# PROBLEMS AND SOLUTIONS

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*Proposed problems and solutions should be sent in duplicate to the* MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before December 31, 2004. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

## PROBLEMS

**11096**. *Proposed by Said Amghibech, Quebec, Canada.* Show that for each positive integer $n$ there exists a polynomial $P_n$ in $\mathbb{C}[x_1, \dots, x_n]$ such that, for every $n \times n$ matrix $A$ over $\mathbb{C}$, $\det A = P_n[\text{Tr } A, \text{Tr } A^2, \dots, \text{Tr } A^n]$. (For example, $x^3/6 - xy/2 + z/3$ will serve for $P_3$.)

**11097**. *Proposed by Kevin Ford, University of Illinois, Urbana, IL.* For any integer $n > 1$, let $\omega(n)$ denote the number of distinct prime factors of $n$, and $d(n)$ the number of divisors. For $1 \le i \le \omega(n)$ let $p_i$ be the $i$th smallest prime factor of $n$, and for $1 \le i \le d(n)$ let $d_i$ be the $i$th smallest positive divisor of $n$. Define $\nu(r)$ by $2^{\nu(r)} \mid r$ and $2^{1+\nu(r)} \nmid r$.
**(a)** Let $n$ be a product of $k$ distinct primes. For $1 \le j \le 2^k - 1$, let $t = \nu(j) + 1$. Prove that $d_{j+1}/d_j \le p_t$.
**(b)** Generalize (a) to the case of an arbitrary positive integer $n$.

**11098**. *Proposed by Christopher Hillar and Darren Rhea, University of California, Berkeley, CA.* Let

$$f(n) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2^i - 1} \binom{n}{i}.$$

Prove that there are constants $c$ and $c'$ such that $c \le f(n)/\log n \le c'$ for sufficiently large $n$ (that is, $f(n) = \Theta(\log n)$).

**11099**. *Proposed by Matthias Beck, San Francisco State University, San Francisco, CA, Richard Ehrenborg, University of Kentucky, Lexington, KY, and Thomas Zaslavsky, State Univ. of New York, Binghamton, NY.* A $3 \times 3$ square array $Q$ of nine distinct integers is *semimagic* if all the row and column sums are equal, and it is *magic* if, in addition, the two diagonals have the same sum as the rows and columns. We make a set of three-sided *row dice* from such a square as follows: the sides of die $i$ are

labelled with the numbers in row $i$. We say that die $i$ *beats* die $j$ if we expect die $i$ to show a larger number than die $j$ more than half the time.
(**a**) Prove that for every $3 \times 3$ magic square each row die beats exactly one other.
(**b**) Prove that the same holds for every $3 \times 3$ semimagic square with entries $1, \dots, 9$.
(**c**) Find a $3 \times 3$ semimagic square not satisfying the conclusion of parts (a) and (b).

**11100**. *Proposed by Călin Popescu, Romanian Academy Institute of Mathematics, Bucharest, Romania.* Given positive integers $m$, $n$, and $p$ satisfying $m \le n \le p$ and positive real numbers $\alpha, x_1, \dots$, and $x_p$, prove that

$$n^{m\alpha}\binom{p}{m}\sum_{I\subseteq\{1,\dots,p\},|I|=n}\left(\sum_{i\in I}x_i\right)^{-m\alpha} \le \binom{p}{n}\sum_{I\subseteq\{1,\dots,p\},|I|=m}\prod_{i\in I}x_i^{-\alpha}.$$

**11101**. *Proposed by Earl F. Skelton, George Washington University, Washington, D.C.* Show that

$$\int_0^\infty \frac{a}{\sqrt{a^2+x^2}}\tan^{-1}\left(\frac{b}{\sqrt{a^2+x^2}}\right)dx = \frac{a\pi}{2}\left[\log\left(b+\sqrt{b^2+a^2}\right)-\log a\right].$$

**11102**. *Proposed by Leroy Quet, Denver, CO.* Let $f(-1) = 1$, and for $m \ge 0$, let $f(m) = \prod_{k=0}^{\lfloor (m-1)/4\rfloor}(m-4k)$. If $a_m = \frac{(-1)^{m+1}(8m^2+1)(f(2m-3))^2}{2m(f(2m-1))^2}$ for $m \ge 1$, show that

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}} = \frac{2}{4-\pi}.$$

# SOLUTIONS

### Comparison of Some Squared Distances

**10970** [2002, 854]. *Proposed by Razvan Satnoianu, City University, London, U.K.* Let $ABC$ be an acute triangle and let $P$ be a point in its interior. Denote by $a, b, c$ the lengths of the triangle's sides, by $d_a, d_b, d_c$ the distances from $P$ to the triangle's sides, and by $R_a, R_b, R_c$ the distances from $P$ to the vertices $A, B, C$ respectively. Show that

$$d_a^2 + d_b^2 + d_c^2 \ge R_a^2\sin^2(A/2) + R_b^2\sin^2(B/2) + R_c^2\sin^2(C/2) \ge \tfrac{1}{3}(d_a+d_b+d_c)^2.$$

When is equality possible?

*Solution by John G. Heuver, Grande Prairie, AB, Canada.* Let $Q_a$, $Q_b$, $Q_c$ be the feet of the perpendiculars from $P$ to $BC, CA, AB$, respectively. The points $Q_b$ and $Q_c$ both lie on the circle with diameter $PA$, so the Extended Law of Sines and the Law of Cosines yield

$$R_a^2 = \frac{(Q_bQ_c)^2}{\sin^2 A} = \frac{d_b^2+d_c^2+2d_bd_c\cos A}{\sin A}.$$

Since $d_b^2 + d_c^2 \ge 2d_bd_c$, with equality if and only if $d_b = d_c$, we have $R_a^2\sin^2 A \le (d_b^2+d_c^2)(1+\cos A)$; hence, $\frac{1}{2}(d_b^2+d_c^2) \ge R_a^2\sin^2(A/2)$. Summing this and the two analogous inequalities yields

$$d_a^2 + d_b^2 + d_c^2 \ge R_a^2\sin^2(A/2) + R_b^2\sin^2(B/2) + R_c^2\sin^2(C/2),$$