The Canonical Vertex Signature and the Cosets of the Complete Binary Cycle Space

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Abstract. We consider two combinatorially simple alternatives to summation with even-degree edge sets for characterizing the class of edge sets E in K_n that have specified odd vertices: replacing a path inside E by a path outside it, or summing with a circle contained in E or in E^c. The latter is equivalent to summation with even-degree edge sets for almost all n, and the former is not quite similarly equivalent. The results help to understand the canonical vertex signature of a signed graph.

One way to transform a graph into another is to take the set sum of its edge set with that of a circle (circuit, cycle) contained in the graph or in its complement. (Call this operation *circle replacement*.) It is well known that a graph with all even degrees is an edge-disjoint union of circles; thus, it can be transformed into any other even-degree graph (on the same vertices) by repeated circle replacement. If we apply the same operation to a graph with some odd-degree vertices, can we get any other graph with the same odd vertices? The answer is that we can—but as is so often true in combinatorial problems, with just a few exceptions.

This transformation rule is a restriction of set summation with an arbitrary circle (*circle addition*), which is known to generate precisely all edge sets with the same odd vertices. The restriction is that we may use only permitted circles, and permission is determined by how the circle interacts with the given edge set E. The restriction in *circle replacement* is that the circle must lie within E or its complement. A somewhat similar restriction is to permit only a circle made up of two paths, one in E and the other outside it; call this *circular path shifting*. With circular path shifting we can get almost, but not quite any other edge set with the same odd vertices; there are now infinitely many exceptions which, fortunately, can be described exactly.

Signed graphs. I asked this question the better to understand the canonical vertex signature of a signed graph. A *signed graph* $\Sigma = (\Gamma, \sigma)$ consists of an underlying graph Γ (all our graphs are simple and undirected) and an edge signature $\sigma : E \to \{+, -\}$. Quite some time ago E. Sampathkumar introduced the idea of marking the vertices with signs derived from the edge signs [4, 5]. He defined the *canonical vertex signature* (or *canonical marking*)[1] associated with Σ, which is $\mu_\sigma : V \to \{+,-\}$ given by

$$\mu_\sigma(v) := \prod_{vw \in E} \sigma(vw).$$

Here are some obvious facts about μ_σ:

1. The number of negative vertices is even [5].
2. The negative vertices are the odd-degree vertices of the negative subgraph, which consists of all the vertices but only the negative edges of Σ.
3. The positive edge set has no effect on μ_σ. Thus, we may assume every signed graph is a signed K_n; the positive edge set is simply the complement of the set of negative edges.
4. Any vertex signature $\mu : V \to \{+,-\}$ that has evenly many negative vertices is canonical with respect to some signed graph whose vertex set is V.
5. There are a great many possible negative subgraphs that yield the same vertex signature μ_σ.

Recently there has been new interest in the canonical vertex signature in connection with deriving other signed graphs from a signed graph, in particular a signed line graph; see (in chronological order) [7, 6, 3, 2]. That led me to wonder how the many negative subgraphs with the same canonical vertex signature are related to each other. There is a well known, simple answer, but for understanding the graph theory behind the canonical vertex signature it is crude and unsatisfactory.

Even-degree edge sets. Instead of a signed graph Σ and its canonical sign function μ_Σ, let us change to the equivalent viewpoint that we have a subgraph of K_n and a vertex subset $T \subseteq V$ (which are related to Σ and μ by $E = \sigma^{-1}(-)$ and $T = \mu^{-1}(-)$).

Write $\partial(E)$ for the set of odd-degree vertices of a subset $E \subseteq E(K_n)$. An edge set E whose odd-degree vertex set is T is known as a *T-join*. For $T \subseteq V$ let $\langle T \rangle := \partial^{-1}(T)$, the class of all T-joins in K_n, and let $[E] := \partial^{-1}\partial(E)$, the class of all $\partial(E)$-joins, i.e., edge sets E' that have the same odd-degree vertices as E. For instance, $\langle \varnothing \rangle = [\varnothing]$ is the class of all even-degree edge sets, or $\varnothing$-joins, in K_n (widely known as the *binary cycle space* of K_n), and $\langle V \rangle$ is the class of V-joins, which are the edge sets with all degrees odd (this class is void if n is even). We call the classes $[E]$ *join classes*; they are the equivalence classes of an equivalence relation on $\mathcal{P}(E(K_n))$ which we call *join equivalence*.

We may now add to the obvious facts about μ_σ, interpreted as its set T of negative vertices, one more observation:

6. The join class $\langle T \rangle$ contains every matching of the vertices of T; indeed, the smallest elements of the join class are the matchings.

Two edge sets E and F are join equivalent if and only if their set sum (i.e., symmetric difference) $E \oplus F$ has even degree at every vertex; therefore,

$$[E] = E \oplus [\varnothing] := \{E \oplus S : S \in [\varnothing]\}. \tag{1}$$

Consequently, join equivalence can be regarded as the relation implied by a particular operation that changes one set $E \in \langle T \rangle$ into another: the operation of replacing E by $E \oplus S$ for any $S \in [\varnothing]$ (*even addition*). The strength of even addition is that it depends not at all

[1] "Marking" in signed graph theory is synonymous with "vertex signature". I am informed [1] that B.D. Acharya suggested the name "canonical", which appeared first in [7].

on the graph structure of E; but this is at the same time a weakness, since even addition gives little insight into the combinatorial relationships amongst the members of the class $[E]$. Besides, in the signed-graph interpretation, since E was given as the set of negative edges of a specific signed graph I want rules for modifying E that pay attention to the structure of the initial set E.

There will be a cost to having such a rule. A modification rule implies an equivalence relation on $\mathcal{P}(E(K_n))$, E and E' being equivalent if the rule permits transforming E into E'. A new rule may imply a new equivalence relation, different from join equivalence, thus losing the close connection with the join classes $\langle T \rangle$. Consequently, if we propose a new transformation rule, we must determine how its equivalence classes compare with the join classes.

Now let us imagine some plausible transformation rules that do involve the graphical structure of E. A guide is the fact that $E \in \langle \varnothing \rangle$ if and only if it is an edge-disjoint union of circles. Thus, a natural new rule is to allow changing E only to $E \oplus C$ where C is any circle in K_n (*circle addition*). This method is obviously equivalent to even addition because every $S \in \langle \varnothing \rangle$ is the set sum of circles; worse, it too has nothing to do with the structure of E. We want an operation that depends on E. We find candidates by restricting the circles that can be summed with a set E.

A first restriction is to replace E by $E \oplus C$ only when C is a circle contained entirely in E or in the complement E^c. This is the *circle replacement* defined in the introduction; its equivalence classes, written $[E]_{\mathcal{C}}$, are the *circle classes* and its equivalence relation is *circle equivalence.*

Circular path shifting, also mentioned in the introduction, sums E with a circle that is composed of two nontrivial paths, one in E and the other in E^c. That is, we replace a path P in E by a path Q in E^c with the same endpoints, under the assumption that $P \cup Q$ is a circle. A liberalized version is *path shifting*, where any path in E can be replaced by any path in E^c with the same endpoints. The equivalence classes $[E]_{\mathcal{P}}$ are *path classes.* Since the parities of the degrees do not change, the path classes refine the join classes; and the circular path classes obviously refine the path classes.

The question we need to answer now is: How different are the equivalence classes of the various methods? We consider path shifting first.

Lemma 1. *The equivalence classes under circular path shifting are the path classes.*

Proof. We prove, by induction on the number of internal points of intersection of the two paths, that path shifting can be expressed as a sequence of circular path shifts.

We shift a uw-path $P \subseteq E$ to a uw-path $Q \subseteq E^c$, the result being $E' = (E \setminus P) \cup Q$. If P and Q are internally disjoint, we have circular path shifting. Suppose, then, that they are not. Let x be the first vertex on Q at which it meets P. Let Q_{ux} be the initial segment of Q, from u to x and let P_{ux} be the part of P from u to x. Replacing P_{ux} by Q_{ux} is a circular path shift. We are left to replace P_{xw} by Q_{xw}, but these two paths have fewer internal common vertices, so by induction the replacement can be done by circular path shifting. $\square$

We now have to determine the exact equivalence classes under path shifting and circle replacement. To do so, we find the reduced members of each equivalence class. With respect to a particular method of modification, a set $E \subseteq E(K_n)$ is *reduced* if no other member of its equivalence class has fewer edges. For instance, $E \in \langle T \rangle$ is reduced with respect to even addition (that is, it is a minimum T-join) if and only if it is a matching of the vertices of T.

Reduced members are, in a sense, canonical representatives of an equivalence class. Finding the reduced edge sets is the main step towards answering the original question.

A path of order k (and length $k-1$) is P_k.

Lemma 2. *An edge set in K_n is reduced in its circle class if and only if it is a matching, or $n=3$ or 4 and the edge set is a P_n.*

Proof. Assume that $E \subseteq E(K_n)$ is reduced under circle replacement. It must be a forest, since any circle in it can be eliminated immediately.

Now we show that circle replacement implies a restricted type of circular path shifting. Suppose $E \supseteq P$, a path with endpoints u, v, and $E^c \supseteq Q, R$, two paths with the same endpoints u, v, and assume that $P \cup Q \cup R$ is a theta graph. We may add $Q \cup R$ to E and then subtract $P \cup R$, with the result that P is replaced by Q in E. This is circular path shifting, with the added requirement that there is an extra path R in E^c with the same endpoints as P and Q and internally disjoint from them.

Suppose E contains a P_3, say uvw; since E is acyclic, it does not contain the edge uw. If there is a vertex z nonadjacent to u and w, then $P = uvw$, $Q = uw$, and $R = uzw$ permit us to shift P to the shorter path Q, thus shrinking E. Therefore, a reduced set E cannot contain a path uvw of length two, unless every other vertex is a neighbor of u or w (but not both, since E is acyclic). The conclusion is that, if E is not a matching, then it contains vertices u, v, w such that $uvw \subseteq E$ and there is a bipartition $V \setminus \{u, v, w\} = Y \cup Z$ (a disjoint union) for which every $y \in Y$ is adjacent to u (but not to v, w) and every $z \in Z$ is adjacent to w (but not to u, v). (Y or Z or both may be empty.)

If Y and Z are nonempty, consider a path vuy where $y \in Y$. Any $z \in Z$ provides us with the extra path $R = vzy \subseteq E^c$ that we need to replace vuy by vy, thereby shrinking E. As E is reduced, that cannot be. Therefore, we may assume $Z = \varnothing$. Now there are three cases for Y: $n=3$ and $Y = \varnothing$, $n=4$ and $|Y| = 1$, and $n \geq 5$ with $|Y| \geq 2$.

If $n = 3$, E is a P_3. There is no possible circle replacement because neither E nor its complement contains a circle.

If $n = 4$, then $n = 4$ and both E and E^c are P_4's. Again, there can be no circle replacement. In both cases, E is reduced (and is its own equivalence class).

If $n \geq 5$, there are distinct vertices $y, y' \in Y$; then the path $P = yuy' \subseteq E$ and the path $R = ywy' \subseteq E^c$ fulfill the conditions necessary for us to replace P by yy' in E. But, again, that is impossible, so E must be a matching.

We have shown that every reduced set is as in the lemma, and every set described in the lemma is reduced. ∎

Lemma 3. *An edge set in K_n is reduced in its path class if and only if it is $\varnothing$, $E(K_n)$, a triangle, or a matching.*

Proof. First, $\varnothing$ and $E(K_n)$ are reduced because no path shifting is possible. They are their own path classes.

Let us assume now that $E \subset E(K_n)$ is reduced and nonvoid; we prove it is a triangle or a matching. Any induced P_3 in E can be transformed by path shifting to a single edge with the same endpoints; the shifting reduces $|E|$. Thus, in E every component is complete. If a component $F = E(K_k)$ with $k \geq 4$, F contains a path P_k, say with endpoints u, w; and there is a vertex $x \notin V(F)$. Replace P_k by $P_3 = uxw$. This reduces $|E|$; therefore, no component of G can have order more than 3. If a component is a triangle, say with vertices x, y, z, and there is another edge uv in E, then replace the path xz by xvz; replace $uvxy$ by uy; and we have fewer edges; thus, E was not reduced.

If E is a matching, clearly it is reduced.

Suppose E is a triangle; we must prove it cannot be reduced to a smaller edge set. One thing path shifting can never do is to eliminate all edges; therefore, whatever the result of operating on E, it has at least one edge and all even degrees. That is only possible if E has at least three edges. Thus, a triangle is irreducible under path shifting. $\square$

Theorem 4. *The circle classes are the join classes, except that when $n = 3, 4$, each join class $\langle T \rangle$ for $|T| = 2$ splits into the following circle classes:*

(i) $[P]_{\mathcal{C}} = \{P\}$*, for each $P \in \langle T \rangle$ that is a P_n with endpoints T, and*

(ii) $\langle T \rangle \setminus \{P \in \langle T \rangle : P \text{ is a } P_n\}$.

When $n = 3$, there is one P_3 and the class in (ii) contains only one edge set, $\{uv\}$ where uv is the edge joining the vertices of T. When $n = 4$, there are two P_4's and the class in (ii) contains several edge sets.

Proof. The proof strategy is to find the equivalences amongst the sets that are reduced under circle replacement.

First, we show that any matching of a particular even set $T \subseteq V$ transforms into any other under circle replacement. We may assume $|T| \geq 4$. Suppose M is one matching of T with edges uv and wx (among others). The paths uv in M and uxv, uwv in M^c permit us to shift uv to uxv, thereby transforming $\{uv, wx\}$ into the star with center x and leaves u, v, w. This star transforms into any matching of u, v, w, x. By repeating this process on pairs of edges, any matching of T does transform into any other.

We conclude from this and Lemma 2 that the class $[E]_{\mathcal{C}} = [E]$ for any edge set E, except when $n = 3$ or 4 and $[E]$ contains a P_n. In that case $[E] = \langle T \rangle$ for some 2-element vertex set $T = \{u, v\}$, and $[E]$ splits into circle classes, each of which must contain an edge set G that is reduced under circle replacement. The only reduced edge sets are $\{uv\}$ and the path uxv if $V = \{u, v, x\}$ ($n = 3$) or paths $uxyv$ and $uyxv$ if $V = \{u, v, x, y\}$ ($n = 4$). Each of the latter three paths is its own class, since both P_n and P_n^c are acyclic. Therefore, the classes $[E]_{\mathcal{C}}$ must be as stated in the theorem. $\square$

Theorem 5. *The equivalence classes under path shifting and circular path shifting are the same. They are the join classes, except that*

(i) *when $n \geq 3$ is odd, $[\varnothing]$ splits into $[\varnothing]_{\mathcal{P}} = \{\varnothing\}$, $[E(K_n)]_{\mathcal{P}} = \{E(K_n)\}$, and (if $n > 3$) $[\varnothing] \setminus \{\varnothing, E(K_n)\}$ (the path class of the triangles of K_n);*

(ii) *when $n \geq 4$ is even, $[\varnothing]$ splits into $[\varnothing]_{\mathcal{P}} = \{\varnothing\}$ and $[\varnothing] \setminus \{\varnothing\}$, and $\langle V \rangle = [E(K_n)]$ splits into $[E(K_n)]_{\mathcal{P}} = \{E(K_n)\}$ and $\langle V \rangle \setminus \{E(K_n)\}$ (the path class of the perfect matchings of K_n).*

Proof. The proof follows the same strategy as that of Theorem 4. We noted in the course of proving Lemma 3 that $\varnothing$ and $E(K_n)$ are their own equivalence classes. The theorem follows if all triangles are equivalent.

Consider two overlapping triangles $E = \{vw, wu, uv\}$ and $E' = \{vw, wx, xv\}$. They differ in a path $wuv \subseteq E$ and a path $wxv \subseteq E^c$. Therefore, they are equivalent under shifting; we conclude that all triangles are equivalent.

The rest of the theorem is a matter of noting that for very small values of n some of the possible path classes are empty. $\square$

Circle replacement strikes me, from the philosophical viewpoint, as the most relevant to the original problem of canonical vertex signatures. The reasons are that the modifying circle is homogeneous (all positive or all negative), and that the modification, changing only a circle, is of a most elementary kind. The next best operation is circular path shifting, because two paths are shifted as a whole between sign classes and they may not intersect except for their endpoints; but the theorems show it gives too weak an equivalence.

None the less it is remarkable that the five methods produce the same equivalence classes when $n \geq 5$ with the sole exception that under path shifting $[\varnothing]$ and (if n is odd) $\langle V \rangle$ both split. I interpret this to mean that circle equivalence is a very good combinatorialization of join equivalence of subsets of $E(K_n)$.

A remark on signed multigraphs. We assumed signed graphs were without loops or multiple edges. However, allowing such edges makes no important difference to our results. The positive edges can be discarded. A negative loop, since it has two ends, contributes two negatives to the canonical sign of its vertex; thus, it has no effect on μ_σ; and it can be eliminated by either circle addition or circle replacement, though not by path shifting. A completely negative digon contribures two negatives at each end, so it also has no effect on μ_σ; it too can be discarded by circle addition or circle replacement, and usually by path shifting (an example is $\{uv, vw, vw\} \to \{uw, vw\} \to \{uv\}$). Thus, in two of the three essentially different methods negative loops and doubled negative edges can be eliminated, reducing $|\Sigma|$ to a simple graph to which our theorems apply.

General ambient graph. Our results are framed within a particular *ambient graph*, K_n. How do the five methods of transforming a T-join compare within an arbitrary ambient graph Γ? It is easy to see that even addition and circle addition remain equivalent; path shifting and circular path shifting also remain equivalent. It is also clear that circle addition can transform any T-join in Γ into any other. But circle replacement and path shifting are sure to be more complicated because the reduced T-joins will not necessarily be matchings when the ambient graph is incomplete.

A first question is this: Might the theorems remain true, with minor changes, within any edge 4-connected, or 4-connected, incomplete ambient graph? I choose 4 because K_5 is, and K_4 is not, edge 4-connected.

Homology. The cosets of $[\varnothing]$ have the flavor of homology. One would like to have a homology of graphs in which the homology classes are the join classes. That requires a chain complex $\mathbf{C} = (C_i(G; \mathbb{F}_2))_i$ such that $Z_1(\mathbf{C}) = \mathcal{P}(E)$ and $B_1(\mathbf{C}) = [\varnothing]$; then $H_1(\mathbf{C})$ will be $\mathcal{P}(E)/[\varnothing]$, the group of join classes.

The usual binary homology theory of a graph G is not suitable. Its chain groups are $C_0 := \mathbb{F}_2^V \cong \mathcal{P}(V)$ and $C_1 := \mathbb{F}_2^E \cong \mathcal{P}(E)$, supplemented by $C_2 = \{0\}$. The boundary mappings are the homomorphisms $\partial_1 := \partial : C_1 \to C_0$ and $\partial_2 : C_2 \to C_1$, which is the zero function. The 1-cycle space $Z_1(G; \mathbb{F}_2)$ is $\operatorname{Ker} \partial_1 = [\varnothing]$ (i.e., it is the binary cycle space); the 0-coboundary space $B_0(G; \mathbb{F}_2)$ is $\operatorname{Im} \partial_1 = \{T \subseteq V : |T| \text{ is even}\}$; and $B_1(G; \mathbb{F}_2)$ is trivial. The first homology group $H_1(G; \mathbb{F}_2)$ is then $Z_1/B_1 = Z_1 = [\varnothing]$. But the cosets of $[\varnothing]$, which are what we want, are not part of this homology theory. They live in the quotient space $C_1/Z_1 \cong \mathcal{P}(E)/[\varnothing]$. Rather than graph homology, it is the general algebraic property that $\operatorname{Dom} \partial_1 / \operatorname{Ker} \partial_1 \cong \operatorname{Im} \partial_1 = B_0$ which "explains" the fact that the cosets of $[\varnothing]$ in C_1 correspond to the even vertex sets, $T \in B_0$. The chain complex necessary to make this into homology eludes me.

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