Characterizations of Signed Graphs*

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ABSTRACT

The possible classes of balanced circles of a signed graph are characterized in two ways.

A signed graph is a graph with arcs signed $+$ or $-$; a circle is balanced if the product of its arc signs is $+$. I give here two characterizations of the possible classes of balanced circles of a signed graph: an elementary one of the balanced portion of an arbitrary subclass of circles, and a stronger one of the entire balanced circle class. The latter characterizes signed graphs among biased graphs (explained in [9]).

Terminology. A *signed graph* Σ consists of an ordinary graph Γ (finite or infinite) with node set N and arc set E, and a mapping $\sigma: E \rightarrow \{+, -\}$, the *sign labeling*. Loops and multiple arcs are allowed (but we omit the half arcs and free loops needed in other parts of signed graph theory [8]). A path has a value obtained by multiplying the signs of its constituent arcs; a circle whose value is $+$ is called *balanced*. An arc set is called balanced when every circle in it is balanced. The class of circles of Γ is denoted $\mathscr{C}(\Gamma)$; the class of circles balanced in Σ is written $\mathscr{B}(\Sigma)$. (Signed graphs and balance were first conceived by Harary [3].) See Figure 1 for illustrations of signed graphs.

First Characterization. When is a class of circles equal to $\mathscr{B}(\Sigma)$ for some Σ? A generalization: given a certain class $\mathscr{D}$ of circles of Γ, when is a subclass $\mathscr{B}$ equal to the balanced subclass of $\mathscr{D}$ in some sign labeling of Γ?

To solve the problem we look first at the binary vector space $\mathscr{P}$ of all subsets of $E(\Gamma)$, whose addition is the symmetric difference Δ. If $\mathscr{D} \subseteq \mathscr{P}$, we can speak of independent and spanning subsets of $\mathscr{D}$ ("spanning" means spanning $\mathscr{D}$). To say a subset $\mathscr{B}$ is *additive in* $\mathscr{D}$ means that whenever $C, C_1, \ldots, C_r \in \mathscr{D}$ and $C = C_1 \Delta \cdots \Delta C_r$, then $C \in \mathscr{B}$ if and only if an even number of $C_1, \ldots, C_r$ are not in $\mathscr{B}$. This is equivalent to saying that $\mathscr{B}$ equals either $\mathscr{D}$ or the intersection of $\mathscr{D}$ with a hyperplane (codimension

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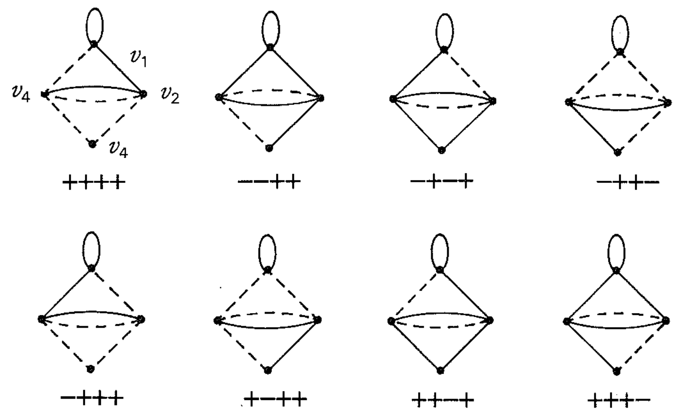

FIGURE 1. *A signed graph and all members of its switching class.* Solid lines are negative arcs, dashed lines are positive. The original graph Σ is in the upper left. Under each graph is a switching function by means of which it is obtained from Σ, displayed as an ordered quadruple $\nu(1)\nu(2)\nu(3)\nu(4)$. (The function $-\nu$ will also do.) The balanced circle class $\mathscr{B}(\Sigma)$ consists only of the triangle $v_1v_2v_4$.

1) of $\mathscr{P}$; in other words that $\mathscr{B}$ is the kernel of a *homomorphism on* $\mathscr{D}$, a map $\delta: \mathscr{D} \to \{+, -\}$ for which

$$\left.\begin{array}{l} C, C_1, \ldots, C_r \in \mathscr{D} \\ C = C_1 \,\Delta\, \cdots \,\Delta\, C_r \end{array}\right\} \Rightarrow \delta(C) = \delta(C_1) \cdots \delta(C_r).$$

The first lemma is now straightforward.

Lemma 1. Let Γ be a graph, $\mathscr{D} \subseteq \mathscr{C}(\Gamma)$, and $\delta: \mathscr{D} \to \{+, -\}$.

(a) δ extends to a homomorphism on $\mathscr{C}(\Gamma)$ if and only if it is a homomorphism on $\mathscr{D}$. It extends uniquely if and only if $\mathscr{D}$ spans $\mathscr{C}(\Gamma)$.

(b) For there to be a signed graph $\Sigma = (\Gamma, \sigma)$ such that $\sigma | \mathscr{D} = \delta$, it is necessary and sufficient that δ be a homomorphism on $\mathscr{D}$. ∎

By setting δ equal to the *signed characteristic function of* $\mathscr{B}$ in $\mathscr{D}$, the mapping $\beta: \mathscr{D} \to \{+, -\}$ defined by $\beta(C) = +$ if $C \in \mathscr{B}$, $\beta(C) = -$ if $C \notin \mathscr{B}$, we obtain a solution of the characterization problem.

Theorem 2. Let Γ be a graph and $\mathscr{B} \subseteq \mathscr{D} \subseteq \mathscr{C}(\Gamma)$. For there to be a signed graph Σ on Γ such that $\mathscr{B} = \mathscr{B}(\Sigma) \cap \mathscr{D}$, it is necessary and sufficient that $\mathscr{B}$ be additive in $\mathscr{D}$. The class $\mathscr{B}(\Sigma)$ is uniquely determined if, and only if, $\mathscr{D}$ spans $\mathscr{C}(\Gamma)$.

It follows that if Σ is a signed graph and $\mathscr{D}$ is a spanning subset of $\mathscr{C}(\Sigma)$, then $\mathscr{B}(\Sigma)$ is determined even if we only know $\mathscr{B}(\Sigma) \cap \mathscr{D}$. If $\mathscr{D}$ is a basis for $\mathscr{C}(\Gamma)$, then any subset $\mathscr{B}$ determines a signed graph Σ such that $\mathscr{B} = \mathscr{B}(\Sigma) \cap \mathscr{D}$, and the class $\mathscr{B}(\Sigma)$ is unique.

In dealing with two-graphs (cf. [5]) and generalizations it is helpful to have criteria for additivity. Call a set of circles $\{C, D_1, \ldots, D_r\}$ a *generating relation for C in $\mathscr{D}'$* if all $D_i \in \mathscr{D}'$ and the set has sum $\emptyset$.

Lemma 3. Let Γ be a graph and let $\mathscr{B}' \subseteq \mathscr{D}' \subseteq \mathscr{D} \subseteq \mathscr{C}(\Gamma)$, where $\mathscr{D}'$ spans $\mathscr{D}$ and $\mathscr{B}'$ is additive in $\mathscr{D}'$. Choose a fixed generating relation in $\mathscr{D}'$ for each $C \in \mathscr{D} \setminus \mathscr{D}'$. Define $\mathscr{B} \subseteq \mathscr{D}$ by $\mathscr{B} \cap \mathscr{D}' = \mathscr{B}'$ and $C \in \mathscr{D} \setminus \mathscr{D}'$ is in $\mathscr{B}$ if and only if an even number of members of its chosen generating relation are in $\mathscr{D}' \setminus \mathscr{B}'$. Then $\mathscr{B}$ is additive in $\mathscr{D}$.

Proof. Suppose $C_1 \Delta \cdots \Delta C_p = \emptyset$, where $C_i \in \mathscr{D}$. Taking the chosen generating relation for each C_i, C_i is a sum $D_{i1} \Delta \cdots \Delta D_{ir_i}$ where the D_{ij} are in $\mathscr{D}'$ and by hypothesis $\beta(C_i) = \Pi_j \beta(D_{ij})$. Respectively summing and multiplying, we have

$$\sum_{i,j} D_{ij} = \emptyset, \quad \prod_i \beta(C_i) = \prod_{i,j} \beta(D_{ij}).$$

That a sum of basis elements equals $\emptyset$ implies that each one appears an even number of times; hence the double product is $+$, which implies $\mathscr{B}$ is additive. ∎

It may be that all the chosen generating relations have in some sense the same form. Then one can throw into the hypotheses the assumption that all relations in $\mathscr{D}$ of that form contain an even number of nonmembers of $\mathscr{B}$. For instance let us say *$\mathscr{D}'$ k-generates $\mathscr{D}$* if every $C \in \mathscr{D} \setminus \mathscr{D}'$ has a generating relation in $\mathscr{D}'$ of size k or less. Call *$\mathscr{B}$ k-additive in $\mathscr{D}$* if it is additive for sums $C = C_1 \Delta \cdots \Delta C_r$ where $r < k$.

Corollary 4. Let $\mathscr{D}' \subseteq \mathscr{D} \subseteq \mathscr{C}(\Gamma)$, where $\mathscr{D}'$ k-generates $\mathscr{D}$. Suppose $\mathscr{B}$ is k-additive in $\mathscr{D}$ and $\mathscr{B} \cap \mathscr{D}'$ is additive in $\mathscr{D}'$. Then $\mathscr{B}$ is additive in $\mathscr{D}$.

Proof. Let $\mathscr{B}' = \mathscr{B} \cap \mathscr{D}'$ and choose a k-generating relation for each $C \in \mathscr{D} \setminus \mathscr{D}'$. Then apply Lemma 3. ∎

Second Characterization. A characterization of all possible $\mathscr{B}(\Sigma)$ follows from Theorem 2 by setting $\mathscr{D} = \mathscr{C}(\Gamma)$. But a stronger result is a consequence of Lemma 5. Say $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$ is *circle additive* if, whenever C_1 and C_2 are circles for which $C_1 \cup C_2$ is a θ graph and C_1 and C_2 are both in $\mathscr{B}$ or both not in $\mathscr{B}$, then $C_1 \Delta C_2 \in \mathscr{B}$.

Lemma 5. A class $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$ is additive in $\mathscr{C}(\Gamma)$ if and only if it is circle additive.

Proof. In the proof of the "if" half (the converse is trivial) we may confine ourselves to the finite case because any counterexample would necessarily be finite. Thus we can assume that any graph smaller than Γ satisfies the lemma.

Let us suppose there were a counterexample in Γ; that is, $C_1 \Delta \cdots \Delta C_r = \emptyset$ but an odd number of the circles $C_1, \ldots, C_r$ are not in $\mathscr{B}$. Let e be an arc in $C_1 \cup \cdots \cup C_r$. The number of $C_1, \ldots, C_r$ which contain e is even, so we can pair them off. We shall prove that each pair, say C and C', can be replaced in the sum by several circles $D_1, \ldots, D_k$, not containing e, such that $D_1 \Delta \cdots \Delta D_k = C \Delta C'$ and the parity of the number of D_i not in $\mathscr{B}$ equals that of the number among C, C' which are not in $\mathscr{B}$. Thus we can reduce the presumed counterexample to one in which no circle contains e. As $\mathscr{B} \cap \mathscr{C}(\Gamma \setminus e)$ is circle additive, this contradicts the induction assumption.

So let C, C' be two circles among $C_1, \ldots, C_r$ which contain e. By Tutte's path theorem (cf. [7], 4.34, or the dual form in [2], Section 15, Theorem 1), there is a chain of circles

$$C = D_0', D_1', \ldots, D_k' = C'$$

such that $e \in D_1', \ldots, D_{k-1}'$ and all $D_{i-1}' \cup D_i'$ are θ graphs. Thus $D_i = D_{i-1}' \Delta D_i'$ is a circle in $\Gamma \setminus e$, and $D_1 \Delta \cdots \Delta D_k = C \Delta C'$. Circle additivity implies that $\beta(D_i) = \beta(D_{i-1}')\beta(D_i')$. Hence $\beta(D_1) \cdots \beta(D_k) = \beta(C)\beta(C')$, which is the parity condition we needed. Thus $D_1, \ldots, D_k$ exist as we required for the proof. ∎

From Lemma 5 applied to Theorem 2 with $\mathscr{D} = \mathscr{C}(\Gamma)$ we have a strong solution to the characterization problem for balanced circles.

Theorem 6. Let Γ be a graph and $\mathscr{B} \subseteq \mathscr{C}(\Gamma)$. There is a signed graph Σ on Γ such that $\mathscr{B}(\Sigma) = \mathscr{B}$ if and only if $\mathscr{B}$ is circle additive.

Switching. Suppose $\nu: N \to \{+, -\}$ is any sign function. *Switching Σ by ν* means forming the switched graph $\Sigma^\nu = (\Gamma, \sigma^\nu)$, whose underlying graph is the same but whose sign function is defined on an arc e: vw by

$$\sigma^\nu(e) = \nu(v)\sigma(e)\nu(w).$$

This is the signed-graphic version of the graph switching originated by van Lint and Seidel [4]. Adapting Seidel's terminology (cf. [5]) we call Σ_1 and Σ_2 *switching equivalent* if there is a switching function ν such that $\Sigma_1 = \Sigma_2^\nu$. The equivalence class of Σ is called its *switching class*. Figure 1 shows a switching class of signed graphs. An easy theorem (proved in [8], Theorem 3.2) is

Theorem 7. Two signed graphs on the same underlying graph are switching equivalent if and only if they have the same list of balanced circles. ∎

A corollary of Theorem 2 by way of Theorem 7 is useful in connection with the combinatorics of two-graphs.

Corollary 8. Let $\mathscr{D}$ be a spanning subset of $\mathscr{C}(\Gamma)$. There is a one-to-one correspondence between switching classes of signed graphs on Γ and additive subsets of $\mathscr{D}$. Two signed graphs on Γ, Σ_1 and Σ_2, are switching equivalent if and only if $\mathscr{B}(\Sigma_1) \cap \mathscr{D} = \mathscr{B}(\Sigma_2) \cap \mathscr{D}$.

Incidentally, the method of Cameron ([1], Sec. 8; or see [6], Sec. 2) establishes the equality for each finite graph Γ of the numbers of switching classes of signed graphs on Γ and of Eulerian subgraphs of Γ. (Eulerian graphs here are graphs with even degree, not necessarily connected. The statement is true for labeled graphs, and also for Γ-automorphism equivalence classes.)

References

[1] P.J. Cameron, Cohomological aspects of two-graphs. *Math. Z.* 157 (1977) 101–119. MR 58 #21779.

[2] H.H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary ed. MIT, Cambridge, Mass. (1968). MR 45 #74.

[3] F. Harary, On the notion of balance of a signed graph. *Michigan Math. J.* 2 (1953–1954) 143–146; Addendum, 2 (1953–1954) preceding p. 1. MR 16, 733.

[4] J.H. van Lint and J.J. Seidel, Equilateral point sets in elliptic geometry. *Proc. K. Ned. Akad. Wet. Ser. A* 69 (*Indag. Math.* 28) (1966) 335–348. MR 34 #685.

[5] J.J. Seidel, A survey of two-graphs, in *Atti Colloq. Internaz. Teorie Combinatorie* (Roma, 1973). Accademia Nazionale Lincei, Rome (1976), Vol. I, pp. 481–511. MR 58 #27659.

[6] J.J. Seidel and D.E. Taylor, Two-graphs, a second survey. In *Proceedings of the International Colloquium on Algebraic Methods in Graph Theory*, Szeged, 1978. To appear.

[7] W.T. Tutte, Lectures on matroids. *J. Res. Natl. Bur. Stand. Sect. B* 69B (1965) 1–47 [in *Selected Works of W.T. Tutte*, D. McCarthy and R. G.

Stanton, Eds. Charles Babbage Research Centre, St. Pierre, Canada (1979), Vol. II, pp. 450–496, with author's preface, pp. 439–449.
[8] T. Zaslavsky, Signed graphs. *Discrete Appl. Math.* 4 (1982). To appear.
[9] T. Zaslavsky, Biased graphs. Unpublished.