

11099



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The American Mathematical Monthly, Vol. 111, No. 7. (Aug. - Sep., 2004), pp. 625-626.

Stable URL:

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Dennis Eichhorn, Tamás Erdélyi, Kevin Ford, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold R. Griggs, Jerrold Grossman, Kiran S. Kedlaya, Andre Kündgen, Frederick W. Luttman, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the inside front cover. Submitted solutions should arrive at that address before December 31, 2004. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk () after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11096. *Proposed by Said Amghibech, Quebec, Canada.* Show that for each positive integer n there exists a polynomial P_n in $\mathbb{C}[x_1, \dots, x_n]$ such that, for every $n \times n$ matrix A over \mathbb{C} , $\det A = P_n[\text{Tr } A, \text{Tr } A^2, \dots, \text{Tr } A^n]$. (For example, $x^3/6 - xy/2 + z/3$ will serve for P_3 .)

11097. *Proposed by Kevin Ford, University of Illinois, Urbana, IL.* For any integer $n > 1$, let $\omega(n)$ denote the number of distinct prime factors of n , and $d(n)$ the number of divisors. For $1 \leq i \leq \omega(n)$ let p_i be the i th smallest prime factor of n , and for $1 \leq i \leq d(n)$ let d_i be the i th smallest positive divisor of n . Define $\nu(r)$ by $2^{\nu(r)} \mid r$ and $2^{1+\nu(r)} \nmid r$.

(a) Let n be a product of k distinct primes. For $1 \leq j \leq 2^k - 1$, let $t = \nu(j) + 1$. Prove that $d_{j+1}/d_j \leq p_t$.

(b) Generalize (a) to the case of an arbitrary positive integer n .

11098. *Proposed by Christopher Hillar and Darren Rhea, University of California, Berkeley, CA.* Let

$$f(n) = \sum_{i=1}^n \frac{(-1)^{i+1}}{2^i - 1} \binom{n}{i}.$$

Prove that there are constants c and c' such that $c \leq f(n)/\log n \leq c'$ for sufficiently large n (that is, $f(n) = \Theta(\log n)$).

11099. *Proposed by Matthias Beck, San Francisco State University, San Francisco, CA, Richard Ehrenborg, University of Kentucky, Lexington, KY, and Thomas Zaslavsky, State Univ. of New York, Binghamton, NY.* A 3×3 square array Q of nine distinct integers is *semimagic* if all the row and column sums are equal, and it is *magic* if, in addition, the two diagonals have the same sum as the rows and columns. We make a set of three-sided *row dice* from such a square as follows: the sides of die i are

labelled with the numbers in row i . We say that die i beats die j if we expect die i to show a larger number than die j more than half the time.

- (a) Prove that for every 3×3 magic square each row die beats exactly one other.
 (b) Prove that the same holds for every 3×3 semimagic square with entries $1, \dots, 9$.
 (c) Find a 3×3 semimagic square not satisfying the conclusion of parts (a) and (b).

11100. Proposed by Călin Popescu, Romanian Academy Institute of Mathematics, Bucharest, Romania. Given positive integers m, n , and p satisfying $m \leq n \leq p$ and positive real numbers α, x_1, \dots , and x_p , prove that

$$n^{m\alpha} \binom{p}{m} \sum_{I \subseteq \{1, \dots, p\}, |I|=n} \left(\sum_{i \in I} x_i \right)^{-m\alpha} \leq \binom{p}{n} \sum_{I \subseteq \{1, \dots, p\}, |I|=m} \prod_{i \in I} x_i^{-\alpha}.$$

11101. Proposed by Earl F. Skelton, George Washington University, Washington, D.C. Show that

$$\int_0^\infty \frac{a}{\sqrt{a^2 + x^2}} \tan^{-1} \left(\frac{b}{\sqrt{a^2 + x^2}} \right) dx = \frac{a\pi}{2} \left[\log \left(b + \sqrt{b^2 + a^2} \right) - \log a \right].$$

11102. Proposed by Leroy Quet, Denver, CO. Let $f(-1) = 1$, and for $m \geq 0$, let $f(m) = \prod_{k=0}^{\lfloor (m-1)/4 \rfloor} (m - 4k)$. If $a_m = \frac{(-1)^{m+1}(8m^2+1)(f(2m-3))^2}{2m(f(2m-1))^2}$ for $m \geq 1$, show that

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}} = \frac{2}{4 - \pi}.$$

SOLUTIONS

Comparison of Some Squared Distances

10970 [2002, 854]. Proposed by Razvan Satnoianu, City University, London, U.K. Let ABC be an acute triangle and let P be a point in its interior. Denote by a, b, c the lengths of the triangle's sides, by d_a, d_b, d_c the distances from P to the triangle's sides, and by R_a, R_b, R_c the distances from P to the vertices A, B, C respectively. Show that

$$d_a^2 + d_b^2 + d_c^2 \geq R_a^2 \sin^2(A/2) + R_b^2 \sin^2(B/2) + R_c^2 \sin^2(C/2) \geq \frac{1}{3}(d_a + d_b + d_c)^2.$$

When is equality possible?

Solution by John G. Heuver, Grande Prairie, AB, Canada. Let Q_a, Q_b, Q_c be the feet of the perpendiculars from P to BC, CA, AB , respectively. The points Q_b and Q_c both lie on the circle with diameter PA , so the Extended Law of Sines and the Law of Cosines yield

$$R_a^2 = \frac{(Q_b Q_c)^2}{\sin^2 A} = \frac{d_b^2 + d_c^2 + 2d_b d_c \cos A}{\sin A}.$$

Since $d_b^2 + d_c^2 \geq 2d_b d_c$, with equality if and only if $d_b = d_c$, we have $R_a^2 \sin^2 A \leq (d_b^2 + d_c^2)(1 + \cos A)$; hence, $\frac{1}{2}(d_b^2 + d_c^2) \geq R_a^2 \sin^2(A/2)$. Summing this and the two analogous inequalities yields

$$d_a^2 + d_b^2 + d_c^2 \geq R_a^2 \sin^2(A/2) + R_b^2 \sin^2(B/2) + R_c^2 \sin^2(C/2),$$