Frustration vs. Clusterability in Two-Mode Signed Networks (Signed Bipartite Graphs)

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Abstract. Mrvar and Doreian recently defined a notion of bipartite clustering in bipartite signed graphs that gives a measure of imbalance of the signed graph, different from previous measures (the "frustration index" or "line index of balance", l, and Davis's clusterability). A *biclustering* of a bipartite signed graph is a pair (π_1, π_2) of partitions of the two color classes; the sets of the partitions are called clusters. The *majority biclusterability index* $M(k_1, k_2)$ is the minimum number of edges that are inconsistent, in a certain definition, with a biclustering, over all biclusterings with $|\pi_1| = k_1$ and $|\pi_2| = k_2$. Theorems: $M(1, k_2) \geq l$, while $M(k_1, k_2) \leq l$ if $k_1, k_2 \geq 2$. For $K_{2,n}$ with $n \geq 2$, $M(2, 2) = l$ in about 1/3 of all signatures. If $n > 2$, then for every signature of $K_{2,n}$ there exists a biclustering with $|\pi_1| = |\pi_2| = 2$ such that $M(\pi_1, \pi_2) = l$. There are many open questions.

Keywords. Two-mode signed network, signed bipartite graph, frustration index, majority clusterability, bipartite clusterability, biclustering.

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1. Introduction

The purpose of this paper is to compare two very different measures, one old and one new, of strain in a signed graph.

1.1. A tale of two clubs.

Not all members of the venerable Acquaintances Club are amicably acquainted. Occasional unfriendly relations put a strain on the social atmosphere by breaking the Grand Rule "The Friend of My Friend", whose "Five Legs" read: "The friend of my friend is my friend, the enemy of my friend is my enemy, the friend of my enemy is my enemy, and the enemy of my enemy is my friend, and so forth unto the nth degree". The amount of strain is measured by the number of relations that break the Five Legs, which is quantified by the smallest number of friendships or enmities that, if ended, would eliminate all violations of the Grand Rule. Cartwright and Harary's theory of structural balance [5] predicts that the relations will evolve towards a state of "balance", in which the members fall into two mutually antagonistic camps, within each of which no unfriendly relations remain.[1] The camps, instead of joining together at the monthly members' meeting, would gather only in separate rooms with separate agendas, motions, and decisions. Irremediable enmity would lead to fission; the Club would split in two.

That is not what happened. In the rain-soaked summer of 2009 the Club, more crowded even than usual, began to splinter into a multitude of factions, each demanding its own meeting room, etc., etc. The Club threatened to disintegrate altogether. Best friends fell out under the pressure of this social collapse. This was in accordance with th prediction of Davis [6], who denied the the fourth Leg—the force of friendship due to mutual enmity. The Club's survival was only ensured by the efforts of a few members of extraordinary wisdom and inspiration.

The Collectors' Club could hardly be more different. Our members, living almost like monks,[2] barely notice each other; their attention is all for the Club's innumerable curiosities, antiquities, natural specimens, objets d'art, and every kind of thing, assembled by the membership over centuries.[3] The Collectors feel strongly about the Club's collections, which may be enjoyed (members only) in comfortably furnished Treasure Rooms. Trouble arises in deciding which objects to display in which of the rooms. No Collector wants objects he or she dislikes out of the Treasure Rooms he or she frequents. When two members both love one item and yet disagree about another, the two objects ought to be kept in separate rooms. However, as there are only so many rooms, incompatibilities must arise. Though some of Burton's favorite objects are in Rooms 12, 16, and 24, the last named are two anatomical specimens that he finds repulsive. In Treasure Room 7, on the other hand, there is one fine piece amongst several ugly or dull ones that leave him reluctant to go there. Smythe, on the other hand, is fond of all the pieces in Room 24 as well as most of those in Room 7. Needless to say that this sort of inconsistency is distressing.

We Collectors have tried many times to eliminate this irritation to our feelings but without general success. Lately, however, Mrvar and Doreian [14] suggested a quantifiable goal. The Club should seek to reduce the sum, over all Collectors, of the number of repulsive objects the Collector encounters in favorite rooms and the number of attractive items in rooms in which

[1] Or, all dislikes would disappear; but that could not be expected.

[2] Not those in Sampson's classic study [16] of factions in a community modelled by a signed graph.

[3] The Collectors' Club is one of the last remaining of the antiquarian societies of the early modern era.

he or she feels uncomfortable. By doing so, they say, we can minimize the disequilibrium amongst the members. This idea has at last given us direction and, we hope, soon will bear fruit, although thus far we have found the necessary calculations difficult.

1.2. **Two-mode networks, or bipartite signed graphs.**

These different ways of measuring strain in a club, by the deviation from balance, clusterability, or the new Mrvar–Doreian measure, aroused my curiosity. I decided to compare them scientifically. The mathematical model of a club is a signed graph; that is, a graph whose edges are marked positive and negative. Let $Q(k)$ denote the *k-clusterability*, the measure suggested by Davis's analysis: this is the minimum number of inconsistent edges possible for a partition of the vertices of the graph into exactly k clusters—an edge being *inconsistent* when it is a positive edge between clusters or a negative edge within a cluster. (This measure suits the situation of a set of acquaintances whose relations may be friendly or unfriendly, and in particular the Acquaintances Club.) The comparison of frustration with k-clusterability is simple. The *frustration index*, the measure suggested by Cartwright and Harary's theory of structural balance, equals $Q(\leq 2)$, i.e., where the number of clusters is at most 2. The relationship between frustration index and k-clusterability is then essentially the behavior of Q as k varies, which was previously studied by Doreian and Mrvar in [9]. (They found that Q decreases weakly as k increases [9, Theorem 4].)

Mrvar and Doreian wanted to capture a distinction that was abandoned in the Cartwright–Harary theory: the difference between "subjects" (or, "actors") and "objects", which play different roles in Heider's original formulation of balance theory [12]. In their new structure, which they call "two-mode signed networks", the relations are only between actors and objects; thus, the graph is bipartite. (That is why this measure is appropriate for the Collectors' Club.) Their idea was to explore clusterability in the context of a signed bipartite graph whose two vertex sets, say U and W, are independently partitioned into clusters, with the aim of making the number of inconsistent edges as small as possible. This model suggested an important change: inconsistency is decided not directly by friendship or enmity but, in a sense to be explained later, by majority decision logic. The properties of this new idea plainly differ from those of Harary and Cartwright's balance and Davis's clustering. I explore here the mathematical relationship between frustration and Mrvar and Doreian's bipartite majority clusterability.

I expected the measure of bipartite majority clusterability to be less than the frustration index almost always, for almost any number of clusters in each of the two classes of vertices, since the majority rule for consistency surely would reduce the number of inconsistent edges in almost any signature. To my considerable surprise, when I examined the question for $K_{2,n}$ that was not the case. The actual proportion of all signatures of $K_{2,n}$ in which the index of majority clusterability with just two clusters in each vertex class is no less than the frustration index is asymptotically a fraction greater than 0. Increasing the number of clusters in the second vertex class to three, the proportion of signatures for which frustration equals biclusterability does become vanishingly small, a mere c/n^2.

2. Background

Here are basic definitions and technical background.

2.1. **Basic concepts.**

All our graphs are *simple*; that is, they have at most one edge joining any two vertices; and they are *bipartite*; that is, V is partitioned into two *color classes*, $U = V_1 = \{u_1, \ldots, u_{n_1}\}$ and $W = V_2 = \{w_1, \ldots, w_{n_2}\}$, and every edge has the form $u_k w_l$. The *complete bipartite graph* K_{n_1,n_2} has every possible edge $u_k w_l$. If X is a subset of vertices of a graph, X^c denotes the complementary subset $V \setminus X$.

A *signed graph* $\Sigma = (V, E, \sigma)$ is a graph (V, E), with vertex set V and edge set E, together with a sign function $\sigma : E \to \{+, -\}$. E^+ and E^- are the sets of positive and negative edges, respectively. The signed degrees of a vertex, $d^+(v)$ and $d^-(v)$, are its degrees in E^+ and E^-, respectively. The *net degree* is $d^\pm(v) := d^+(v) - d^-(v)$.

The signed graph Σ is *balanced* if in every circle ("cycle", "circuit") the product of the edge signs is positive. Harary's Balance Theorem [10] states that Σ is balanced if and only if V has a bipartition $\{X, Y\}$ (that is, $X \cup Y = V$ and $X \cap Y = \varnothing$; X or Y may be void) such that the negative edges are exactly those with one endpoint in X and one in Y.

The *frustration index* $l(\Sigma)$ is the smallest number of edges whose deletion (or sign reversal [11]) makes Σ balanced. Abelson and Rosenberg introduced this measure, calling it the "complexity" [1]; Harary called it the "line index of balance" [11].[4] Obviously, $l(\Sigma) \leq |E^-(\Sigma)|$. Finding the exact frustration index is a hard problem; Roth and Viswanathan [15] proved NP-hardness just for the special case where the underlying graph is a complete bipartite graph with the same number of vertices in each color class.

Clusterability is an extension of balance introduced by Davis [6], based on Harary's characterization of balance via bipartitions. We say Σ is *clusterable* if there is a partition of V into subsets, called *clusters* (or *plus-sets*), such that every positive edge is within a cluster and each negative edge goes between clusters. The partition is called a *clustering* of Σ. We say Σ is *k-clusterable* if it has a clustering with k clusters, and ($\leq k$)-clusterable if it has a clustering into at most k clusters. Harary's theorem says a signed graph is balanced if and only if it is (≤ 2)-clusterable.

The *clusterability index* $Q(\Sigma)$ is the smallest number of edges whose deletion (or negation [8]) makes Σ clusterable. Similarly, the k-clusterability index $Q(\Sigma; k)$ is the least number of edges whose deletion makes Σ k-clusterable.

2.2. **Bimodality.**

From now on, we assume we have a bipartite signed graph Σ without multiple edges.

A *biclustering* is a pair (π_1, π_2) where $\pi_1 = \{U_1, \ldots, U_{k_1}\}$ partitions U and $\pi_2 = \{W_1, \ldots, W_{k_2}\}$ partitions W. A *bicut* is an edge set

$$E_{ij} := E(U_i, W_j) := \{u_k w_l \in E : u_k \in U_i,\ w_l \in W_j\};$$

one may think of it as a cut induced by the biclustering in an induced subgraph.[5] E_{ij} is a *null bicut* if it has no edges. It is *homogeneous* if it has edges of only one sign.

Each bicut of (π_1, π_2) gets a sign by majority vote of its edges. If the number of positive edges is at least half the total number of edges, it is called positive. Otherwise, it is negative. It is *neutral* if it is not null but it has equally many edges of each sign. Neutral and null

[4]The term "frustration" (which Harary enjoyed) comes from Toulouse [17], writing on physics, but is well suited to social psychology.

[5]Mrvar and Doreian called a bicut a "block", but that conflicts with the normal use of the term "block" in graph theory. Their term refers to the fact that in the $U \times W$ bipartite adjacency matrix $B(\Sigma)$ of Σ, in which the (u_k, w_l) entry is the sign of $u_k w_l$, or 0 if there is no edge $u_k w_l$, the biclustering (π_1, π_2) induces a block structure in which each block is a $U_i \times W_j$ submatrix of $B(\Sigma)$.

bicuts may be given either sign. (Mrvar and Doreian define neutral bicuts to be positive [14].) An edge is *consistent* with (π_1, π_2) if its sign agrees with that of the bicut that contains it; the choice of sign of a neutral or null bicut does not affect the number of inconsistent edges in it.

The *biclusterability*, in full the *bipartite majority clusterability*, of (π_1, π_2) in Σ is

$$M(\pi_1, \pi_2) := M(\Sigma; \pi_1, \pi_2) := \text{the number of inconsistent edges.}$$

(This is twice Mrvar and Doreian's "criterion function" P with their usual value $\alpha = .5$.) The (k_1, k_2)-*biclusterability index* of Σ is Mrvar and Doreian's adaptation of Davis's idea of clusterability to the bipartite situation of persons and objects. It is defined as

$$M(k_1, k_2) := M(\Sigma; k_1, k_2) := \min\{M(\pi_1, \pi_2) : |\pi_1| = k_1 \text{ and } |\pi_2| = k_2\}.$$

A (k_1, k_2)-biclustering with minimal biclusterability is called a *minimal* (k_1, k_2)-*biclustering*. When considering $M(k_1, k_2)$ I assume $|U| \geq k_1$ and $|W| \geq k_2$. (We can also define an overall *biclusterability index* $M(\Sigma) := \min_{k_1,k_2} M(k_1, k_2)$, but the work of Mrvar and Doreian suggests it is less interesting.)

A partition π' is *finer* than π, $\pi' \leq \pi$, if they partition the same set and the clusters of π' are subsets of the clusters of π. (A partition is considered finer than itself.) The refinement ordering of biclusterings is componentwise: we say $(\pi_1', \pi_2') \leq (\pi_1, \pi_2)$ if $\pi_1' \leq \pi_1$ and $\pi_2' \leq \pi_2$. A useful lemma is drawn from Mrvar and Doreian's proof of [14, Theorem 4].

Lemma 1. *Suppose (π_1, π_2) and (π_1', π_2') are biclusterings of Σ such that π_1' is finer than π_1 and π_2' is finer than π_2. Then $M(\pi_1', \pi_2') \leq M(\pi_1, \pi_2)$.*

The full theorem is an immediate consequence.

Theorem 1 (Mrvar and Doreian [14, Theorem 4])**.** *$M(k_1, k_2)$ is a weakly decreasing function of k_1 and k_2. That is, if $n_1 \geq k_1' \geq k_1$ and $n_2 \geq k_2' \geq k_2$, then $M(k_1', k_2') \leq M(k_1, k_2)$.*

Corollary 2. *If $k_1 \leq n_1$ and $k_2 \leq n_2$, then $M(k_1, k_2) = M(\leq k_1, k_2) = M(k_1, \leq k_2) = M(\leq k_1, \leq k_2)$.*

Proof. Refine a minimal $(\leq k_1, \leq k_2)$-biclustering to a (k_1, k_2)-biclustering. $\square$

Corollary 2 is useful when we try to calculate exact clusterability indices, as in Section 5. If we have a general construction for a minimal (k_1, k_2)-biclustering, sometimes one of the clusters may happen to be empty; then the empty cluster is not counted as one of the clusters in the partition. Corollary 2 tells us that this possibility does not invalidate the use of that biclustering to determine $M(k_1, k_2)$.

2.3. **Switching.**

Switching a vertex set means reversing the signs of all edges that have one endpoint in that set. If X is the set switched, the switched signed graph is written Σ^X. Switching X and its complement have the same effect. Switching one of the color classes, say U, in a bipartite graph negates every edge: $\Sigma^U = -\Sigma$. The essential property of switching is that it preserves the signs of all circles; consequently, it preserves the frustration index. In fact, it has the stronger property that

$$l(\Sigma) = \min_X |E^-(\Sigma^X)|. \tag{1}$$

(This formula was stated first by Abelson and Rosenberg [1] .)

Arbitrary switching does not preserve biclusterability indices, but controlled switching does.

Lemma 3. *Given a biclustering (π_1, π_2), switching one or more clusters of one or both partitions does not change $M(\pi_1, \pi_2)$.*

The proof is easy. Call a set X that is a union of clusters *compatible* with the biclustering. Thus, compatible switching preserves the biclusterability index.

2.4. **The indicator signed graph.**

From a biclustering of type (k_1, k_2) we construct a bipartite graph of order (k_1, k_2) whose vertices are the clusters of $\pi_1 \cup \pi_2$, with an edge for each bicut that is neither null nor neutral, whose sign is the sign of that bicut. This is the *indicator signed graph*, $I(\pi_1, \pi_2)$.

Switching in the indicator corresponds to switching clusters in Σ. Vertices that are switched in $I(\pi_1, \pi_2)$ correspond to clusters to be switched in Σ. By Lemma 3, this switching in Σ does not change the value of $M(\pi_1, \pi_2)$. That is what makes the indicator graph important.

Lemma 4. *If the indicator signed graph is all positive, then $M(\pi_1, \pi_2) = |E^-(\Sigma)|$.*

If the indicator is balanced, then

$$M(\pi_1, \pi_2) = \min\{|E^-(\Sigma^X)| : X \textit{ is compatible with } (\pi_1, \pi_2)\} \geq l(\Sigma).$$

Proof. If each bicut is positive, obviously $M(\pi_1, \pi_2) = |E^-|$.

By switching vertices in the indicator and the corresponding clusters of (π_1, π_2) we may assume the indicator is all positive. Now every bicut is positive (or null or neutral), so $M(\pi_1, \pi_2) = |E^-| \geq l(\Sigma)$ by Equation (1). $\square$

Lemma 5. *Suppose $Z \subseteq V(I(\pi_1, \pi_2))$ corresponds to $X \subseteq V(\Sigma)$. If switching Z in $I(\pi_1, \pi_2)$ makes a negative edge positive but does not make a positive edge negative, then $|E^-(\Sigma^X)| < |E^-(\Sigma)|$.*

The *proof* is obvious. $\square$

3. The Critical Type

Our basic result shows that $M(2, 2)$ is, in an intuitive sense, a critical value for the biclusterability function.

Theorem 2. *For $1 \leq k_1 \leq n_1$ and $1 \leq k_2 \leq n_2$, the biclusterability indices $M(k_1, k_2)$ satisfy:*

(1) $M(1, 1) = \big||E^+| - |E^-|\big| \geq l(\Sigma)$.

(2) *Concerning $k_i \geq 2$,*

$$l(\Sigma) \leq M(1, 2) = M(1, 3) = \cdots = M(1, n_2) = \sum_{w \in W} \min(d^+(w), d^-(w))$$

and

$$l(\Sigma) \leq M(2, 1) = M(3, 1) = \cdots = M(n_1, 1) = \sum_{u \in U} \min(d^+(u), d^-(u)).$$

(3) $M(2, 2) \leq l(\Sigma)$. *Equality and strict inequality are both possible.*

Proof of (1) *and* (2)*, for type-*$(1, k_2)$ *biclusterings.* Consider a $(1, k_2)$-biclustering for general $k_2 \geq 1$. The bicuts are $E(U, W_j)$. Since the indicator is a forest, hence balanced, we can switch every W_j of a negative bicut; then all bicuts are positive and the number of negative edges equals the number of inconsistent edges. This shows that $M(\pi_1, \pi_2) \geq l(\Sigma)$, hence $M(1, k_2) \geq l(\Sigma)$ for every $k_2 \geq 1$.

For fixed $k_2 \geq 2$, take any partition $\pi_2 = \{W_1, \ldots, W_{k_2}\}$ of W. The number of inconsistent edges in $E(U, W_j)$ is

$$\min\Big(\sum_{w\in W_j} d^+(w), \sum_{w\in W_j} d^-(w)\Big).$$

The total over all clusters is

$$M(\{U\}, \pi_2) = \sum_{j=1}^{k_2} \min\Big(\sum_{w\in W_j} d^+(w), \sum_{w\in W_j} d^-(w)\Big).$$

This total is minimized if the sets W_j are chosen so that in each one all $d^\pm(w)$ have the same sign. We can get this result by choosing π_2 to be the total partition (the partition with n_2 singleton clusters), or at the other extreme by collecting all vertices with $d^\pm(w) \geq 0$ into W_1 and the remainder into W_2. Thus, $M(1, 2) = M(1, n_2)$. Furthermore,

$$M(\{U\}, \{W_1, W_2\}) = \sum_{w\in W_1} d^-(w) + \sum_{w\in W_2} d^+(w) = \sum_{w\in W} \min(d^+(w), d^-(w)),$$

which, incidentally, equals $|E^-(\Sigma^{W_2})|$. That proves $M(1, {\leq}2) = \sum_w \min(d^+(w), d^-(w))$. By Corollary 2, this is $M(1, 2)$. Now it is easy to see that $M(1, k_2) = M(1, 2) = M(1, n_2)$. □

Proof of (3). Switch by $X \subseteq V$ so $|E^-(\Sigma^X)| = l(\Sigma)$. Then $(\pi_1, \pi_2) := (\{U \cap X, U \setminus X\}, \{W \cap X, W \setminus X\})$ is a $({\leq}2, {\leq}2)$ biclustering with which X is compatible; thus $M(\Sigma; \pi_1, \pi_2) = M(\Sigma^X; \pi_1, \pi_2)$. In Σ^X, $M(\pi_1, \pi_2) = l(\Sigma)$ if the indicator signed graph $I := I(\Sigma; \pi_1, \pi_2)$ is balanced, by Lemma 4 and the fact that $|E^-(\Sigma^X)| = l(\Sigma)$. Contrariwise, if the indicator graph is unbalanced Σ^X will have $M(\pi_1, \pi_2) < l(\Sigma)$, because then $M(\pi_1, \pi_2) < |E^-(\Sigma^X)|$.

We have a $({\leq}2, {\leq}2)$ biclustering, not necessarily a $(2, 2)$ biclustering. If, say, $X \cap U = \varnothing$ or $X \supseteq U$, we can use as π_1 any partition of U into two clusters, thereby forming a $(2, {\leq}2)$ biclustering.

Later examples, especially signatures of $K_{2,n}$ (see Corollary 11), show concretely that $M(2, 2)$ and $l(\Sigma)$ may be equal or unequal. □

In the componentwise partial ordering of ordered pairs of integers, $M(k_1, k_2) \leq l(\Sigma)$ if $(k_1, k_2) \geq (2, 2)$ and otherwise $M(k_1, k_2) \geq l(\Sigma)$. For that reason I call $M(2, 2)$ the *critical value* of M and $(2, 2)$ the *critical type.*

4. A Raft of Questions

Theorem 2 raises three main questions.

4.1. **Is frustration index a biclusterability index?**

This is two questions.

Question 1. For which signed bipartite graphs does $M(1, 2) = l(\Sigma)$?

I believe this is an essentially difficult problem. Let us start with a bipartite signed graph Σ that has $l(\Sigma)$ negative edges (we say it is *reduced*; cf. Bowlin [4] where frustration in signed bipartite graphs gets a thorough treatment), and switch any subset of W. Then we get $M(1, \leq 2) = l(\Sigma)$. This suggests that detecting when $M(1, \leq 2) = l(\Sigma)$ is similar in difficulty to detecting when Σ is reduced. As it is a nontrivial matter to decide the latter (due to the difficulty of finding the frustration index), one would expect the same of the former. I shall not attempt to answer Question 1, except in the simple case of a signed $K_{2,n}$ (see Proposition 12).

Question 2. For which signed bipartite graphs does $M(2, 2)$ equal the frustration index $l(\Sigma)$?

The answer to this question for $K_{2,n}$ (in Proposition 9) led to the surprising conclusion that equality is frequent. This has a geometrical explanation that I discuss briefly in Section 5.2.

4.2. **Realizing the frustration index through clustering.**

Since $M(2, 2)$ can easily be less than $l(\Sigma)$, I wonder whether at least it is possible to find a biclustering that gives $l(\Sigma)$ exactly. Thus, the last questions are "realizability" versions of the preceding ones: I ask whether the frustration index is realizable by any $(2, 2)$-biclustering, not necessarily a minimal one.

Question 3. When does there exist a biclustering of given type (k_1, k_2) with $M(k_1, k_2) = l(\Sigma)$? Especially, (a) for $(k_1, k_2) = (1, k_2)$, (b) for $(k_1, k_2) = (2, 2)$, and (c) $(k_1, k_2) = (2, \leq 2)$. (A positive answer in(c) would be weaker than one to (b) because it would have little significance for the critical type $(2, 2)$.)

Call a biclustering with $M = l(\Sigma)$ *red* if of type $(2, 2)$ and *pink* if of type $(2, \leq 2)$ (so "pink" includes "red"). We verify in the next example that a red biclustering does not always exist.

Example 1. Consider Σ consisting of $K_{2,2}$ with various signatures. Define $K_{2,2}(1)$ to be $K_{2,2}$ with one negative edge, and $K_{2,2}(3)$ to have three negative edges. (The numbers 1 and 3 determine the signed graph up to symmetry.)

If $\Sigma = K_{2,2}(1)$, its frustration index is 1. The unique type-$(2, 2)$ biclustering, where every cluster is a singleton, has $M = 0 < l(\Sigma)$. There is (up to symmetry) one $(2, 1)$-biclustering; it has $M = 1 = l(\Sigma)$.

The switching-equivalent signature $K_{2,2}(3)$ behaves the same way.

With an even number of negative edges $K_{2,2}$ is balanced and the unique $(2, 2)$ biclustering gives $M = 0 = l(\Sigma)$ (as it must, by Theorem 2).

Example 2. For $K_{3,3}$ I found that just one choice of signs (and its negative) fails to have a red biclustering; and worse, it has no pink biclustering. Table 1 shows all signatures with up to four negative edges. This makes a complete list because a signed $K_{3,3}$ with more negative edges, switched by U (which negates all edges), has $9 - |E^-| \leq 4$ negative edges, while $l(\Sigma)$ and all $M(\pi_1, \pi_2)$ remain the same. Define $K_{3,3}(G)$ to be $K_{3,3}$ with negative subgraph G.

Each partition of U or W has one singleton cluster and one doubleton cluster. Consider how this works for $K_{3,3}(C_4)$, where the negative edge set is a circle of length 4. Say the C_4 is $u_2w_2u_3w_3u_2$. Up to symmetries of $K_{3,3}$ there are three ways to choose $U_1 = \{u\}$ and $W_1 = \{w\}$: (u, w) may be (u_1, w_1), (u_1, w_2), or (u_2, w_2). The biclusterabilities are 0, 2, and 3, respectively; none equals the frustration index, $l = 1$. That proves $K_{3,3}(C_4)$ has no red biclustering.

The signature $K_{3,3}(C_4)$ has no pink biclustering, either. With $U_1 = U$ and $W_1 = \{w\}$ we get $M = 2, 3$ when $w = w_1, w_2$.

Negative subgraph G	$\lvert E^-\rvert$	$l(\Sigma)$	X	U_1'	W_1'	$M(2,2)$	U_1	W_1
$\varnothing$	0	0	$\varnothing$	u_1	w_1	0	u_1	w_1
$P_2 \cup P_2 = w_2u_1w_3 \cup u_2w_1u_3$	4	0	u_1w_1	u_1	w_1	0	u_1	w_1
$K_{1,3}$	3	0	u_1	u_1	w_1	0	u_1	w_1
$P_1 = u_1w_1$	1	1	$\varnothing$	u_1	w_3	0	u_1	w_1
$P_2 = w_2u_1w_3$	2	1	u_1	u_1	w_2	0	u_1	w_1
$P_2 \cup P_1 = u_2w_2u_3 \cup u_1w_3$	3	1	u_1w_2	u_1	w_2	1	u_1	w_2
$K_{1,3} \cup_{w_1} P_1$	4	1	u_2	u_2	w_2	1	u_2	w_1
$C_4 = u_2w_2u_3w_3u_2$	4	1	$u_1w_2w_3$	**none**		0	u_1	w_1
$P_3 \cup P_1 = u_1w_3u_2w_1 \cup u_3w_2$	4	1	$u_1u_2w_2$	u_3	w_2	1	u_3	w_3
$P_1 \cup P_1 = u_1w_1 \cup u_2w_2$	2	2	$\varnothing$	u_3	w_3	1	u_1	w_1
$P_1 \cup P_1 \cup P_1 = u_1w_1 \cup u_2w_2 \cup u_3w_3$	3	2	$u_1u_2w_3$	u_3	w_3	2	u_3	w_3
$P_3 = u_1w_3u_2w_1$	3	2	u_2	u_2	w_3	1	u_3	w_2
$P_4 = w_1u_1w_2u_2w_3$	4	2	u_1u_2	u_3	w_3	2	u_3	w_3

TABLE 1. The signed $K_{3,3}$'s with their properties. $K_{3,3}(G)$ switches by the switching set X to $K_{3,3}(G')$ with $l(K_{3,3}(G')) = |E^-(G')|$. P_k is a path with k edges. $K_{1,3}$ has center u_2. $K_{1,3} \cup_{w_2} P_1$ has an extra edge w_2u_1. The singleton clusters U_1' and W_1' give a biclustering of type $(2,2)$ with $M(\pi_1, \pi_2) = l(\Sigma)$. The singleton clusters U_1 and W_1 give the minimum number $M(2,2)$ of inconsistent edges.

The examples suggest that $K_{2,2}$ and $K_{3,3}$ may simply be too small to give a positive answer to Question 3(b). This is certainly true in the family K_{2,n_2}; see Theorem 4.

Conjecture 1. Fix $n_1 \geq 2$. Every signed bipartite graph with $|U| = n_1$ and $|W|$ sufficiently large (how large depends on n_1) has a red biclustering. The same holds true for biclusterings of type $(2, \leq 2)$, but with many fewer exceptions.

The conjecture may gain plausibility from Bowlin's calculation of the maximum frustration index [4]. He found that $\max_\sigma l(K_{n_1,n_2}, \sigma)$, as a function of n_2 with n_1 fixed, settles down to a regular formula for sufficiently large n_2. This suggests there may be enough regularity in signatures of K_{n_1,n_2} as $n_2 \to \infty$ to enable a proof.

5. The case of $K_{2,n}$

Further evidence comes from a complete solution of the general example of signed K_{2,n_2}'s. Here, where $n_1 = 2$, Conjecture 1 is true, and moreover, both $M(2,2) = l(\Sigma)$ and $M(2,2) < l(\Sigma)$ are common amongst all signatures.

Every signed K_{2,n_2} has a certain simple structure. Of the n_2 vertices in W, some number a are negatively adjacent to both u_1 and u_2, $d_1 = b_1 + a$ are negative neighbors of u_1, and

$d_2 = b_2 + a$ are negative neighbors of u_2. Let $c := n_2 - (a + b_1 + b_2)$; this is the number of vertices that are positively adjacent to both u_1 and u_2. Obviously, $a, b_1, b_2, c \geq 0$ and $a + b_1 + b_2 + c = n_2$. Call this signed graph $K_{2,n_2}(a, b_1, b_2, c)$. It has $2c + b_1 + b_2$ positive edges and $2a + b_1 + b_2$ negative edges. Let

$$\begin{aligned}
A &:= \{w : \sigma(u_1w) = \sigma(u_2w) = -\}, && \text{so } |A| = a,\\
B_1 &:= \{w : \sigma(u_1w) = -,\ \sigma(u_2w) = +\}, && \text{so } |B_1| = b_1,\\
B_2 &:= \{w : \sigma(u_1w) = +,\ \sigma(u_2w) = -\}, && \text{so } |B_2| = b_2,\\
C &:= \{w : \sigma(u_1w) = \sigma(u_2w) = +\} = W \setminus [A \cup B_1 \cup B_2], && \text{so } |C| = c.
\end{aligned}$$

Remember that switching U (equivalently, W) has no effect on $l(\Sigma)$ or any $M(\pi_1, \pi_2)$. For our purposes, then, Σ^U is totally equivalent to Σ.

Proposition 6. *The signed graph $K_{2,n_2}(a, b_1, b_2, c)$ has frustration index equal to* $\min(b_1 + b_2, a + c)$. *It is balanced if and only if $b_1 = b_2 = 0$ or $a = c = 0$.*

Proof. Switching u_1 yields $K_{2,n_2}(b_2, c, a, b_1)$. Switching u_2 gives $K_{2,n_2}(b_1, a, c, b_2)$. Switching both we get $K_{2,n_2}(c, b_2, b_1, a)$. The effect of switching a vertex $w_j \in W$ is to reduce $|E^-|$ by 2 if w_j is a negative neighbor of u_1 and u_2, and to reduce it not at all otherwise. Therefore, by switching all vertices in W whose negative degree is 2 we reduce the number of negative edges in $K_{2,n_2}(a, b_1, b_2, c)$ to $b_1 + b_2$. By combining switching operations we can reduce it to $\min(b_1 + b_2, a + c)$ and no less. By Equation (1) this is $l(K_{2,n_2}(a, b_1, b_2, c))$. □

Call two vertices x and y *similar* if they have the same positive neighbors and the same negative neighbors. Thus, the similarity classes in W are A, B_1, B_2, C.

A crucial observation is that, if (π_1, π_2) is a biclustering and if Σ_j denotes the subgraph induced by $U \cup W_j$, and, then

$$M(\pi_1, \pi_2) = \sum_{W_j \in \pi_2} M(\Sigma_j; \pi_1, \{W_j\}). \tag{2}$$

5.1. **The biclusterability indices.**

The first question is the relationship between frustration and biclusterability. To answer that we find exact formulas for all the biclusterability indices $M(k_1, k_2)$.

Theorem 3. *A signed graph $\Sigma = K_{2,n_2}(a, b_1, b_2, c)$, where $n_2 \geq 2$, has*

$$\begin{aligned}
M(1,1) &= \min(2a + b_1 + b_2, 2c + b_1 + b_2),\\
M(1,k_2) &= b_1 + b_2 && \textit{for } 2 \leq k_2 \leq n_2,\\
M(2,1) &= \min(a + b_1, c + b_2) + \min(a + b_2, c + b_1),\\
M(2,2) &= \min(a + c, b_1 + b_2, a + b_1, a + b_2, c + b_1, c + b_2),\\
M(2,3) &= \min(a, b_1, b_2, c) && \textit{if } 3 \leq n_2,\\
M(2,k_2) &= 0 && \textit{for } 4 \leq k_2 \leq n_2.
\end{aligned}$$

Proof. The proof is mostly via a series of propositions which provide more detailed information.

Clearly, $M(1,1)$ is simply $\min(|E^+|, |E^-|)$. The only other $M(1,k_2)$ that needs examination is $M(1,2)$.

Proposition 7. *$M(1,2) = b_1 + b_2$. The difference $M(1,2) - l(\Sigma) = \min(b_1 + b_2 - a - c, 0)$; so $M(1,2) = l(\Sigma)$ if and only if $a + c \le b_1 + b_2$.*

Proof. Taking $W_1 = A$ and $W_2 = W \setminus A$ gives $M(\pi_1, \pi_2) = b_1 + b_2$. No other two-part partition π_2 can do better, because every vertex in $B_1 \cup B_2$ must contribute one edge to $M(\pi_1, \pi_2)$. $\square$

Proposition 8. *$M(2,1) = \min(a + b_1, c + b_2) + \min(a + b_2, c + b_1)$. The difference*

$$\begin{aligned} &M(2,1) - l(\Sigma) \\ &= \max\big[\min(a - b_2, c - b_1) + \min(a - b_1, c - b_2),\ \min(b_2 - a, b_1 - c) + \min(b_1 - a, b_2 - c)\big] \\ &= \max\big[\min S, -\max S\big], \end{aligned}$$

where $S := \{2a - b_1 - b_2, 2c - b_1 - b_2, a + c - 2b_1, a + c - 2b_2\}$. This difference is positive if and only if either $a, c > \frac{1}{2}(b_1 + b_2)$ and $\frac{1}{2}(a + c) > b_1, b_2$ (which can occur only when $a + c > b_1 + b_2$) or $a, c < \frac{1}{2}(b_1 + b_2)$ and $\frac{1}{2}(a + c) < b_1, b_2$ (which can occur only when $a + c < b_1 + b_2$).

Proof. The bicuts are $E(\{u_i\}, W)$ for $i = 1, 2$. The first has $a + b_1$ negative edges and $c + b_2$ positive ones. The second is similar. The minima of these pairs of numbers, added together, give $M(\pi_1, \pi_2)$.

To find the condition under which $M(2,1) > l(\Sigma)$ we reformulate α and β. First,

$$\alpha = \min\big(2a - b_1 - b_2, 2c - b_1 - b_2, a + c - 2b_1, a + c - 2b_2\big).$$

When $\alpha > 0$, necessarily $a + c > b_1 + b_2$, so $M(2,1) - l(\Sigma) = \alpha > 0$. That gives one case of inequality. The other follows similarly from taking $\beta > 0$. $\square$

The most important question is how the frustration index compares to the critical value $M(2,2)$. Both equality and inequality occur often.

Proposition 9. *$K_{2,n_2}(a, b_1, b_2, c)$ with $n_2 \ge 2$ has*

$$M(2,2) = \min(a + c, b_1 + b_2, a + b_1, a + b_2, c + b_1, c + b_2). \tag{3}$$

A minimal $(2, \le 2)$-biclustering has

$$\pi_2 = \begin{cases} \{A \cup B_1, C \cup B_2\} & \text{if } M(2,2) = \min(a + c, b_1 + b_2, a + b_2, c + b_1), \\ \{A \cup B_2, C \cup B_1\} & \text{if } M(2,2) = \min(a + c, b_1 + b_2, a + b_1, c + b_2). \end{cases}$$

Furthermore, $M(2,2) < l(\Sigma)$ if and only if, for $i = 1$ or 2, we have

(i$_i$) *$a < b_{3-i}$, $b_i < c$, $a \le c$, and $b_i \le b_{3-i}$; then $M(2,2) = a + b_i$ and*

$$l(\Sigma) - M(2,2) = \min(c - b_i, b_{3-i} - a);$$

or

(ii$_i$) *$c < b_{3-i}$, $b_i < a$, $c \le a$, and $b_i \le b_{3-i}$; then $M(2,2) = c + b_i$ and*

$$l(\Sigma) - M(2,2) = \min(a - b_i, b_{3-i} - c).$$

(These four cases are not mutually exclusive.)

Equation (3), in conjunction with Proposition 6, explains, differently than does the proof of Theorem 2(3), why $M(2,2) \le l(\Sigma)$ for a signed K_{2,n_2}: the former is a minimum over more numbers than is the latter.

When $a + b_i = 0$ or $c + b_{3-i} = 0$ for $i = 1$ or 2, the minimal biclustering mentioned in the proposition has type $(2, 1)$. Then one gets a minimal $(2, 2)$-biclustering by partitioning W into any two proper subsets.

Proof. In all the biclusterings we consider here, $\pi_1 = \{\{u_1\}, \{u_2\}\}$ and $\pi_2 = \{W_1, W_2\}$. Define A_j, B_{ij}, C_j to be the intersections with W_j of A, B_i, C, respectively, and $a_j := |A_j$, etc.

Write μ for the asserted value of $M(2, 2)$. The latter is no more than μ because of two biclusterings:

$$\begin{aligned} M(\pi_1, \{A \cup B_1, C \cup B_2\}) &= \min(a, b_1) + \min(c, b_2) \\ &= \min(a + c, a + b_2, b_1 + c, b_1 + b_2), \\ M(\pi_1, \{A \cup B_2, C \cup B_1\}) &= \min(a, b_2) + \min(c, b_1) \\ &= \min(a + c, a + b_1, b_2 + c, b_2 + b_1). \end{aligned}$$

The minimum of these two expressions is μ; thus, $M(2, 2) \leq \mu$.

We now prove μ is a lower bound for $M(\pi_1, \pi_2)$ over all $(2, 2)$-biclusterings. The first step is to analyze the various possible indicator graphs $I(\pi_1, \pi_2)$ of biclusterings. If the indicator is balanced, $M(\pi_1, \pi_2) \geq l(\Sigma)$ due to Lemma 4; that is, Σ cannot generate a case with $M(2, 2) < l(\Sigma)$.

Thus, we may assume the indicator graph is a signed $K_{2,2}$ with an odd number of negative edges. By switching only vertices in U we may ensure that in $I(\pi_1, \pi_2)$ each U vertex $\{u_i\}$ has negative degree at most 1 and that the negative edge is E_{i2}. As these switchings involve only u_1 and u_2, they are compatible with any $(2, 2)$-biclustering and therefore cannot affect the value of either $M(\pi_1, \pi_2)$ or $l(\Sigma)$, by Equation (1) and Lemma 3.

Since only E_{i2} is negative, $M(\pi_1, \pi_2) = |E^-| - |E^-_{i2}| + |E^+_{i2}| = |E^-| + (b_{3-i,2} + c_2) - (a_2 + b_{i2})$. Clearly, this number is decreased by moving all of $B_{3-i,2} \cup C_2$ from W_2 to W_1 and all of $A_1 \cup B_{i2}$ from W_1 to W_2, giving a new partition π'_2 with $W'_1 = A \cup B_{3-i}$ and $W'_2 = B_i \cup C$. We already know that $M(\pi_1, \pi'_2) \geq \mu$; thus, we have established the exact value of $M(2, 2)$.

Now we examine how $M(2, 2)$ can be less than $l(\Sigma)$. One way is to have $\mu = a + b_1 < l(\Sigma) = \min(a + c, b_1 + b_2)$. That immediately implies $b_1 < c$ and $a < b_2$. It also implies that $a + b_1 \leq a + b_2, c + b_1, c + b_2$, which are true if and only if $b_1 \leq b_2$, $a \leq c$, and $a + b_1 \leq c + b_2$. The last inequality is redundant; thus the conditions for $\mu = a + b_1 < l(\Sigma)$ are as stated in Case (i_1). The other four cases are exactly similar.

The value of $l(\Sigma) - M(2, 2)$ is a routine computation. □

Proposition 10. *We have $M(2, 3) = \min(a, b_1, b_2, c)$. One way to form a minimal $(2, \leq 3)$-biclustering is by taking the clusters of π_2 to be unions of similarity classes: clusters A, B_i, and $C \cup B_{3-i}$ if $M(2, 3) = a$ or b_i, or $A \cup B_{3-i}$, B_i, and C if $M(2, 3) = c$ or b_i (but omitting empty clusters in each case).*

Proof. For $M(2, \leq 3)$, partition W into $A \cup B_1$, B_2, and C. (By Corollary 2 it does not matter if any of these sets is empty.) Then $M(\pi_1, \pi_2) = \min(a, b_1)$ since the only possibly inhomogeneous bicut is $E(\{u_2\}, A \cup B_1)$, which has b_1 positive and a negative edges. Reversing the roles of u_1 and u_2 gives $\pi_2 = \{A \cup B_2, B_1, C\}$ with $M(\pi_1, \pi_2) = \min(a, b_2)$. Thus, $M(2, 3) \leq \min(a, b_1, b_2)$. Switching U interchanges the roles of a and c; hence, $M(2, 3) \leq \min(a, b_1, b_2, c)$. Two other partitions whose clusters are unions of similarity classes are $\{A, B_1 \cup B_2, C\}$ and $\{A \cup C, B_1, B_2\}$. The biclusterabilities in these cases are $\min(a, c)$ and $\min(b_1, b_2)$.

We now show that no $(2,3)$-biclustering can have lesser biclusterability. The proof is by the method of descent applied to $\nu(\pi_2) :=$ the number of (nonempty) clusters in the common refinement of π_2 and κ, where κ partitions W into its nonempty similarity classes. Evidently, $\nu(\pi_2) \geq \nu(\kappa)$, with equality if and only if every cluster in π_2 is a union of similarity classes in W.

Choose a minimal $\pi_2 = \{W_1, W_2, W_3\}$. If $\nu(\pi_2) > \nu(\kappa)$, some cluster, say W_1, is not a union of similarity classes. Then some vertex in W_1 is similar to a vertex of another cluster, say W_2. In other words, there is a similarity class in which both W_1 and W_2 have vertices.

Set $\Sigma' := \Sigma \setminus W_3$. Then Σ' has the $(2,2)$-biclustering $(\pi_1, \{W_1, W_2\})$. Define A', B_i', C' for Σ' like A, B_i, C for Σ, and notice that $A' = A \setminus W_3$, $B_i' = B_i \setminus W_3$, and $C' = C \setminus W_3$. Take π_2' as the $(2,\leq 2)$-biclustering $\{W_1', W_2'\}$ where $W_1' = A' \cup B_i'$ and $W_2' = C' \cup B_{3-i}'$ for $i = 1$ or 2. By Proposition 9, $M'(\pi_1, \pi_2')$ attains the $(2,\leq 2)$-biclusterability index $M'(2,2)$.

Now similar vertices in $W \setminus W_3$ are in the same cluster of π_2'. By Equation (2),

$$\begin{aligned} M(\pi_1, \pi_2) &= M'(\pi_1, \{W_1, W_2\}) + M(\Sigma_3; \pi_1, \{W_3\}) \\ &\geq M'(\pi_1, \{W_1', W_2'\}) + M(\Sigma_3; \pi_1, \{W_3\}) = M(\pi_1, \{W_1', W_2', W_3\}). \end{aligned}$$

Since $M(\pi_1, \pi_2) = M(2,3)$, equality holds throughout; consequently, $(\pi_1, \{W_1', W_2', W_3\})$ is another minimal biclustering. If we replace π_2 by $\{W_1', W_2', W_3\}$, any two similar vertices in $W \setminus W_3$ will belong to the same cluster. That is, we have decreased the value of $\nu(\pi_2)$.

It follows that there exists a minimal $(2,\leq 3)$-biclustering in which each cluster is a union of similarity classes. Two clusters W_j are similarity classes and one is the union of two similarity classes. We checked every such partition π_2 at the beginning of the proof and found that their biclusterability indices are not less than the expression in the proposition. $\square$

To conclude the proof of Theorem 3 it remains to evaluate $M(2, k_2)$ for $k_2 \geq 4$. For $M(2,4)$, partition W into the sets A, B_1, B_2, and C. Each bicut is homogeneous, so $M(\pi_1, \pi_2) = 0$. When $k_2 > 4$, $M(2, k_2) = 0$ by Theorem 1. $\square$

5.2. **The frequency of equality.**

The data in the propositions lets us estimate the proportion of all isomorphism types of signed K_{2,n_2}'s for which any $M(k_1, k_2)$ equals the frustration index.

From the viewpoint of our analysis, it seems reasonable to distinguish only signatures of K_{2,n_2} that differ in their parameters a, b_1, b_2, c. Signatures that have the same parameters are isomorphic under isomorphisms that fix u_1 and u_2. I refer to the different isomorphism types of signature under these isomorphisms as *signature types*. We shall estimate the proportion of all signature types that have $M(2,2) < l(\Sigma)$ or satisfy a similar inequality or equation. That equality holds in many cases may be due to some still hidden relationship between the critical value and the frustration index.

The total number of signature types is the number of solutions of $a + b_1 + b_2 + c = n_2$ in nonnegative integers, which is $\binom{n_2+3}{3} \approx n_2^3/3!$.

Corollary 11. *Approximately* $2/3$ *of all signature types of* K_{2,n_2} *have* $M(2,2) < l(\Sigma)$.

Proof. For large n_2, almost all signature types have a unique minimum in the formula (3) for the value of $M(2,2)$. By the symmetry of the expressions in that formula, each expression is the unique minimum in about 1/6 of the cases. The cases in which $M(2,2) < l(\Sigma)$ are precisely those in which neither $a + c$ nor $b_1 + b_2$ is minimal. That happens in about 4/6 of signature types. $\square$

Corollary 12. *For a signed K_{2,n_2} with $n_2 \geq 2$, approximately half of all signature types $K_{2,n_2}(a, b_1, b_2, c)$ have $M(1,2) = l(\Sigma)$. $M(2,1) = l(\Sigma)$ in approximately $12/n_2^2$ of all signature types of K_{2,n_2}.*

Proof. The estimate for when $M(1,2) = l(\Sigma)$ is obvious from Proposition 7, since out of all non-negative solutions of $(a + c) + (b_1 + b_2) = n_2$ for large n_2, approximately half the time $a + c < b_1 + b_2$ and approximately half the time the reverse holds.

That $M(2,1) = l(\Sigma)$ if and only if the signed graph is balanced, is obvious from Proposition 6 and the value of $M(2,1)$ from Theorem 3.

Asymptotically, there are about $n_2^3/6$ signature types of which about $2n_2$ are balanced; thus, the approximate proportion with $M(2,1) = l(\Sigma)$ is $12/n_2^2$. □

Thus, a great many signature types have frustration index equal to $M(1,2)$, but vanishingly few have it equal to $M(2,1)$. Formally, $M(1,2)$ and $M(2,1)$ have symmetrical arguments. I suggest the difference is due to the smallness of n_1; for larger fixed n_1, as $n_2 \to \infty$ there should be a smaller but still positive proportion of signature types for which $M(1,2) = l(\Sigma)$.

Corollary 13. *The proportion of signature types $K_{2,n_2}(a, b_1, b_2, c)$ for which $M(2,3) = l(\Sigma)$ is approximately $12/n_2^2$.*

Proof. From Proposition 6 we see that $M(2,3) = l(\Sigma)$ if and only if the signature is balanced. The proportion of signature types for which that is true is $12/n_2^2$, as shown in the proof of Corollary 12. □

The geometry of indices. The fact that the frustration index and the $(2,2)$-biclusterability index are equal in a substantial proportion of signature types of $K_{2,n}$ has an explanation in geometry. Think of a signature type as a quadruple (a, b_1, b_2, c) of nonnegative integers; then it is an integer point in $\mathbb{R}^4$ which lies in the simplex defined by $x_1 + x_2 + x_3 + x_4 = n$ and all $x_i \geq 0$. The conditions for $l(\Sigma) = M(2,2)$ are linear inequalities in $\mathbb{R}^4$. By Ehrhart's theory of counting integer points (cf. e.g. [3]), the asymptotic proportion of integer points in the region where $l(\Sigma) = M(2,2)$ equals the proportion of the volume of the whole simplex that satisfies the linear inequalities.

5.3. **Realizing frustration index with a biclustering.**

The third main question was whether a signed K_{2,n_2} has a biclustering whose biclusterability equals the frustration index. That is almost invariably so. Thus we have in this case a deeper explanation of the inequality $M(2,2) \leq l(\Sigma)$.

Theorem 4. *With the exception of $K_{2,2}(1)$ and $K_{2,2}(3)$, every signed K_{2,n_2} with $n_2 \geq 2$ has a biclustering (π_1, π_2) of type $(2,2)$ such that $M(\pi_1, \pi_2) = l(\Sigma)$.*

Proof. The partition π_1 is necessarily $\big\{\{u_1\}, \{u_2\}\big\}$, with two singleton clusters. The problem is to produce a partition $\pi_2 = \{W_1, W_2\}$ of W such that $M(\pi_1, \pi_2) = l(\Sigma)$, when $M(2,2) < l(\Sigma)$. By Lemma 4, $M(\pi_1, \pi_2) = l(\Sigma)$ if the indicator graph $I(\pi_1, \pi_2)$ is balanced.

By Proposition 9 we may assume $M(2,2) = a + b_i$ or $c + b_i$. By switching U if necessary we may assume $M(2,2) = a + b_i$; by choice of notation we may assume $i = 1$. Thus, $a + b_1 < a + c, b_1 + b_2$; that is, $a < b_2$ and $b_1 < c$.

Let $W_1 = \{w, w'\}$ where $w \in C$ and $w' \in B_2$. Then E_{11} is neutral so the indicator is balanced. The only difficulty is that W_2 might be void; then the signed graph is the exceptional graph $K_{2,2}(3)$. □

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