Topological Hyperplane Arrangements

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Topological hyperplane (*topoplane*):

$$Y \subset X \text{ such that } (X, Y) \cong (\mathbb{R}^n, \mathbb{R}^{n-1}).$$

$\mathcal{A}$: finite set of topoplanes.

Intersection semilattice:

$$\mathcal{L} := \{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{A} \text{ and } \bigcap \mathcal{S} \neq \varnothing\},$$

partially ordered (as is customary) by reverse inclusion.

Flat: An intersection (an element of $\mathcal{L}$).

Main definition:

$\mathcal{A}$ is an *arrangement of topoplanes* if:

$\forall\ H \in \mathcal{A}$ and $\forall\ Y \in \mathcal{L}$, either

$$Y \subseteq H$$

or

$$H \cap Y = \varnothing$$

or

$$H \cap Y \text{ is a topoplane in } Y.$$

Main Examples:

- Arrangement of real hyperplanes in $\mathbb{R}^n$ (homogeneous or affine). (Winder
  Vergnas)
- Arrangement of affine pseudohyperplanes representing an oriented matroid
- Intersection of a real hyperplane arrangement with a convex set. (Alexand
  Zaslavsky)

Induced arrangement in a flat Y:

$$\mathcal{A}^Y := \{Y \cap H : H \in \mathcal{A} \text{ and } Y \not\subseteq H \text{ and } Y \cap H \neq \varnothing\}.$$

Region: Connected component of complement $X \smallsetminus \bigcup \mathcal{A}$.

Face: Region of any $\mathcal{A}^Y$.

Theorem (Zaslavsky, 1977):

$$\# \text{ regions of } \mathcal{A} = \sum_{Y \in \mathcal{L}} |\mu(X, Y)|, \tag{1}$$

where μ is the Möbius function of $\mathcal{L}$,
assuming the side condition that every region is a topological cell.

Primary Question:
Is this really new? Can we finagle it out of something 'simpler'? Las Vergna
pseudohyperplane arrangements (oriented matroids)?

I.e.: $\exists \mathcal{A}'$ such that $\bigcup \mathcal{A}' = \bigcup \mathcal{A}$ and $\mathcal{A}'$ is a pseudohyperplane arrangement?

Answer:

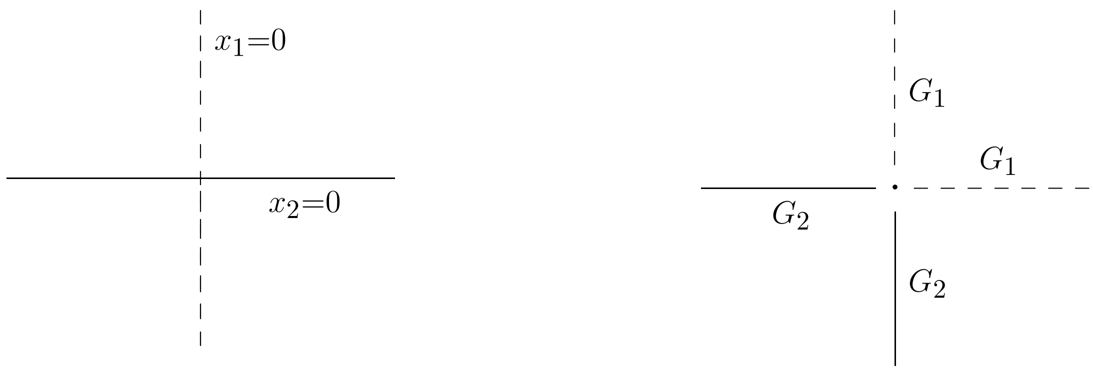
No !

Intersecting topoplanes may have the topology of two crossing hyperplanes,

(2) $$(X, H_1, H_2, H_1 \cap H_2) \cong (\mathbb{R}^n, x_1 = 0, x_2 = 0, x_1 = x_2 = 0),$$

or of two noncrossing 'flat' topoplanes,

(3) $$(X, H_1, H_2, H_1 \cap H_2) \cong (\mathbb{R}^n, G_+, G_-, x_1 = x_2 = 0).$$

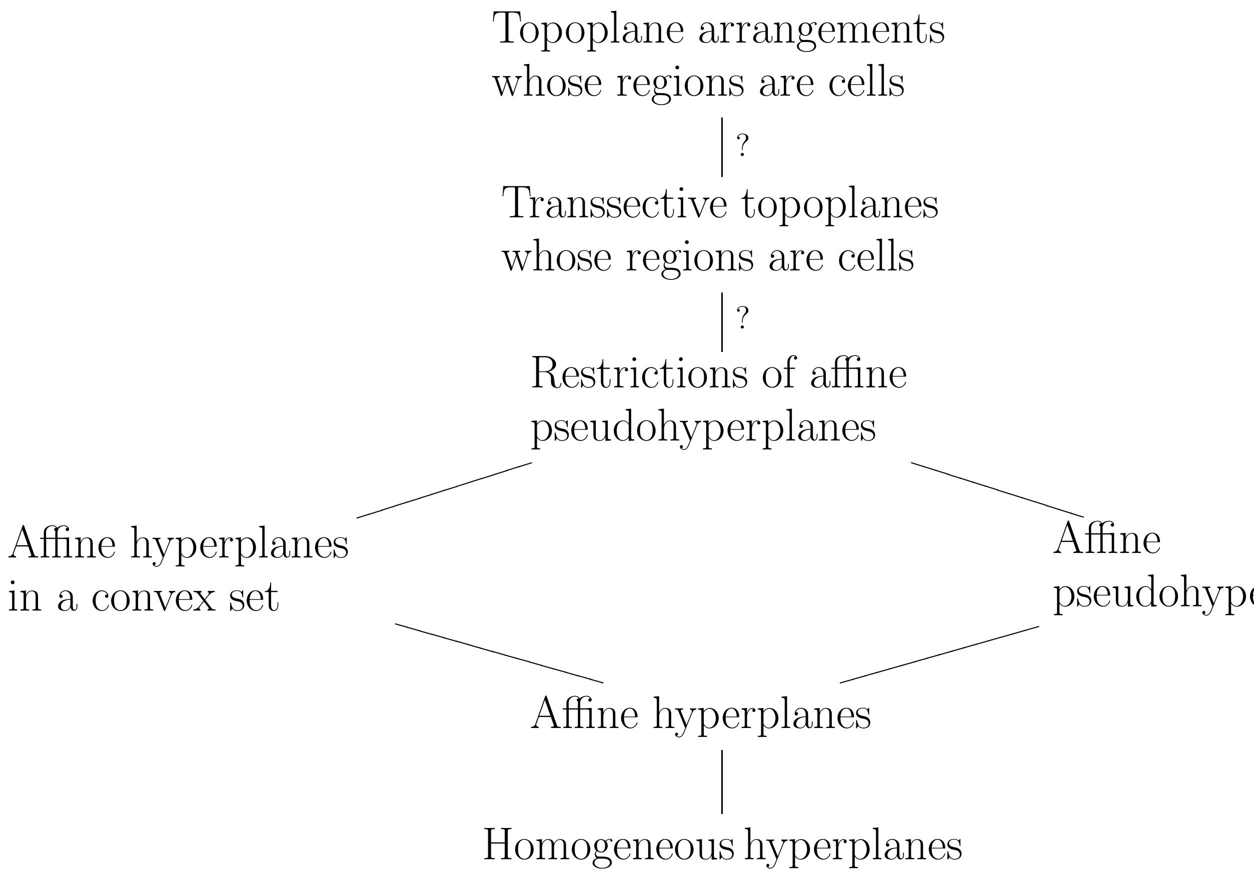


(2) (3)

Definition: $\mathcal{A}$ is *transsective* if every intersecting pair of topoplanes cr

Fact: An arrangement of (affine) pseudohyperplanes is transsective.

Types of Topoplane Arrangement



Reglueing

This means there is another arrangement, $\mathcal{A}'$, that has the same faces as $\mathcal{A}$:

$$\bigcup \mathcal{A}' = \bigcup \mathcal{A}.$$

In the Plane:

Theorem 9. *For any arrangement of topolines, there is a transsective every topoline intersection is a crossing).*

(Proof by construction.)

Higher Dimensions:

Theorem 10. *For a simple topoplane arrangement in which every r there is a transsective reglueing.*

(Proof by construction.)

Theorem 10$'$. *For a nonsimple such arrangement, there need not be reglueing.*

(Proof by example.)

Elementary properties

(1) If $\mathcal{A}$ is an arrangement of topoplanes and Y is a flat, then the induced
an arrangement of topoplanes.
(2) For an arrangement of topoplanes, each interval in $\mathfrak{L}$ is a geometric la
given by codimension.
(3) Suppose every region is a cell. Topoplanes H_1 and H_2 cross if and only
each other and each of the regions they form has boundary that meets bo
$H_2 \smallsetminus H_1$.
(4) In a topoline arrangement every face is a cell.

Lemma 4. *Suppose every region is a cell.* H_1 *and* H_2 *cross iff they inte*
region they form has boundary that meets both $H_1 \smallsetminus H_2$ *and* $H_2 \smallsetminus H_1$*.*

Proof: Easy.

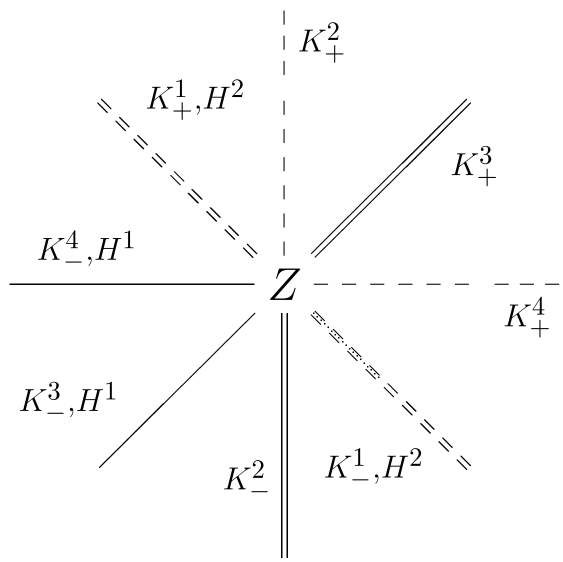
Lemma 6. *Suppose every region is a cell. If* H_1 *and* H_2*, cross, then* $Y \cap$
cross in $\mathcal{A}^Y$ *for each* $Y \in \mathfrak{L}$ *such that* $Y \cap H_1, Y \cap H_2$ *are distinct topopl*

Proof: Not as easy as you might think.

Reglueing in the Plane

Theorem 9. *For any arrangement of topolines, there is an arrangeme*
same faces, and in which every intersection is a crossing.

Proof Sketch. We apply the method of descent to the number of noncrossing pa
ing topolines. Suppose noncrossing topolines H^1, H^2 have intersection point Z



$\mathcal{A}' := \{K^1, K^2, K^3, K^4\}$.

$\mathcal{A}'$ has the same faces. Must check: $\mathcal{A}'$ is an arrangement of topolines (tak
with fewer noncrossing pairs (nearly obvious).

Since there are fewer noncrossing topoline pairs in the new arrangement, by
process we get a transsective arrangement.

Reglueing Fails in Three Dimensions

Proof of Theorem $10'$ by a counterexample of five topoplanes in $\mathbb{R}^3$:

$$\begin{aligned} H_1 &= \{x : x_1 = 0\}, \\ H_2 &= \{x : x_2 = 0\}, \\ H_3 &= \{x : x_2 = |x_1|\}, \\ H_4 &= \{x : x_3 = 0\}, \\ H_5 &= \{x : x_2 + x_3 = 0\}. \end{aligned}$$

Every pair crosses except H_2 and H_3. The common point of all topoplanes is

The 1-dimensional flats are:

$$\begin{aligned} Z := H_1 \cap H_2 \cap H_3 &= \{x : x_1 = x_2 = 0\}, \\ H_1 \cap H_4 &= \{x : x_1 = x_3 = 0\}, \\ H_1 \cap H_5 &= \{x : x_1 = 0, x_2 + x_3 = 0\}, \\ Y := H_2 \cap H_4 \cap H_5 &= \{x : x_2 = x_3 = 0\}, \\ H_3 \cap H_4 &= \{x : x_2 = |x_1|, x_3 = 0\}, \\ H_3 \cap H_5 &= \{x : x_2 = |x_1| = -x_3\}. \end{aligned}$$

The only two 1-dimensional flats that lie in three topoplanes are Z and Y.

Fact: It is impossible to have a transsective arrangement whose regions are th
of this one.

Simple Arrangements Reglue

$\mathcal{A}$ is *simple* if every flat is the intersection of the fewest possible topoplanes. *multiple.*

Theorem 10. *For a simple topoplane arrangement in which every region is an arrangement that has the same faces, and in which every topoplane a crossing.*

Proof Sketch. Similar to the planar proof: the method of descent on the numbe intersecting pairs of topoplanes. The construction is the same. The complicati but not too bad.

To show that $\mathcal{A}'$ is an arrangement of topoplanes we consider the intersection H and a flat Y of the reglued arrangement $\mathcal{A}'$. This is the hard part of the p four cases, depending mostly on whether either H or Y is a topoplane or flat arrangment $\mathcal{A}$.

Topoplanes vs. pseudohyperplanes

Projective pseudohyperplane arrangement:
A finite set $\mathcal{P}$ of subspaces in $\mathbb{RP}^n$ such that

- each $(\mathbb{RP}^n, H) \cong (\mathbb{RP}^n, \mathbb{RP}^{n-1})$,
- the intersection Y of any members of $\mathcal{P}$ is a $\mathbb{RP}^d$, and
- for any other H, either $Y \subseteq H$ or $H \cap Y$ is a pseudohyperplane in Y.

Known: Every region is an open cell and its closure is a closed cell.

Affine pseudohyperplane arrangement:

$$\mathcal{P}_0 := \{H \setminus H_0\} \text{ in } \mathbb{R}^n = \mathbb{RP}^n \setminus H_0.$$

$\mathcal{A}$ is *projectivizable* if it is homeomorphic to a $\mathcal{P}_0$.

Two topoplanes are *parallel* if they are disjoint.

Lemma 11. *If a topoplane arrangement is projectivizable then it is tra*
allelism is an equivalence relation on topoplanes, and every region is a ce

Proof: Easy.

Look at a transsective topoplane arrangement in which parallelism is an equiv

1. Disconnection:

$\mathcal{A}$ is *connected* if $\bigcup \mathcal{A}$ is connected.

Disconnected $\mathcal{A}$ may be a pseudohyperplane arrangement. But:

Counterexample:
Put $\mathcal{A}_1$ and $\mathcal{A}_2$ in the right and left halfspaces of $\mathbb{R}^n$. Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is discon

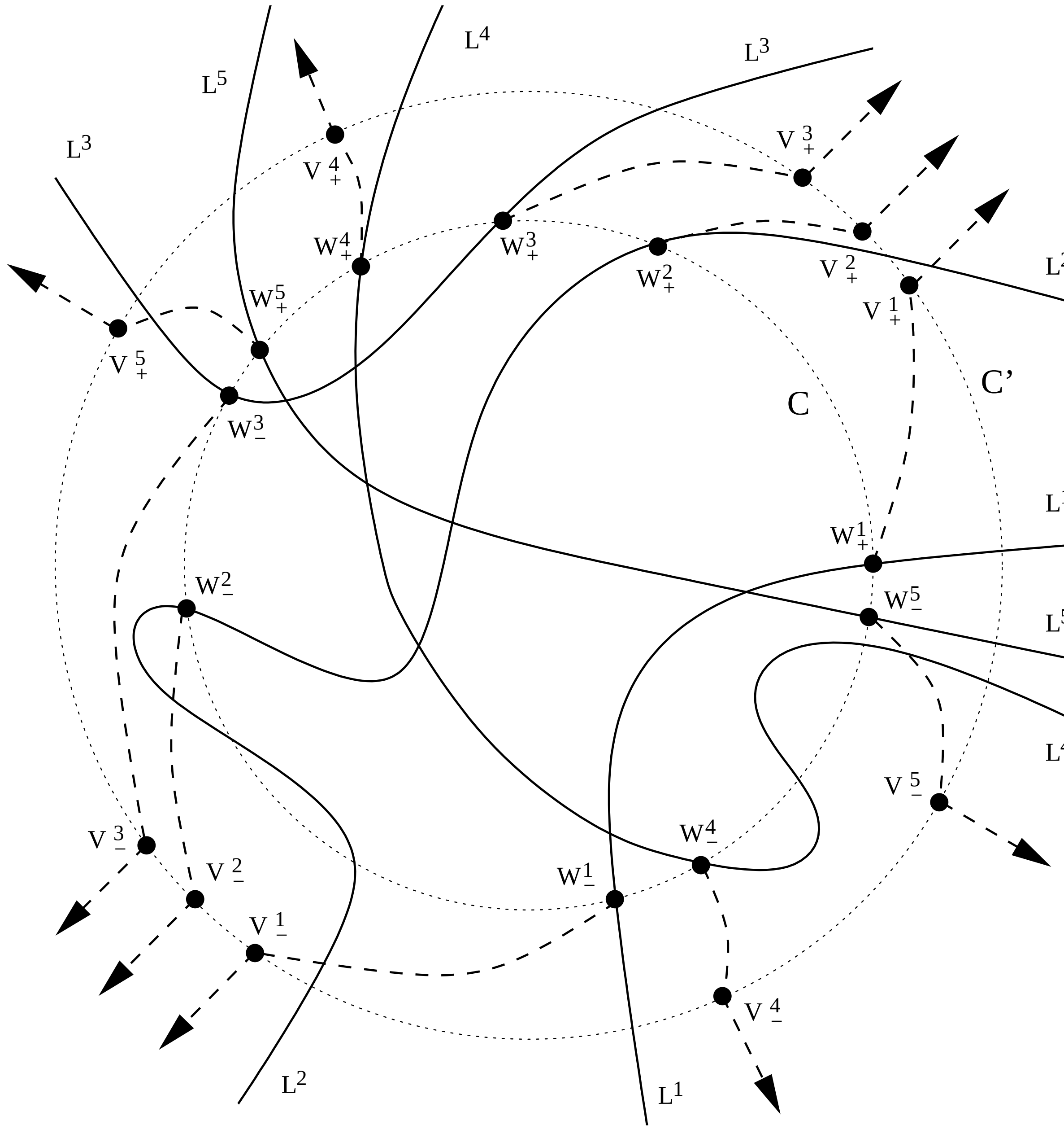
Proposition 12. *If $\mathcal{A}_1$ has a pair of intersecting topoplanes, $\mathcal{A}_1 \cup \mathcal{A}_2$ tivizable.*

Proof: Easy.

In the Plane:

Theorem 13. *A topoline arrangement is projectivizable iff it is transs allelism is an equivalence relation.*

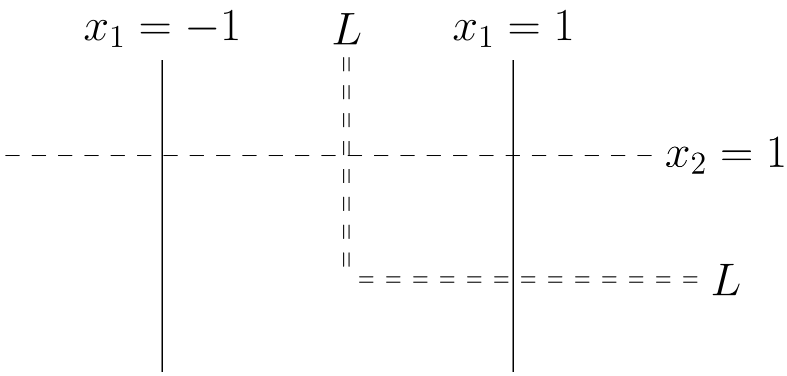
The diagram (next) shows the construction.



L3
L5
L4
L3
V +4
W+4
W+3
V +3
V +2
W+2
V +1
L2
W+5
V +5
W−3
C
C'
W+1
L1
W−2
W−5
L5
L4
V −5
W−4
V −3
V −2
W−1
V −1
V −4
L2
L1

2. Connection:

Counterexample:



$$L := \{x : x_1 x_2 = 0 \text{ and } x_1, x_2 \geq 0\}.$$

Parallelism is not transitive. Connected, transsective, but not projectivizable

Question:

In higher dimensions, is intransitivity of parallelism the only obstruction?

3. Restriction to a Domain

Restriction of $\mathcal{A}$ to a domain:

$$\mathcal{A}^D := \{\text{components of } H \cap D : H \in \mathcal{A} \text{ and } H \cap D \neq \varnothing\},$$

where $D \subseteq \mathbb{R}^n$ is a cellular domain and $\mathcal{A}^D$ is a topoplane arrangement in D. (Alexanderson and Wetzel, Zaslavsky)

Properties:

- $\mathcal{A}^D$ is transsective if $\mathcal{A}$ is transsective.
- Parallelism could be transitive in $\mathcal{A}$ but not in $\mathcal{A}^D$.

Theorem (Las Vergnas, unpublished). *Any transsective topoline* *the restriction to a cellular domain of a projectivizable arrangement.*

Question:

In higher dimensions, is failure of transsectivity the only obstruction to being t a projectivizable arrangement?

Las Vergnas has an apparent counterexample in dimension 3, being studied Alfonsín.

Open Questions

(1) Is the condition that every region be a cell ever superfluous?

(2) Are there simple properties that imply all intersecting topoplanes cross? there are enough topoplanes?

(3) Complexify!

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