The Dimension of the Negative Cycle Vectors of a Signed Graph

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Abstract

A *signed graph* is a graph Γ where the edges are assigned sign labels, either "+" or "−". The sign of a cycle is the product of the signs of its edges. Let $\mathrm{SpecC}(\Gamma)$ denote the list of lengths of cycles in Γ. We equip each signed graph with a vector whose entries are the numbers of negative k-cycles for $k \in \mathrm{SpecC}(\Gamma)$. These vectors generate a subspace of $\mathbb{R}^{\mathrm{SpecC}(\Gamma)}$. Using matchings with a strong permutability property, we provide lower bounds on the dimension of this space; in particular, we show for complete graphs, complete bipartite graphs, and a few other graphs that this space is all of $\mathbb{R}^{\mathrm{SpecC}(\Gamma)}$.

1 Introduction

A *signed graph* Σ is a graph Γ whose edges have sign labels, either "+" or "−". The sign of a cycle in the graph is the product of the signs of its edges. Write $c_l^-(\Sigma)$ for the number of negative cycles of length l in Σ and collect these numbers in the *negative cycle vector* $c^-(\Sigma) = (c_3^-, c_4^-, \ldots, c_n^-) \in \mathbb{R}^{n-2}$, where n is the order of Σ. We are interested in the structure of the collection $\mathrm{NCV}(\Gamma)$ of all negative cycle vectors of signings of a fixed underlying simple graph Γ.

There are (at least) three natural questions raised by the existence of these collections of vectors. Most simply, what is their dimension? This is the question we address here. The *cycle spectrum* $\mathrm{SpecC}(\Gamma)$ is the list of lengths of cycles in Γ; $\mathrm{NCV}(\Gamma)$ is a subset of $\mathbb{R}^{\mathrm{SpecC}(\Gamma)}$ and generates an affine subspace (which is a linear subspace since the negative cycle vector $c^-(+\Gamma)$ corresponding to the all-positive signing is the zero vector). We develop a general approach to the dimension question in terms of "permutable matchings" (see Section 2.3) that allows us to prove for $\Gamma = K_n$, $K_{m,n}$, and the Petersen graph that $\mathrm{NCV}(\Gamma)$ has dimension $|\mathrm{SpecC}(\Gamma)|$; it also gives us a lower bound for the Heawood graph and one other graph family. (We also solve a few examples with an *ad hoc* method.)

Secondly, what is their convex hull? In [3] and [5], Popescu and Tomescu gave inequalities bounding the numbers of negative cycles in a signed complete graph, which is a step towards the answer for K_n. A related question: Do the facets of the convex cone generated by $\mathrm{NCV}(\Gamma)$ have combinatorial meaning?

Finally, which vectors in the convex hull are actually the vectors of signed graphs? Recently Kittipassorn and Mészáros [1] gave strong restrictions on the number of negative triangles in a signed K_n. Again, this provides a step towards that answer.

Our work was originally motivated by the complete graph and a natural extension to complete bipartite graphs. Those cases and others led to the following plausible conjecture.

Conjecture 1.1 (Schaefer, 2017). For any graph Γ, $\dim \mathrm{NCV}(\Gamma) = |\mathrm{SpecC}(\Gamma)|$.

2 Background

2.1 *Graphs*

A *graph* is a pair $\Gamma = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is a (finite) set of *vertices* and E is a (finite) set of unordered pairs of vertices, called *edges*. Our graphs are all unlabeled, simple, and undirected. Thus, all cycle lengths are between 3 and n.

The number of cycles of length l in Γ is $c_l = c_l(\Gamma)$. The cycle vector of Γ is $c(\Gamma) = (c_3, c_4, \ldots, c_n)$; sometimes we omit the components that correspond to lengths l not in the cycle spectrum.

2.2 *Signed graphs*

A *signed graph* is a triple $\Sigma = (V, E, \sigma)$ where $\Gamma = (V, E)$ is a graph (the *underlying graph* of Σ) and $\sigma : E \to \{+, -\}$ is the *sign function*. The *sign of a cycle* is the product of the signs of its edges; a signed graph in which every cycle is positive is called *balanced*. The *negative edge set* E^- is the set of negative edges of Σ and the *negative subgraph* is $\Sigma^- = (V, E^-)$, the spanning subgraph of negative edges. We sometimes write Γ_N for Γ signed so that N is its set of negative edges.

Switching Σ means choosing a vertex subset $X \subseteq V$ and negating all the edges between X and its complement. Switching yields an equivalence relation on the set of all signings of a fixed underlying graph. If Σ_2 is isomorphic to a switching of Σ_1, we say that Σ_1 and Σ_2 are *switching isomorphic*. This relation is an equivalence relation on signed graphs; we denote the equivalence class of Σ by $[\Sigma]$. A signed graph is balanced if and only if it is switching isomorphic to the all-positive graph. Signed graphs that are switching isomorphic (like those in Figure 1) have the same negative cycle vector.

As with $c(\Gamma)$, we may omit the components of $c^-(\Sigma)$ that correspond to lengths l not in the cycle spectrum. Also, we may write either $c^-(\Sigma)$ or $c^-(\sigma)$, the latter when only the signature σ is varying.

Figure 1: Two switching equivalent signings of K_6, with the same negative cycle vector $(10, 18, 36, 36)$. Solid lines are positive, dashed lines are negative.

The *negation* of Σ is $-\Sigma = (V, E, -\sigma)$, in which the sign of every edge is negated.

Sometimes Σ and $-\Sigma$ are switching isomorphic, e.g., when Σ is bipartite or when it is a signed complete graph whose negative subgraph is self-complementary.

2.3 *Permutable matchings*

A *matching* in Γ is a set M of pairwise nonadjacent edges; it is *perfect* if $V(M) = V$. A matching M (or any other edge set) is *permutable* if the automorphism group of Γ acts on the edges of M as the symmetric group $S_{|M|}$. We base our results largely on permutable matchings, after Zaslavsky noticed their utility in proving our results for complete and complete bipartite graphs. The advantage of permutability is that, in counting negative cycles using a permutable matching, any two equicardinal subsets belong to the same number of negative cycles of each length. That makes it feasible to calculate the numbers in the vectors we use to estimate the dimension of $\mathrm{NCV}(\Gamma)$.

Our introduction of permutable matchings led to the question: Which graphs have permutable matchings? That has been investigated by Schaefer and Swartz in [4]; they find large families of examples. On the other hand, there are only a few kinds of graph with permutable perfect matchings; Schaefer and Swartz determine them all.

3 Rank and Dimension

The dimension of $\mathrm{NCV}(\Gamma)$ is the rank of the matrix whose rows are the negative cycle vectors of all signatures of Γ. (The columns of this matrix may be regarded as corresponding to all lengths $k \in \{3, 4, \ldots, n\}$, or only the lengths in $\mathrm{SpecC}(\Gamma)$, depending on which is more convenient. The column of $k \notin \mathrm{SpecC}(\Gamma)$, if included, is all zero.) We know the rank cannot be greater than $|\,\mathrm{SpecC}(\Gamma)|$, the number of nonzero columns, so if we produce a submatrix of that rank we have proved that $\dim \mathrm{NCV}(\Gamma) = |\,\mathrm{SpecC}(\Gamma)|$. That is what we now endeavor to do with the aid of a permutable matching.

Even if permutable matchings fail to reach the spectral upper bound, they imply a lower bound. However, we are happy to say that in our three main examples, permutable matchings solve the dimension problem.

The rank of a matrix A is written $\mathrm{rk}(A)$.

3.1 *Any negative edge set*

We begin with the most general calculation. Given a signed graph Γ_N with an arbitrary negative edge set $N \subseteq E$, how many negative cycles are there of each length? For $X \subseteq N$ let $f_l(X) :=$ the number of l-cycles that intersect N precisely in X. We get a formula for f by Möbius inversion from $g_l(X) :=$ the number of l-cycles that contain X, since

$$g_l(X) = \sum_{X \subseteq Y \subseteq N} f_l(Y),$$

which implies that

$$f_l(X) = \sum_{X \subseteq Y \subseteq N} (-1)^{|Y|-|X|} g_l(Y).$$

The number of negative l-cycles is the number of l-cycles that intersect N in an odd number of edges; therefore,

$$\begin{aligned} c_l^-(\Gamma_N) = \sum_{X \subseteq N,\, |X| \text{ odd}} f_l(X) &= \sum\sum_{X \subseteq Y \subseteq N,\, |X| \text{ odd}} (-1)^{|Y|-|X|} g_l(Y) \\ &= \sum_{Y \subseteq N} g_l(Y) \sum_{X \subseteq Y,\, |X| \text{ odd}} (-1)^{|Y|-|X|} \\ &= \sum_{\emptyset \neq Y \subseteq N} (-2)^{|Y|-1} g_l(Y). \end{aligned} \tag{3.1}$$

This applies to every underlying graph Γ.

3.2 *A matrix calculation*

Now assume we have a graph Γ of order n and m unbalanced sign functions $\sigma_1, \ldots, \sigma_m$ in addition to the all-positive function $\sigma_0 \equiv +$. To avoid redundancy we want the associated signed graphs to be switching nonisomorphic. For instance, choosing more than half the edges at a vertex to be negative is switching equivalent to choosing fewer than half, so we would not want the negative edge set to contain more than $\frac{1}{2}(\deg(v) - 1)$ of the edges incident with any vertex v.

For the present assume n is even. Here is the matrix of the negative cycle vectors of all signings σ_s and their negatives, with columns segregated by parity. The rows are one for $+\Gamma$ ($\sigma_0 \equiv +$), then m rows for the unbalanced signatures σ_s, $0 < s \leq m$, then $-\Gamma$ (the signature $-\sigma_0 \equiv -$), then the m negations $-\sigma_s$. The relationship between the upper and lower halves is that

$$c_l^-(-\sigma_s) = \begin{cases} c_l - c_l^-(\sigma_s) & \text{if } l \text{ is odd.} \\ c_l^-(\sigma_s) & \text{if } l \text{ is even.} \end{cases}$$

The resulting matrix is

$$\left(\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ c_3^-(\sigma_1) & c_5^-(\sigma_1) & \cdots & c_{n-1}^-(\sigma_1) & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots & c_n^-(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_3^-(\sigma_m) & c_5^-(\sigma_m) & \cdots & c_{n-1}^-(\sigma_m) & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots & c_n^-(\sigma_m) \\ c_3 & c_5 & \cdots & c_{n-1} & 0 & 0 & \cdots & 0 \\ c_3 - c_3^-(\sigma_1) & c_5 - c_5^-(\sigma_1) & \cdots & c_{n-1} - c_{n-1}^-(\sigma_1) & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots & c_n^-(\sigma_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ c_3 - c_3^-(\sigma_m) & c_5 - c_5^-(\sigma_m) & \cdots & c_{n-1} - c_{n-1}^-(\sigma_m) & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots & c_n^-(\sigma_m) \end{array}\right). \tag{3.2}$$

Row operations reduce this matrix to

$$\left(\begin{array}{cccc|cccc}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
c_3^-(\sigma_1) & c_5^-(\sigma_1) & \cdots & c_{n-1}^-(\sigma_1) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
c_3^-(\sigma_m) & c_5^-(\sigma_m) & \cdots & c_{n-1}^-(\sigma_m) & 0 & 0 & \cdots & 0 \\
c_3 & c_5 & \cdots & c_{n-1} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots & c_n^-(\sigma_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots & c_n^-(\sigma_m)
\end{array}\right). \tag{3.3}$$

Ignoring the first row of zeroes, this is a block matrix

$$A := \begin{pmatrix} U & O \\ c_{\mathrm{odd}}(\Gamma) & \mathbf{0} \\ O & R \end{pmatrix}.$$

The middle row $c_{\mathrm{odd}}(\Gamma)$, consisting of the odd-cycle numbers of Γ, corresponds to $-\Gamma$. The upper left block U is the matrix of negative odd-cycle vectors of the unbalanced signatures σ_s, and the lower right block R is the matrix of negative even-cycle vectors of the same signatures. We infer the fundamental fact that:

Lemma 3.1. *The rank of the negative cycle matrix* (3.2) *equals the sum of the ranks of* $\begin{pmatrix} U \\ c_{\mathrm{odd}}(\Gamma) \end{pmatrix}$ *and* R*.*

Lemma 3.1 is written for even n but by putting into U a column for c_{n+1}^- we include the odd cycles of order $n+1$ (n still being even). This can be handled by the same computation. The reduced matrix in this case is

$$\left(\begin{array}{ccccc|cccc}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
c_3^-(\sigma_1) & c_5^-(\sigma_1) & \cdots & c_{n-1}^-(\sigma_1) & c_{n+1}^-(\sigma_1) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
c_3^-(\sigma_m) & c_5^-(\sigma_m) & \cdots & c_{n-1}^-(\sigma_m) & c_{n+1}^-(\sigma_m) & 0 & 0 & \cdots & 0 \\
c_3 & c_5 & \cdots & c_{n-1} & c_{n+1} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & c_4^-(\sigma_1) & c_6^-(\sigma_1) & \cdots & c_n^-(\sigma_1) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & c_4^-(\sigma_m) & c_6^-(\sigma_m) & \cdots & c_n^-(\sigma_m)
\end{array}\right). \tag{3.4}$$

For a bipartite graph $U = O$ and $c_{\mathrm{odd}} = \mathbf{0}$, so only R needs to be considered.

3.3 *Permutable negative matchings*

Henceforth we assume we have chosen a fixed permutable matching M_m of m edges in Γ. For each $s = 1, 2, \ldots, m$ we choose a submatching $M_s \subseteq M_m$ of s edges and we define the signature σ_s as that of the signed graph Γ_{M_s}. (It does not matter which M_s we use, because M_m is permutable.) This generates a matrix of negative cycle vectors as in (3.2).

In particular, in K_n the biggest permutable edge set is a perfect or near-perfect matching. This turns out to be "perfect" for our purposes. (An almost equally big set is half the edges incident to one vertex, but we found that to be useless since then the entire matrix (3.2) has rank 1.)

Permutability implies that $g_l(Y)$ depends only on $|Y|$ so we may define $G_l(k) = g_l(Y)$ for any one k-edge subset $Y \subseteq M_m$. Then (3.1) becomes

$$c_l^-(\Gamma_{M_s}) = \sum_{k=1}^{s} (-2)^{k-1} \binom{s}{k} G_l(k) = \sum_{k=1}^{n} (-2)^{k-1} \frac{G_l(k)}{k!} (s)_k, \tag{3.5}$$

where $(x)_k$ denotes the falling factorial, $(x)_k = x(x-1)\cdots(x-[k-1])$. Formula (3.5) gives $c_l^-(\Gamma_{M_s})$ as a polynomial function $p_l(s)$ without constant term, of degree d_l where d_l as the largest integer k for which $G_l(k) > 0$; that is, d_l is the largest size of a submatching of M_m that is contained in some cycle of length l. (We leave d_l undefined if no l-cycle intersects M_m.) Clearly, $d_l \leq m$.

(Our reasoning works equally well for subsets of any permutable edge set N in any graph. It is easy to see that there are only three possible kinds of permutable set: a matching, a subset of the edges incident to a vertex, and the three edges of a triangle. We mentioned that a permutable set of edges at a vertex is useless for K_n. We have not seen a graph where a triangle's edges might help find the dimension.)

We illustrate our calculations with K_n as a running example. The data is from Section 4.1. Let $m = \lfloor n/2 \rfloor$. The number of l-cycles in K_n that intersect a maximum matching M_m in a fixed set of k edges is

$$G_l(k) = \binom{n-2k}{l-2k} (l-k-1)! \cdot 2^{k-1}.$$

A column of U or R is not all zero if and only if it corresponds to a cycle length l for which there exists an l-cycle in Γ that intersects M_m. Such a column contains m values of the polynomial $p_l(s)$. Since p_l has degree at most m and no constant term, these values determine p_l completely.

Now a nonzero column in U or R for cycle length l looks like this:

$$\begin{pmatrix} p_l(1) \\ p_l(2) \\ \vdots \\ p_l(m) \end{pmatrix} = \begin{pmatrix} \alpha_l 1^{d_l} + \cdots \\ \alpha_l 2^{d_l} + \cdots \\ \vdots \\ \alpha_l m^{d_l} + \cdots \end{pmatrix}, \tag{3.6}$$

since p_l is a polynomial of degree d_l; here $\alpha_l = (-2)^{d_l-1} G_l(d_l)/d_l!$.

Suppose the set $\Delta_{\text{odd}} := \{d_3, d_5, \ldots, d_{n-1}\}$ has δ_{odd} (distinct) elements and the set $\Delta_{\text{even}} := \{d_4, d_6, \ldots, d_n\}$ has δ_{even} elements. The number of polynomial degrees represented in the columns of U is δ_{odd} (which may be less than the number of nonzero columns), and similarly for R.

In K_n with a maximum matching, $\Delta_{\text{odd}} = \{3, 5, \ldots\}$ (odd numbers up to n) and $\Delta_{\text{even}} = \{4, 6, \ldots\}$ (even numbers up to n).

Lemma 3.2. *The rank of U is at least δ_{odd} and that of R is at least δ_{even}.*

The rank of $\begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix}$ is $\operatorname{rk}(U) + 1$ if there is an odd length l such that an l-cycle exists in Γ but no l-cycle intersects M_m.

Proof. In U choose one column of each different degree d_l. Divide by the leading coefficient α_l; this does not affect the rank. Now add columns of the form $\left(l^d\right)_{s=1}^{m}$ for every $d = 1, 2, \ldots, m$ that is not in Δ_{odd}. Column operations allow us to eliminate the lower-degree terms of the column (3.6), leaving a Vandermonde matrix with 1^d in the top row and m^d in the bottom row of column d for each $d = 1, 2, \ldots, m$. The rank of is m. Now reverse the column operations; the rank remains the same, so the columns of U must have full column rank.

The same reasoning applies to R.

The extra 1 in the rank of $\begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix}$ arises from the fact that, under the assumption, it has a column that is zero in U but is nonzero in c_{odd}. $\square$

By this lemma, for K_n the ranks of U and R are $\lceil n/2 \rceil - 1$ and $\lfloor n/2 \rfloor - 1$, respectively, which sum to $n - 2$.

3.4 *Theorems*

Lemma 3.2 yields our principal general theorem. Given a matching M_m and a cycle length $l \in \operatorname{SpecC}(\Gamma)$, define

$$\mu(l) := \max_{C_l} |C_l \cap M_m|,$$

maximized over all l-cycles C_l.

Theorem 3.3. *Let M_m be a permutable m-matching in Γ. Then*

$$\begin{aligned} |\{\mu(l) : \textit{odd } l \in \operatorname{SpecC}(\Gamma)\}| + |\{\mu(l) > 0 : \textit{even } l \in \operatorname{SpecC}(\Gamma)\}| \\ \leq \dim \operatorname{NCV}(\Gamma) \leq |\operatorname{SpecC}(\Gamma)|. \end{aligned}$$

Suppose that every even cycle length, and all odd cycle lengths with at most one exception, are values of $\mu(l)$. Then $\operatorname{NCV}(\Gamma)$ spans $\mathbb{R}^{|\operatorname{SpecC}(\Gamma)|}$.

Proof. The value of $\mu(l)$ is the degree d_l of the polynomials $p_l(s)$ if such a polynomial exists. The polynomial exists and d_l is defined if and only if some $C_l \cap M_m \neq \emptyset$, in other words if and only if $\mu(l) > 0$. Thus, there is a value $\mu(l) = 0$ for some odd $l \in \operatorname{SpecC}(\Gamma)$ if and only if $\operatorname{rk} \begin{pmatrix} U \\ c_{\text{odd}} \end{pmatrix} = \operatorname{rk}(U) + 1$. $\square$

There is a simpler statement that applies to graphs with a sufficiently omnipresent permutable matching. Given m, define $\nu_{\rm odd}(m) :=$ the number of odd lengths $l < 2m$ in $\mathrm{SpecC}(\Gamma)$, $+1$ if there is an odd cycle length $l \geq 2m$, and define $\nu_{\rm even}(m) :=$ the number of even lengths $l < 2m$ in $\mathrm{SpecC}(\Gamma)$, $+1$ if there is an even cycle length $l \geq 2m$.

Theorem 3.4. *Suppose M_m is a permutable m-matching in Γ and for every length $l \in \mathrm{SpecC}(\Gamma)$ there exists a cycle C_l such that $|C_l \cap M_m| = \min(m, \lfloor l/2 \rfloor)$. Then $\dim \mathrm{NCV}(\Gamma) \geq \nu_{\rm odd}(m) + \nu_{\rm even}(m)$.*

The hypothesis can be lessened since, if there is any cycle length $l \geq 2m$, it suffices to have one length $l \geq 2m$ for which there is a C_l with $|C_l \cap M_m| = m$.

Proof. The hypotheses imply that

$$d_l = \begin{cases} \lfloor l/2 \rfloor & \text{if } l \leq 2m, \\ m & \text{if } l \geq 2m. \end{cases}$$

We count the number of distinct values d_l for odd and even cycle lengths. For odd l we get $(l-1)/2$ if $l \in \mathrm{SpecC}(\Gamma)$ and $l < 2m$, and we get m if and only if there exists a cycle length $l \geq 2m$. The total is $\nu_{\rm odd}$. The computation of $\nu_{\rm even}$ is similar.

The values of $\mu(l)$ in Theorem 3.3 are the same as those of d_l unless there is a cycle length for which no l-cycle intersects M_m; but that is ruled out by our hypotheses. Theorem 3.4 follows. □

A graph is *bipancyclic* if it is bipartite and has a cycle of every even length from 4 to n.

Corollary 3.5. *Assume Γ is pancyclic and has a permutable m-matching M_m, and for every l with $3 \leq l \leq n$ there is an l-cycle C_l with $|C_l \cap M_m| = \min(m, \lfloor l/2 \rfloor)$. Then $\dim \mathrm{NCV}(\Gamma) = n-2$ if $2m \geq n-1$, and $n-2 \geq \dim \mathrm{NCV}(\Gamma) \geq 2m-1$ if $2m \leq n-2$.*

Assume Γ is bipancyclic and has vertex class sizes p, q with $p \leq q$, and it has a permutable m-matching M_m such that for every k with $2 \leq k \leq p$ there is a $2k$-cycle C_{2k} with $|C_{2k} \cap M_m| = \min(m, k)$. Then $\dim \mathrm{NCV}(\Gamma) = p-1$ if $m = p$, and $p-1 \geq \dim \mathrm{NCV}(\Gamma) \geq m-1$ if $m \leq p-1$.

The hypotheses can be lessened in the same way as those of Theorem 3.4.

Proof. If Γ is pancyclic, $\nu_{\rm odd}$ counts all the numbers $3, 5, \ldots, 2m-1$ plus 1 for $2m+1$ if $n > 2m$, and $\nu_{\rm even}$ counts the numbers $4, 6, \ldots, 2m-2$ plus 1 for $2m$ since $n \geq 2m$. Thus

$$\nu_{\rm odd} + \nu_{\rm even} = \begin{cases} (m) + (m-1) = 2m-1 & \text{if } n > 2m, \\ (m-1) + (m-1) = 2m-2 & \text{if } n = 2m. \end{cases}$$

The conclusion follows easily.

If Γ is bipancyclic, then $\nu_{\rm even} = m-1$ and the conclusion follows easily. □

The two most complete graphs are easy consequences of any of the preceding results, but especially of Corollary 3.5.

Corollary 3.6. *For a complete graph K_n with $n \geq 3$,* $\dim \mathrm{NCV}(K_n) = n - 2$.

For a complete bipartite graph $K_{p,q}$ with $p, q \geq 2$, $\dim \mathrm{NCV}(K_{p,q}) = \min(p, q) - 1$.

4 Examples

4.1 *The Compleat Complete Graph*

We need to supply a missing computation for K_n. But first, let us see the negative cycle vectors of all signings of small complete graphs.

The vectors for K_3 are

$$(0),\ (1)$$

(from the balanced and unbalanced triangle). The vectors for K_4 are

$$(0,0),\ (2,2),\ (4,0)$$

(the all-positive graph, one negative edge, and two nonadjacent negative edges). Here are the vectors for K_5:

$$(0,0,0),\ (3,6,6),\ (4,8,8),\ (5,10,6),\ (6,8,4),\ (7,6,6),\ (10,0,12);$$

and for K_6:

$$\begin{array}{llll}
(0,0,0,0), & (4,12,24,24), & (6,18,36,36), & (8,20,32,24), \\
(10,18,36,36), & (8,24,40,32), & (10,22,36,28), & (12,24,24,32), \\
(10,26,36,28), & (8,24,48,32), & (14,18,36,36), & (12,24,32,32), \\
(12,20,40,24), & (10,30,36,20), & (16,12,48,24), & (20,0,72,0).
\end{array}$$

The number of switching isomorphism classes of complete graphs grows super-exponentially [2]. Since two signed graphs which yield different vectors must belong to different classes, one naturally wonders about the converse property, that the vector uniquely identifies a switching class. This is true up through K_7 but false for K_8: see Figure 4.1 below (found by Gary Greaves, whose assistance we greatly appreciate). Thus when $n \geq 8$ there are (certainly when $n = 8$ and surely also for all larger orders) fewer vectors than classes, but in general there will still be a very large number.

Now we carry out the missing computation of the function G_l of Section 3.3. Consider the signed K_n's whose negative edges are s nonadjacent edges, for $0 \leq s \leq \lfloor n/2 \rfloor$. It is straightforward to compute g_l. For a fixed $k \geq 1$ and set Y with $|Y| = k$, we need to form an l-cycle using Y and $l - k$ other edges. (Since Y is a matching, we know that $l \geq 2k$.) So we choose $l - 2k$ of the remaining $n - 2k$ vertices, and then create our cycle as follows: imagine contracting the edges in Y; the resultant vertices, together with the other $l - 2k$ vertices, will form an $l - k$-cycle in the contracted graph

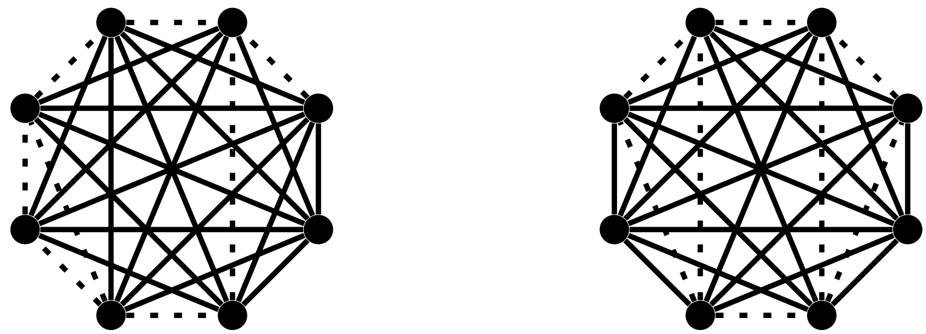


Figure 2: Two switching inequivalent signings of K_8 with the same negative cycle vector $(28, 108, 336, 848, 1440, 1248)$.

(which will eventually give an l-cycle in K_n). Cyclically order these $l-k$ "vertices"; this orders the vertices in our actual cycle while ensuring the edges from Y remain. There are $\frac{(l-k-1)!}{2}$ ways to do this. Then, we expand the contracted edges to regain them; there are 2 ways to do this for each edge. So we have

$$g_l(Y) = \binom{n-2k}{l-2k}(l-k-1)! \cdot 2^{k-1},$$

whence

$$G_l(k) = \binom{n-2k}{l-2k}(l-k-1)! \cdot 2^{k-1}.$$

By Equation (3.5), $c_l^-(s)$ is a polynomial in s of degree $d_l = \lfloor l/2 \rfloor$ and the general formula is

$$c_l^-(s) = \sum_{k=1}^{n} \binom{s}{k}(-4)^{k-1}\binom{n-2k}{l-2k}(l-k-1)!,$$

For example, $c_3^-(s) = s(n-2)$ and $c_4^-(s) = s(n^2+5n+8) - 2s^2$. This formula for $c_l^-(s)$ demonstrates that the degrees d_l of the odd polynomials are all distinct, and the same for the even polynomials; consequently our main theorem 3.3 itself implies that the matrix of negative cycle vectors $c^-(s)$ has full rank $n-2$.

4.2 *Complete Bipartite Graphs*

We move along to $K_{p,q}$, which always has $p \leq q$. We use a maximum matching M_p, i.e., we set $m = p$.

To get $c_{2l}^-(K_{p,q})$ we compute g_{2l} (where the subscript is now $2l$ because all cycles have even length). Call the two independent vertex sets $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$. For a fixed k-edge set $Y = \{a_{i_1}b_{j_1}, \ldots, a_{i_k}b_{j_k}\} \subseteq M_p$, where $k \leq l$, we need to form a $2l$-cycle using Y and $2l-2k$ other vertices. Fix one edge $y_1 \in Y$, say $y_1 = a_{i_1}b_{j_1}$. Choose $l-k$ of the remaining $p-k$ vertices from A, in order, in one of $(p-k)_{l-k}$ ways; $l-k$ of the remaining $q-k$ vertices from B, also in order, in one of $(q-k)_{l-k}$ ways; and shuffle the sequences together as $(a_{i_{k+1}}, b_{j_{k+1}}, \dots, a_{i_l}, b_{j_l})$. Insert Y into this $2(l-k)$-sequence by inserting y_1 before $a_{i_{k+1}}$ (which we may do because each Y edge must be between an A vertex and a B vertex), treating the resulting sequence as cyclically ordered (which can be done in only one way since the A neighbor of y_1 appears after y_1); then ordering $Y \setminus \{y_1\}$ in one of $(k-1)!$ ways as $(y_2, \dots, y_k)$; and finally inserting $y_2, \dots, y_k$ anywhere into the cycle in one of

$$\binom{[2(l-k)+1]+[k-1]-1}{[2(l-k)+1]-1} = \binom{2l-k-1}{k-1}$$

ways. (When those edges are inserted into the cycle, there is only one way to orient each edge.) The net result is that

$$G_{2l}(k) = g_{2l}(Y) = (p-k)_{l-k}(q-k)_{l-k} \cdot (k-1)! \binom{2l-k-1}{k-1}.$$

Then by Equation (3.5), for $2 \leq l \leq p$,

$$c^-_{2l}(s) = \sum_{k=1}^{p} (s)_k \frac{(-2)^{k-1}}{k} (p-k)_{l-k}(q-k)_{l-k} \binom{2l-k-1}{k-1}.$$

This explicit formula for the negative cycle vectors $c^-(s)$, with Theorem 3.3, implies that $\dim \mathrm{NCV}(K_{p,q}) = p = \min(p,q)$.

4.3 *The Petersen graph*

Next we consider the Petersen graph P, which has four cycle lengths, 5, 6, 8, and 9, so $\dim \mathrm{NCV}(P) \leq 4$. It lacks a permutable 4-matching. In fact:

Theorem 4.1. *A 3-regular graph that is arc transitive cannot have a permutable 4-matching.*

Proof. By [4, Theorem 1.1] an arc-transitive graph with a permutable m-matching, where $m \geq 4$, must have degree at least m. $\square$

The Petersen graph does have a permutable 3-matching, in fact, two kinds.

The first kind consists of alternate edges of a C_6. In the language of Theorem 3.3, we must compute $\mu(l) = |\max\{C_l \cap M_3\}|$ for each cycle length. We find with little difficulty that $\mu(5) = 2$, $\mu(6) = 3$, $\mu(8) = 2$, and $\mu(9) = 3$. Therefore $|\Delta_{\mathrm{odd}}| = 2$ and $|\Delta_{\mathrm{even}}| = 2$, whence, despite only having a 3-matching, we can deduce that $\dim \mathrm{NCV}(P) = 4$. We even know the negative cycle vectors corresponding to negative 0-, 1-, 2-, and 3-submatchings and the negated signatures; they are (in order of matching size)

$$\begin{array}{cccc} (0,0,0,0), & (4,4,8,12), & (6,6,8,10), & (6,10,0,10) \\ (12,0,0,20), & (8,4,8,8), & (6,8,8,10), & (6,10,0,10). \end{array}$$

The bottom vector in each column corresponds to the negated signing.

The second kind of permutable 3-matching consists of three edges at distance 3. The first matching type also is three equally spaced edges in a C_9, but not every such subset of a C_9 is also a set of alternating edges of a C_6; the other such subsets are 3-matchings of the second kind. This second kind generates negative cycle vectors from negative submatchings and the corresponding negated sign functions whose dimension is only 3, not 4. (With this matching the negated signatures are switching isomorphic to unnegated signatures.) This shows that not all permutable m-matchings in a graph are equally useful.

4.4 *The Heawood graph*

The Heawood graph H is bipartite and has five cycle lengths, 6, 8, 10, 12, and 14, so $\dim \mathrm{NCV}(H) \leq 5$. It has a permutable 3-matching, indeed three different kinds, for instance alternate edges of a 6-cycle. Using that 3-matching we find that $\mu(6) = 3$ (obviously), $\mu(8) = 2$, $\mu(10) = 3$, $\mu(12) = 3$, and $\mu(14) = 3$. These are two different values, thus $\dim \mathrm{NCV}(H) \geq 2$. The results for the other two kinds of permutable 3-matching are the same except that $\mu(6) = 2$. In every case μ has two values.

Our matching method, in principle, cannot prove more because H has no permutable 4-matching (see Theorem 4.1). Nonetheless we suspect the dimension equals $|\operatorname{SpecC}(H)|$.

4.5 *Other graphs with permutable perfect matchings, and the cube*

Schaefer and Swartz found all graphs that have a permutable perfect matching. Besides K_n and $K_{p,p}$ they are the hexagon C_6, the octahedron graph O_3, and three general examples: the join $K_p \vee \overline{K}_p$ of a complete graph with its complement, the *matching join* $K_p \vee_M K_p$ obtained from two copies of K_p by inserting a perfect matching between the two copies, and the matching join $K_p \vee_M \overline{K}_p$, obtained by hanging a pendant edge from each vertex of K_p.

Our treatment of them leads us to one other family, the cyclic prisms $C_p \,\square\, K_2$.

4.5.1 The simple four

Trivially, $\dim \mathrm{NCV}(C_6) = 1 = |\operatorname{SpecC}(C_6)|$.

It is easy to verify by hand that O_3 satisfies the conditions of Corollary 3.5, so $\dim \mathrm{NCV}(O_3) = |\operatorname{SpecC}(O_3)| = 4$.

As for $K_p \vee_M \overline{K}_p$, since the pendant edges contribute nothing to cycles,

$$\operatorname{SpecC}(K_p \vee_M \overline{K}_p) = \operatorname{SpecC}(K_p) \text{ and } \mathrm{NCV}(K_p \vee_M \overline{K}_p) = \mathrm{NCV}(K_p);$$

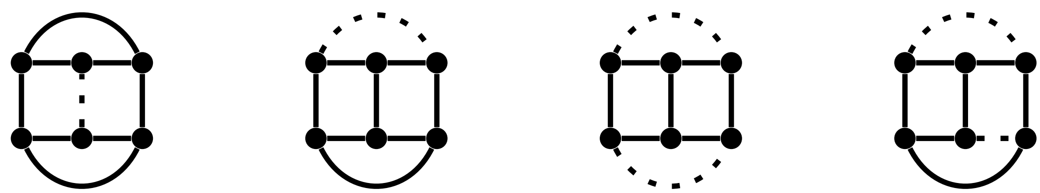
thence $\dim \mathrm{NCV}(K_p \vee_M \overline{K}_p) = |\operatorname{SpecC}(K_p \vee_M \overline{K}_p)| = p$.

It is also easy to show that $K_p \vee \overline{K}_p$ satisfies the conditions of Corollary 3.5. Thus, $\dim \mathrm{NCV}(K_p \vee \overline{K}_p) = |\operatorname{SpecC}(K_p \vee \overline{K}_p)| = 2p$.

4.5.2 The matching join $K_p \vee_M K_p$

This graph is pancyclic, but its permutable matchings are peculiar. One kind is any matching in a K_p. A maximum matching $M_{\lfloor p/2 \rfloor}$ in K_p, for which $\mu(l) = \min(p, \lfloor l/2 \rfloor)$, hence $\dim \mathrm{NCV}(K_p \vee_M K_p) \geq p$ by reasoning similar to that for K_p. The matching $M_p^\vee$ that joins the copies of K_p also prevents a permutable matching from having edges in both copies. The only other permutable matchings are subsets of $M_p^\vee$. This matching only generates $\lfloor p/2 \rfloor$ switching nonisomorphic signatures since negating a subset of $M_p^\vee$ switches to negating the complementary subset. By itself, therefore, choosing our grand matching M_m to be $M_p^\vee$ does not give a better lower bound than p. Nonetheless we feel the dimension is likely to be $n - 2 = 2p - 2$.

The smallest case, $K_3 \vee_M K_3$, is the triangular prism. The cycle count vector is $(c_3, c_4, c_5, c_6) = (2, 3, 6, 3)$. There are four unbalanced signatures; see Figure 3. The negative cycle vectors are linearly independent so $\dim \mathrm{NCV}(K_3 \vee_M K_3) = |\operatorname{SpecC}(v)|$, in agreement with Conjecture 1.1.



(0, 2, 4, 2) (1, 1, 3, 2) (2, 0, 6, 0) (2, 2, 2, 2)

Figure 3: The four unbalanced switching classes of the prism $K_3 \vee_M K_3$ and their negative cycle vectors.

4.5.3 Prisms, with cube

The triangular prism lends support to our belief that $\dim \mathrm{NCV}(K_p \vee_M K_p) = 2p - 2$. However, it is atypical since it is also a prism, $C_p \,\square\, K_2$ with $p = 3$. (Prisms with $p > 3$ do not have permutable perfect matchings but they make good examples.) The next prism is the cube, $Q_3 = C_4 \,\square\, K_2$. It is bipartite and has only three cycle lengths: 4, 6, and 8. Three unbalanced signatures whose negative cycle vectors are linearly independent are

- σ_1, with one negative edge, e. It has $c^-(\sigma_1) = (2, 8, 4)$;
- σ_2, with a second negative edge, parallel to e and sharing a quadrilateral with it. It has $c^-(\sigma_2) = (2, 12, 4)$;
- σ_3, with a second negative edge, also parallel to e but not in a common quadrilateral. It has $c^-(\sigma_3) = (2, 4, 2)$.

Thus, $\dim \mathrm{NCV}(Q_3) = |\operatorname{SpecC}(Q_3)|$, again agreeing with Conjecture 1.1.

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