f - f_{W_k}$ such that $\varepsilon(v_k, e_{k+1}) = -\varepsilon(v_k, e_k)$. Extend W_k to $W_{k+1} := W_k e_{k+1} v_{k+1}$, where v_{k+1} is the other endpoint of e_{k+1}.

If W_k is closed and positive, we have a positive, directed closed walk, by Lemma 1, such that $f_{W_k} \geq 0$. If $f - f_{W_k} \neq 0$, consider the restrictions of $f - f_{W_k}$ to the components of its support. Each restriction f' is a nonnegative flow that has a corresponding positive, directed closed walk and has $\|f'\| < \|f\|$. Each of these walks has a vertex in common with W_k, so we can assemble them into a single positive, directed closed walk W such that $f_W = f$. □

Lifted flows. Consider a function $f : \tilde{E} \to \mathbb{Z}$ defined on the edge set of the signed covering graph $\tilde{\Sigma}$. The *projection* is the function $p(\tilde{f}) : E \to \mathbb{Z}$ defined by $p(\tilde{f})(e) = \tilde{f}(\tilde{e}) + \tilde{f}(\tilde{e}^*)$. A *lift* of an integral flow on (Σ, ε) to the signed covering graph is a flow $\tilde{f}$ on $(\tilde{\Sigma}, \tilde{\varepsilon})$ such that $p(\tilde{f}) = f$.

Lemma 5. *A nonnegative integral flow on (Σ, ε) lifts to a nonnegative integral flow on $(\tilde{\Sigma}, \tilde{\varepsilon})$.*

Proof. Construct a corresponding walk W to the flow. Lift W to $\tilde{W}$. By construction, $\tilde{W}$ is a directed closed walk on the signed covering graph. Thus, $f_{\tilde{W}}$ is nonnegative. □

If we have an integral flow that is nonnegative, we apply the lemma after reorienting (Σ, ε) so f is nonnegative.

Irreducible flows. An integral flow f on Σ is called *reducible* if it is nontrivial and it can be represented as the sum of two other integral flows, $f = f_1 + f_2$, each of which is nontrivial and conforms to the sign pattern of f. It is easy to see that irreducibility is equivalent to *minimality*: the only nontrivial flow that conforms to the sign pattern of f and satisfies $0 \leq f' \leq f$ is f itself.

It is well known that the irreducible flows on an unsigned graph are the circle flows.

Given a signed graph Σ, construct an integral flow by the following process.

1. Choose a connected subgraph Σ' such that
   (a) each block is a circle or an edge,
   (b) each end block is a circle,
   (c) each cutpoint is incident with exactly two blocks, and
   (d) the sign of a circle block equals $(-1)^p$, where p is the number of cutpoints on the circle.
2. Orient Σ' so that
   (a) within each circle block, each non-cutpoint is coherent and each cutpoint is incoherent, and
   (b) there are no sources or sinks.
3. Assign flow values 1 to each circle edge and 2 to each isthmus.

Let us call the results of Step 1 *essential signed graphs*, the results of Step 2 *essential orientations*, the flows of Step 3 *essential flows*, and the corresponding closed walks *essential walks*. There are other ways to describe the essential orientations.

Proposition 6. *Let ε be an orientation of an essential signed graph Σ. The following properties of ε are equivalent:*

(i) *It is essential.*
(ii) (Σ, ε) *has a coherently oriented essential walk.*
(iii) *Every essential walk in* (Σ, ε) *is coherently oriented.*

Proof. Clearly, (iii) implies (ii) and (ii) implies (i). We show that (i) implies (iii).

Let W be an essential walk; we show it is coherent at every vertex v. This is obvious if v is not a cutpoint, since it has degree 2 and is neither a sink nor a source. If it is a cutpoint, v divides W into segments W_1 and W_2. Each segment, by (i), has its ends at v directed similarly: both towards v, or both away from v. Also by (i), W_1 and W_2 are directed differently at v: one is towards v and the other away from it. The expression $W = vW_1vW_2v$ shows that W is coherent at v. □

Theorem 7. *A flow is irreducible if and only if it is essential.*

Proof. We have two things to prove: that an essential flow is irreducible, and that every irreducible flow is essential. We begin with the latter.

The idea of the proof is to turn an irreducible flow into a positive, directed closed walk W, which we lift into the signed covering graph, where it becomes a circle $\tilde{W}$. Then we have to find out which circles project to irreducible flows. Self-intersections in the base graph correspond to vertex pairs $+v, -v$—that is, vertex fibers—in $\tilde{W}$; hence there are no triple self-intersections in W, and each half of W separated by a self-intersection is a negative closed walk. The self-intersection is a cutpoint because, if the two halves intersected at any other vertex, $\tilde{W}$ could be adjusted to become non-simple while still projecting to W; then

$\tilde{W}$, hence W, would be reducible. It follows that every block is a circle or an isthmus, and the sign of a circle block follows from the negativity of each half of W when split by a cutpoint.

The proof itself is a series of lemmas. The irreducible flow is f and a corresponding positive, directed closed walk (which exists, by Proposition 4, because irreducibility implies that $\operatorname{supp} f$ is connected) is $W = v_0 e_1 v_1 e_2 \cdots e_l v_l$. W lifts to a walk $\tilde{W} = v_0^{\alpha_0} \tilde{e}_1 v_1^{\alpha_1} \tilde{e}_2 \cdots \tilde{e}_l v_l^{\alpha_l}$ in the signed covering graph $\tilde{\Sigma}$. A vertex in W is a *double point* if W passes through it exactly twice. A double point divides W into two closed subwalks. We can think of W as a subgraph, namely, as the subgraph induced by the edges of W; as a subgraph it has cutpoints and blocks.

Lemma 8. *A flow in Σ that lifts to a reducible flow in $\tilde{\Sigma}$ is reducible.*

Proof. Let f be the flow in Σ, assumed nonnegative by choosing the right orientation of Σ. Suppose f lifts to $\tilde{f}$, which is the sum of nontrivial flows $\tilde{f}_1$ and $\tilde{f}_2$. Then the projections $f_1 = p\tilde{f}_1$ and $f_2 = p\tilde{f}_2$ are nontrivial, nonnegative flows in Σ (in the chosen orientation) whose sum is f. □

Lemma 9. *$\tilde{W}$ is a circle.*

Proof. Lemma 8 implies that the lift of an irreducible flow is irreducible. $\tilde{\Sigma}$ is an unsigned graph. An irreducible flow in an unsigned graph is a circle flow. □

Lemma 10. *W is a positive walk.*

Proof. Since the lift $\tilde{W}$ is closed, W must be positive. □

Lemma 11. *Any self-intersection vertex of W is a double point and a cutpoint of W. The two closed subwalks into which it divides W are negative walks.*

Proof. Suppose $v_i = v_j = v$ with $i < j$. Let W_1 be the segment of W from v_i to v_j and let W_2 be the segment from v_j through v_0 to v_i. Neither $\tilde{W}_1$ nor $\tilde{W}_2$ can be closed, so $v_i^{\alpha_i} \neq v_j^{\alpha_j}$. Consequently, $\alpha_j = -\alpha_i$, so the segments are negative walks. Since v is covered by only two vertices of $\tilde{\Sigma}$, it cannot be repeated again in W.

Now suppose there is a vertex w, other than v, that appears in both W_1 and W_2. Then $\tilde{W}_1$ has w^β as a vertex, and $\tilde{W}_2$ has a vertex $w^{-\beta}$. If we replace $\tilde{W}_2$ in $\tilde{W}$ by $(\tilde{W}_2^*)^{-1}$ (reversing the direction so it goes from v^{α_i} to $v^{-\alpha_i}$, just like $\tilde{W}_2$), we get a new walk $\tilde{W}'$ that still is a lift of W but has a self-intersection at w^β. Thus, by Lemma 8, f is reducible, contrary to the assumption. This shows that v is a cutpoint of W. □

Lemma 12. *Each block of W is a circle or an isthmus. An end block is a circle.*

Proof. No block can have a vertex of degree greater than 2. An end block which is an isthmus makes an edge with flow 0; this edge would not have been in W in the first place. □

Lemma 13. *The sign of a circle block of W equals $(-1)^p$ where p is the number of cutpoints of W on that circle.*

Proof. Let C be a circle block with cutpoints $v_1, \ldots, v_k$. Each v_i separates W into a half walk that contains the edges of C and another half walk, which we call W_i. We know W_i is negative, W is positive, and $\sigma(W) = \sigma(C) \prod_i \sigma(W_i)$. The lemma follows at once. □

Lemma 14. *The edges of W are oriented so that no vertex is a source or a sink in its underlying oriented signed graph. The orientation of a circle block of W is incoherent at each cutpoint and coherent at each other vertex.*

The last statement is equivalent to saying that at a cutpoint v, which is necessarily incident with two blocks, the edges of one block are directed into v and the edges of the other block are directed away from v. This is true for both circle and isthmus blocks.

Proof. Since W is coherently oriented, no vertex can be a source or a sink. A cutpoint v divides W into segments W_1 and W_2; the edges of each segment at v belong to different blocks, B_1 and B_2 respectively. Each segment is negative, by Lemma 16, and coherent, so its ends at v are directed similarly: both towards v, or both away from v; this shows v is incoherent in each B_i. By coherence of W, W_1 and W_2 are directed differently at v: one is towards v and the other away from it. □

Lemma 15. *The flow values on edges of W are 1 for an edge in a circle block and 2 for an isthmus.*

Proof. These are the values obtained by projecting the circle flow from $\tilde{\Sigma}$. A doubly covered edge e has to have flow value $1+1$ or $1+(-1)=0$, but the latter case is impossible since then e would not have been in W in the first place. □

This completes the proof that every irreducible flow is essential. We still have to prove that any essential flow is irreducible. We require a lemma.

Lemma 16. *In an essential walk, a segment separated by a cutpoint is negative.*

Proof. A cutpoint separates the walk W into two segments. Let W' be the one in question. The sign of W' is the product of the signs of its circle blocks. Thus, the sign is the number of times a cutpoint appears on a circle block of W'.

Note that isthmi appear in paths that connect two circle blocks, since each end block is a circle. Every cutpoint appears twice in a circle block, with the following exceptions: A cutpoint between two isthmi appears no times and consequently does not contribute to the sign of W'. A cutpoint that connects a circle block to an isthmus contributes -1 to the sign but pairs with the cutpoint at the other end of the path to which that isthmus belongs; the two cutpoints contribute a total of $+1$ to the sign. Finally, v appears only once, so it contributes -1. The sign $\sigma(W')$ is the product of these signs, hence is -1. □

Suppose f is an essential flow which is reducible, say $f = f_1 + f_2 + \cdots$ where f_i corresponds to a walk W_i. Then each $\operatorname{supp} f_i$ forms a connected subgraph of $\operatorname{supp} f$ which is joined to the rest of $\operatorname{supp} f$ only at cutpoints of $\operatorname{supp} f$. There must be a $\operatorname{supp} f_i$ that is joined at only one cutpoint v. Then W_i is a closed segment of W. By Lemma 16, W_i has negative sign, but the corresponding walk of a flow is positive. Therefore, f cannot be reducible. □

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